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CORONA TYPE DECOMPOSITION IN SOME BESOV SPACES

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Abstract.

Let $g_1, \dots, g_m$ be holomorphic functions on the unit ball of $\mathbf{C}^n$ and consider the map $M_g: H(B) \times \dots \times H(B) \to H(B)$ defined by

$$M_g(f)(z) = \sum_{j=1}^{m} g_j(z) f_j(z).$$

Let us denote by $\mathscr{E}$ either the usual Bloch space or the Sobolev space W_1^1 intersection the space of holomorphic functions.

In tnis paper we prove that necessary and sufficient conditions on $g_1, \dots, g_m$ such that M_g maps $\mathscr{E} \times \dots \times \mathscr{E}$ onto $\mathscr{E}$ are that the functions g_j are multipliers of the space $\mathscr{E}$ and $\sum_{j=1}^{m} |g_j(z)|^2 \geqq \delta > 0$.

1. Introduction.

Let $g = (g_1, \dots, g_m)$ be a vector valued holomorphic function in the unit ball B of $\mathbf{C}^n$. We consider the application $M_g: H(B) \times \dots \times H(B) \to H(B)$ defined by $M_g(f) = \sum_{j=1}^{m} g_j f_j$. It is well known that if the functions g_j have not common zeroes then the map M_g is surjective. Many authors have considered this problem replacing the space of holomorphic functions on B by other subspaces and the unit ball for other domains. The motivating work of our subject is the so called H^p-corona problem:

Let g_j be bounded functions satisfying

$$|g(z)|^2 = \sum_{j=1}^{m} |g_j(z)|^2 \geqq \delta > 0.$$

The problem consists to show that the map

$$M_g: H^p(B) \times \dots \times H^p(B) \to H^p(B)$$

is surjective. ([A], [AN], [AN-CA 1,2], [LI], [LIN]).

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Of course, the most interesting and difficult case is $p = \infty$ which corresponds to the classical corona problem, that up to now has been solved only for dimension 1.

Our goal is to replace the Hardy spaces for some Besov spaces closely related with them. We consider the spaces of holomorphic functions $A^p_{p,1}(B)$, $1 \leqq p \leqq \infty$, with norm

$$\|f\|_{p,p,1} = \left(\int_B |(I + R)f(z)|^p (1 - |z|^2)^{p-1}\, dV(z)\right)^{\frac{1}{p}}, \quad 1 \leqq p < \infty$$

$$\|f\|_{\infty,\infty,1} = \sup\{|(I + R)f(z)|(1 - |z|^2); z \in B\}$$

where I denotes the identity operator, R the radial derivative $R = \sum_{j=1}^n z_j \dfrac{\partial}{\partial z_j}$ and $dV(z)$ the normalized volume element.

We recall that if $p \leqq 2$ then $A^p_{p,1} \subset H^p$ and that if $p \geqq 2$ then $H^p \subset A^p_{p,1}$. In particular we have $H^2 = A^2_{2,1}$.

Also, observe that for $p = \infty$ the space $A^\infty_{\infty,1}$ is the usual Bloch space. We will write $\mathscr{B}$ instead $A^\infty_{\infty,1}$ and $\|f\|_{\mathscr{B}}$ instead $\|f\|_{\infty,\infty,1}$.

Then it is natural to consider the following problem for these spaces:

PROBLEM. *Find necessary and sufficient conditions on g such that M_g maps $A^p_{p,1} \times \ldots \times A^p_{p,1}$ onto $A^p_{p,1}$.*

Clearly, a necessary condition is that every function g_j be a multiplier of $A^p_{p,1}$. We will denote this space of multipliers by $\mathscr{M}_p$. An important difference between our case and the Hardy case is that the spaces of multipliers $\mathscr{M}_p$ are different for each p, whereas the space of multipliers of the Hardy spaces coincides with the space of bounded holomorphic functions on B for all $1 \leqq p \leqq \infty$.

We have the following characterizations of the spaces $\mathscr{M}_p$.

For $p = \infty$, i.e. the Bloch space, K. Zhu [Z] showed that a holomorphic function g belongs to $\mathscr{M}_\infty$ if and only if it is bounded on B and satisfies $|\partial g(z)| \leqq \dfrac{c}{1 - |z|^2} \log^{-1} \dfrac{2}{1 - |z|^2}$.

For $1 \leqq p \leqq \infty$ the space $\mathscr{M}_p$ coincides with the space of bounded holomorphic functions g on B such that $|\partial g(z)|^p (1 - |z|^2)^{p-1}\, dV(z)$ is a $A^p_{p,1}$-Carleson measure, i.e.

$$\left(\int_B |f(z)|^p |\partial g(z)|^p (1 - |z|^2)^{p-1}\, dV(z)\right)^{\frac{1}{p}} \leqq c\|f\|_{p,p,1}$$

for all $f \in A^p_{p,1}$.

It is easy to show that these spaces of multipliers satisfy

$$\mathcal{M}_1 \subset \mathcal{M}_{p_1} \subset \mathcal{M}_{p_2} \subset \mathcal{M}_2 = H^\infty, \quad 1 \leqq p_1 \leqq p_2 \leqq 2$$

$$\mathcal{M}_\infty \subset \mathcal{M}_{q_1} \subset \mathcal{M}_{q_2} \subset \mathcal{M}_2 = H^\infty, \quad \infty \geqq q_1 \geqq q_2 \geqq 2.$$

Here we have three extremal cases: $p = 1$, $p = 2$ and $p = \infty$.

In this paper, we will obtain a solution of our problem for the extreme cases $p = \infty$ and $p = 1$. The case $p = 2$ corresponds to the known H^2-corona problem. To be precise we will prove the following two theorems:

THEOREM A. *Let $g = (g_1, \ldots, g_m)$ be a vector valued holomorphic function on B. Then the operator*

$$M_g(f)(z) = \sum_{j=1}^{m} f_j(z) g_j(z)$$

is onto from $\mathcal{B} \times \ldots \times \mathcal{B}$ to $\mathcal{B}$ iff the functions g_j are multipliers of the Bloch space and satisfy the condition $\sup_{z\in B}\{|g(z)|^2\} \geqq \delta > 0$.

THEOREM B. *Let $g = (g_1, \ldots, g_m)$ be a vector valued holomorphic function on B. Then the operator*

$$M_g(f)(z) = \sum_{j=1}^{m} f_j(z) g_j(z)$$

is onto from $A^1_{1,1} \times \ldots \times A^1_{1,1}$ to $A^1_{1,1}$ iff the functions g_j are multipliers of the Besov space $A^1_{1,1}$ and satisfy the condition $\sup_{z\in B}\{|g(z)|^2\} \geqq \delta > 0$.

The paper is organized in the following way. In section 2 we obtain the necessary conditions. In section 3 we recall the division formulas of B. Berndtsson and we obtain the estimates that we will use in section 4 to prove the decomposition theorems.

2. Necessary conditions.

NECESSARY CONDITIONS IN THEOREM A. By the closed graph Theorem we have that M_g is continuous and thus, every g_j is a multiplier of $\mathcal{B}$.

Let us prove that $|g|^2 \geqq \delta > 0$. By the open map Theorem, for every function f of $\mathcal{B}$ there exist functions f_i of $\mathcal{B}$ such that

$$\text{i) } f = \sum_{j=1}^{m} f_i g_i$$

$$\text{ii) } \|f_i\|_{\mathcal{B}} \leqq c\, \|f\|_{\mathcal{B}}.$$

Using $|f_i(\zeta)| \leqq c\, \|f\|_{\mathcal{B}} \log \dfrac{2}{1 - |\zeta|^2}$ for $\zeta \in B$, we obtain

$$|f(\zeta)| \leqq \sum_{j=1}^{m} |f_j(\zeta)|\,|g_j(\zeta)| \leqq c\|f\|_{\mathscr{B}} \log\frac{2}{1-|\zeta|^2} \sum_{j=1}^{m} |g_j(\zeta)|.$$

Taking the functions $f_z(\zeta) = \dfrac{1}{2}\log\dfrac{2}{1-\bar{z}\zeta}$ we have $\|f_z\|_{\mathscr{B}} \leqq 1$ and

$$\log\frac{2}{1-|z|^2} \leqq c\log\frac{2}{1-|z|^2}\sum_{j=1}^{m}|g_j(z)|$$

which proves the result.

REMARK. It is clear that every holomorphic Lipschitz function on B is a multiplier of the Bloch space. However there exist functions in $\mathscr{M}_\infty$ such that are not continuous on $\bar{B}$. For example the function $f(z_1, z_2) = \dfrac{\log\dfrac{12}{1-z_1^2}}{\log\dfrac{12}{1-z_1^2-z_2^2}}$ defined in the unit ball of $\mathbf{C}^2$ satisfies these conditions.

NECESSARY CONDITIONS IN THEOREM B. To prove the theorem we need some preliminary results. The first is a well known representation formula for holomorphic functions on B.

THEOREM 2.1. *Let f be a holomorphic function on B such that* $\sup_{z\in B}\{|f(z)|(1-|z|^2)^{s_0}\} \leqq c$. *Then,*

$$f(z) = P(f)(z) := c_s \int_B f(\zeta)\frac{(1-|\zeta|^2)^s}{(1-\bar{\zeta}z)^{n+1+s}}\,dV(\zeta),$$

with $s > s_0$, and $c_s = \dbinom{n+s}{s}$

The following estimates are also well known.

LEMMA 2.2. *For $\delta > 0$ we have*

$$\int_{\partial B}\frac{(1-|\zeta|^2)^s}{|\delta+1-\bar{\zeta}z|^{n+t}}\,dV(\zeta) \leqq \begin{cases} c & \text{if } s > t,\\ c\log\dfrac{2}{1-|z|^2+\delta} & \text{if } s = t,\\ c(1-|z|^2+\delta)^{s-t} & \text{if } s < t\end{cases}$$

The next lemma is a kind of integration by parts formula.

LEMMA 2.3. *If $\varphi(\zeta, z)$ is a function of class $\mathscr{C}^1(\bar{B}\times\bar{B})$ and $s > 0$, then*

$$c_s \int_B (1 - |\zeta|^2)^s \varphi(\zeta, z) dV(\zeta) = c_{s+1} \int_B (1 - |\zeta|^2)^{s+1} R_s \varphi(\zeta, d) dV(\zeta)$$

where $R_s := I + \dfrac{1}{n+1+s} R$ *with* $R = \sum_{i=1}^n \zeta_i \dfrac{\partial}{\partial \zeta_i}$.

PROOF. The proof is a direct application of Stokes Theorem.

We can now prove the necessary conditions of theorem B.

As in the proof of theorem A, it is clear that the functions g_j are multipliers of $A^1_{1,1}$. To prove that $|g(z)|^2 \geqq \delta > 0$ for every function f of $A^1_{1,1}$, we consider functions f_i of $A^1_{1,1}$ such that

$$\text{i) } f = \sum_{j=1}^m f_j g_j$$

$$\text{ii) } \|f_j\|_{1,1,1} \leqq c \|f\|_{1,1,1}.$$

The representation formula of theorem 2.1 and the integration by parts formula of lemma 2.3 give

$$|f_j(z)| \leqq c \int_B |R_s f_j(\zeta)| \frac{(1 - |\zeta|^2)^{s+1}}{|1 - \bar{\zeta} z|^{n+1+s}} dV(\zeta) \leqq c \|f_j\|_{1,1,1} (1 - |z|^2)^{-n}$$

for $z \in B$. Thus, using ii),

$$|f(\zeta)| \leqq c \sum_{j=1}^m |f_j(\zeta)| \, |g_j(\zeta)| \leqq c \|f\|_{1,1,1} \frac{1}{(1 - |\zeta|^2)^n} \sum_{j=1}^m |g_j(\zeta)|.$$

Finally, we can apply these inequalities to the functions $f_z(\zeta) = \dfrac{1 - |z|^2}{(1 - \bar{z}\zeta)^{n+1}}$ to obtain

$$\frac{1}{(1 - |z|^2)^n} \leqq c \frac{1}{(1 - |z|^2)^n} \sum_{j=1}^m |g_j(z)|$$

which proves the result.

REMARK. For $1 \leqq p \leqq 2$ the condition that $|\partial g(z)|^p (1 - |z|^2)^{p-1} dV(z)$ be a $A^p_{p,1}$-Carleson measure is equivalent to $(\partial g(z)|^p (1 - |z|^2)^{p-1} dV(z)$ be a Carleson measure in the ordinary sense. ([M-S] pag. 177). In particular $\mathcal{M}_1$ is the space of bounded holomorphic functions on B such that $|\partial g(z)| dV(z)$ is a Carleson measure.

3. Division formulas and estimates.

In the proof of the decomposition theorems we will follow the scheme of M. Andersson and H. Carlsson [AN-CA 2] based in the division formulas of B. Berndtsson [B].

Let $g = (g_1, \ldots, g_m)$ be holomorphic functions on B satisfying the conditions $|g(z)|^2 \geqq c > 0$ and $\sup\{|\partial g(z)|(1 - |z|^2)^{s_0}; z \in B\} < \infty$ for some $s_0 > 0$. For every $j = 1, \ldots, m$ we consider a Hefer decomposition of g_j, $g_j(z) - g_j(\zeta) = \sum_{i=1}^n (z_i - \zeta_i) h_{j,i}(\zeta, z)$ and the $(1,0)$-forms

$$\tilde{h}_j(\zeta, z) = \sum_{i=1}^n h_{j,i}(\zeta, z)\, d\zeta_i,$$

$$\tilde{Q}(\zeta, z) = \sum_{j=1}^m \frac{\bar{g}_j(\zeta)}{|g(\zeta)|^2} \tilde{h}_j(\zeta, z).$$

Then for $s > s_0$ large enough and $k_0 = \min\{n, m\}$ we have the following representation formula for holomorphic functions on B which satisfy an adequate growth condition:

$$f(z) = \sum_{k=0}^{k_0} c_k \int_B f(\zeta) \left(\frac{g(z)\bar{g}(\zeta)}{|g(\zeta)|^2}\right)^{k_0+1-k} \left(\frac{1 - |\zeta|^2}{1 - \bar{\zeta}z}\right)^{n+1+s+k} \times \left(\bar{\partial}\tilde{Q}\right)^k \wedge \left(\bar{\partial}\left(\frac{\partial|\zeta|^2}{1 - |\zeta|^2}\right)\right)^{n-k}. \tag{3.1}$$

It is clear that the operators

$$(T_j f)(z) := \sum_{k=0}^{k_0} c_k \int_B f(\zeta) \left(\frac{g(z)\bar{g}(\zeta)}{|g(\zeta)|^2}\right)^{k_0-k} \frac{\bar{g}_j(\zeta)}{|g(\zeta)|^2} \left(\frac{1 - |\zeta|^2}{1 - \bar{\zeta}z}\right)^{n+1+s+k} \times \left(\bar{\partial}\tilde{Q}\right)^k \wedge \left(\bar{\partial}\left(\frac{\partial|\zeta|^2}{1 - |\zeta|^2}\right)\right)^{n-k}. \tag{3.2}$$

give an explicit solution of the decomposition problem

$$f(z) = \sum_{j=1}^m g_j(z)\, T_j(f)(z).$$

Note that the above formula depends of the choose of the Hefer decompositions. We need to find an adequate Hefer decomposition of the functions g_j in order to obtain good estimates of the operators T_j.

To do so, we take a positive integer s and use the representation formula of lemma 2.1 to obtain

$$g_j(z) - g_j(\zeta) = c_s \int_B g_j(\eta)\left(\frac{(1-|\eta|^2)^s}{(1-\bar{\eta}z)^{n+1+s}} - \frac{(1-|\eta|^2)^s}{(1-\bar{\eta}\zeta)^{n+1+s}}\right) dV(\eta)$$

$$= \sum_{i=1}^{n} (z_i - \zeta_i)\, c_s \int_B g_j(\eta) \sum_{l=0}^{n+s} \frac{\bar{\eta}_i(1-|\eta|^2)^s}{(1-\bar{\eta}z)^{n+1+s-l}(1-\bar{\eta}\zeta)^{l+1}}\, dV(\eta)$$

$$=: \sum_{i=1}^{n} (z_i - \zeta_i)\, h_{j,i}(\zeta, z).$$

From now on we fix the above Hefer decompositions with s large enough in the representation formula.

To obtain precise estimates of the kernels T_j we need to consider the following decompositions of $h_{j,i}$.

DEFINITION 3.1. We write

$$h_{j,i}(\zeta, z) = h_{j,i}^0(\zeta, z) + \bar{\zeta}_i h_j^1(\zeta, z)$$

where

$$h_{j,i}^0(\zeta, z) := c_s \int_B g_j(\eta)(1-|\eta|^2)^s \sum_{l=0}^{n+s} \frac{\bar{\eta}_i - \bar{\zeta}_i}{(1-\bar{\eta}z)^{n+1+s-l}(1-\bar{\eta}\zeta)^{l+1}}\, dV(\eta),$$

$$h_j^1(\zeta, z) := c_s \int_B g_j(\eta)(1-|\eta|^2)^s \sum_{l=0}^{n+s} \frac{1}{(1-\bar{\eta}z)^{n+1+s-l}(1-\bar{\eta}\zeta)^{l+1}}\, dV(\eta),$$

Note that

$$\tilde{h}_j(\zeta, z) = \tilde{h}_j^0(\zeta, z) + h_j^1(\zeta, z)\,\partial(1-|\zeta|^2) \tag{3.3}$$

with $\tilde{h}_j(\zeta, z) = \sum_{i=1}^{n} h_{j,i}^0(\zeta, z)\, d\zeta_i$.

We also need the following decomposition of $\bar{\partial}\left(\dfrac{\bar{g}_j(\zeta)}{|g(\zeta)|^2}\right)$.

$$\bar{\partial}\left(\frac{\bar{g}_j(\zeta)}{|g(\zeta)|^2}\right) = \sum_{i=1}^{n}\left(\frac{\partial}{\partial\bar{\zeta}_i} - \zeta_i \bar{R}\right)\frac{\bar{g}_j(\zeta)}{|g(\zeta)|^2}\, d\bar{\zeta}_i - \bar{R}\frac{\bar{g}_j(\zeta)}{|g(\zeta)|^2}\,\bar{\partial}(1-|\zeta|^2) \tag{3.4}$$

$$=: \omega_j^0(\zeta) - \omega_j^1(\zeta)$$

and also the identity

$$\left(\bar{\partial}\frac{\partial(1-|\zeta|^2)}{1-|\zeta|^2}\right)^{n-k} = \frac{(\bar{\partial}\partial(1-|\zeta|^2))^{n-k}}{(1-|\zeta|^2)^{n-k}}$$

$$- (n-k)\frac{(\bar{\partial}\partial(1-|\zeta|^2))^{n-k-1} \wedge \bar{\partial}(1-|\zeta|^2) \wedge \partial(1-|\zeta|^2)}{(1-|\zeta|^2)^{n-k+1}}$$

Using elementary computations we obtain:

$$(3.5)\qquad (T_j f)(z) = a'_0 \int_B f(\zeta) \frac{\bar{g}_j(\zeta)}{|g(\zeta)|^2} \left(\frac{g(z)\bar{g}(\zeta)}{|g(\zeta)|^2} \right)^{k_0} \frac{(1-|\zeta|^2)^s}{(1-\bar{\zeta}z)^{n+1+s}} dV(\zeta)$$

$$+ \sum_{k=1}^{k_0} a'_k \int_B f(\zeta) \frac{\bar{g}_j(\zeta)}{|g(\zeta)|^2} \left(\frac{g(z)\bar{g}(\zeta)}{|g(\zeta)|^2} \right)^{k_0-k} \frac{(1-|\zeta|^2)^{1+s+2k}}{(1-\bar{\zeta}z)^{n+1+s+k}}$$

$$\times \sum_{\substack{|K|+|I|=|K'|+|I'|=k \\ |I|,|I'|\leqq 1}} \bar{G}^0_K \wedge \bar{G}^1_I \wedge H^0_{K'} \wedge H^1_{I'} \wedge (\bar{\partial}\partial(1-|\zeta|^2))^{n-k}$$

$$+ \sum_{k=1}^{k_0} (n-k) a''_k \int_B f(\zeta) \frac{\bar{g}_j(\zeta)}{|g(\zeta)|^2} \left(\frac{g(z)\bar{g}(\zeta)}{|g(\zeta)|^2} \right)^{k} \frac{(1-|\zeta|^2)^{s+2k}}{(1-\bar{\zeta}z)^{n+1+s+k}}$$

$$\times \sum_{|K|,|K'|=k} \bar{G}^0_K \wedge H^0_{K'} \wedge \bar{\partial}(1-|\zeta|^2) \wedge \partial(1-|\zeta|^2) \wedge (\bar{\partial}\partial(1-|\zeta|^2))^{n-k-1}$$

where H^i_K, $i = 0, 1$ are $(|K|, 0)$ forms in ζ which are linear combination of forms of type $\wedge_{|K|=l} \tilde{h}^i_{K_j}$, and G^i_K, $i = 0, 1$ are $(0, |K|)$ forms in ζ which are linear combinations of forms of type $\wedge_{|K|=l} \omega^i_{K_j}$.

The next step will be to obtain estimates of these kernels and of their derivatives.

It will be convenient to establish some preliminary notations and lemmas.

DEFINITION 3.2. Let $d(\zeta, z) = |\bar{z}(z-\zeta)| + |\bar{\zeta}(\zeta - z)|$ be the non isotropic pseudodistance and c_d a constant such that

$$d(\zeta, z) \leqq c_d(d(\zeta, w) + d(w, z)).$$

Given ζ, z of B we consider the following partition of B:

$$\Omega_1 = \left\{ \eta \in B;\ d(\eta, z) \leqq \frac{d(\zeta, z)}{2c_d} \right\}$$

$$\Omega_2 = \left\{ \eta \in B;\ d(\eta, \zeta) \leqq \frac{d(\zeta, z)}{2c_d} \right\}$$

$$\Omega_3 = \left\{ \eta \in B;\ \frac{d(\zeta, z)}{2c_d} < d(\eta, z) \leqq d(\eta, \zeta) \right\}$$

$$\Omega_4 = \left\{ \eta \in B;\ \frac{d(\zeta, z)}{2c_d} < d(\eta, \zeta) \leqq d(\eta, z) \right\}$$

LEMMA 3.3. *With the above notations we have*

$$\begin{aligned} |1-\bar{\eta}z| &\leqq c|1-\bar{\zeta}z| \leqq c|1-\bar{\eta}\zeta|, \quad \eta \in \Omega_1 \\ |1-\bar{\eta}\zeta| &\leqq c|1-\bar{\zeta}z| \leqq c|1-\bar{\eta}z|, \quad \eta \in \Omega_2 \\ |1-\bar{\zeta}z| &\leqq c|1-\bar{\eta}z| \leqq c|1-\bar{\eta}\zeta|, \quad \eta \in \Omega_3 \\ |1-\bar{\zeta}z| &\leqq c|1-\bar{\eta}\zeta| \leqq c|1-\bar{\eta}z|, \quad \eta \in \Omega_4 \end{aligned}$$

PROOF. The results follow trivially from the definitions of Ω_j and the fact that

$$|1 - \bar{\zeta} z| \approx 1 - |\zeta|^2 + d(\zeta, z) \approx 1 - |z|^2 + d(\zeta, z)$$

for every ζ, z of B.

LEMMA 3.4. *Assume that* $0 \leqq t_0, t_1 < s < t_0 + t_1$. *Then*

$$\int_B \frac{(1-|\eta|^2)^s}{|1-\bar{\eta} z|^{n+1+t_0}|1-\bar{\eta}\zeta|^{t_1}} \log^k \frac{2}{1-|\eta|^2}\, dV(\eta) \leqq c \frac{1}{|1-\bar{\zeta} z|^{t_0+t_1-s}} \log^k \frac{2}{|1-\bar{\zeta} z|}$$

PROOF. Using the partition of B given in definition 3.2 we obtain

$$\begin{aligned} I := & \int_B \frac{(1-|\eta|^2)^s}{|1-\bar{\eta} z|^{n+1+t_0}|1-\bar{\eta}\zeta|^{t_1}} \log^k \frac{2}{1-|\eta|^2}\, dV(\eta) \\ \leqq & \, c \int_{\Omega_1} \frac{(1-|\eta|^2)^s}{|1-\bar{\eta} z|^{n+1+t_0}(|1-\bar{\zeta} z| + 1 - |\eta|^2)^{t_1}} \log^k \frac{2}{1-|\eta|^2}\, dV(\eta) \\ & + c \int_{B\setminus\Omega_1} \frac{(1-|\eta|^2)^{s-t_1}}{(|1-\bar{\zeta} z| + |1-\bar{\eta} z|)^{n+1+t_0}} \log^k \frac{2}{1-|\eta|^2}\, dV(\eta) \end{aligned}$$

We integrate in polar coordinates and use the usual estimates of lemma 2.2 to obtain

$$\begin{aligned} I \leqq & \, c \int_0^1 \frac{(1-r^2)^s}{(1-|z|^2+1-r^2)^{1+t_0}(|1-\bar{\zeta} z| + 1 - r^2)^{t_1}} \log^k \frac{2}{1-r^2}\, dr \\ & + c \int_0^1 \frac{(1-r^2)^{s-t_1}}{(|1-\bar{\zeta} z| + 1 - r^2)^{1+t_0}} \log^k \frac{2}{1-r^2}\, dr \\ \leqq & \frac{c}{|1-\bar{\zeta} z|^{t_0+t_1-s}} \log^k \frac{2}{|1-\bar{\zeta} z|} \end{aligned}$$

and thus the lemma is proved.

LEMMA 3.5. *Assume that* $0 \leqq t_1 < s < t_0$. *Then*

$$\int_B \frac{(1-|\eta|^2)^s}{|1-\bar{\eta} z|^{n+1+t_0}|1-\bar{\eta}\zeta|^{t_1}} \log^k \frac{2}{1-|\eta|^2}\, dV(\eta) \leqq \frac{c}{(1-|z|^2)^{t_0-s}|1-\bar{\zeta} z|^{t_1}} \log^k \frac{2}{1-|z|^2}.$$

PROOF. The estimate follows using the pattern of the proof of the above lemma.

We need to recall the following representation formula given by P. Charpentier [CH] (see also B. Berndtsson and M. Anderson [B-AN]) and some well known estimates.

THEOREM 3.6. *The operator*

$$P(f)(z) = c_s \int_B f(\zeta) \frac{(1-|\zeta|^2)^s}{(1-\bar{\zeta}z)^{n+1+s}} dV(\zeta)$$

defined in Theorem 2.1 verifies $(I-P)(f) = K(\bar{\partial} f)$ *where K is an integral operator which solves the* $\bar{\partial}$*-equation and* $K(\zeta, z) = K_0(\zeta, z) + K_1(\zeta, z) \wedge \bar{\partial}(1-|\zeta|^2)$ *with*

$$\int_{\{\zeta; d(\zeta,z)<t\}} |K_0(\zeta, z)| dV(\zeta) \leqq ct,$$

$$\int_{\{\zeta; d(\zeta,z)<t\}} |K_1(\zeta, z)| dV(\zeta) \leqq ct^{\frac{1}{2}}.$$

LEMMA 3.7. *There exists a constant c such that for every bounded holomorphic function f on B we have*

$$\text{i) } I_1 := \left| c_s \int_B f(\eta) \frac{(1-|\eta|^2)^s}{(1-\bar{\eta}z)^{n+s}(1-\bar{\eta}\zeta)} dV(\eta) \right| \leqq c \|f\|_\infty.$$

$$\text{ii) } I_2 := \left| c_s \int_B f(\eta) \frac{(1-|\eta|^2)^s}{(1-\bar{\eta}z)^{n+1+s}(1-\bar{\eta}\zeta)} dV(\eta) \right| \leqq c \frac{\|f\|_\infty}{|1-\bar{\zeta}z|}.$$

PROOF. First note that if we replace ζ by z in i) we obtain $I_1 = |f(z)| \leqq \|f\|_\infty$. We point pout that if we put modulus inside the integral we would not obtain the required estimate.

Let us consider $z \neq \zeta$. Following the same notations of definition 3.2, we define

$$\Omega_1' = \left\{ \eta \in B;\ d(\eta, z) \leqq \frac{d(\zeta, z)}{4c_d} \right\} \subset\subset \Omega_1,$$

and we consider a $\mathscr{C}^\infty$ function $\chi(\eta)$ such that (see lemma 2.3 of [BR-OR] for a construction):

a) $0 \leqq \chi \leqq 1$

b) $\chi = 0$ in Ω_1' and $\chi = 1$ in $\mathbf{C}^n \setminus \Omega_1$

c) $|\bar{\partial}\chi(\eta)| \leqq \dfrac{c}{d(\zeta, z)}$

d) $|\bar{\partial}(1-|\eta|^2) \wedge \bar{\partial}\chi(\eta)| \leqq \dfrac{c}{d(\zeta, z)^{\frac{1}{2}}}$

where the constants which appear in the expression are independent of ζ and z.

Also note that

$$\frac{1}{1-\bar{\eta}\zeta} = \frac{1}{1-\bar{\eta}z} + \frac{\bar{\eta}(\zeta - z)}{(1-\bar{\eta}\zeta)(1-\bar{\eta}z)}.$$

Therefore

$$I_1 \leqq c \int_{\Omega_1} |f(\eta)| \frac{(1-|\eta|^2)^s}{|1-\bar{\eta}z|^{n+s}|1-\bar{\eta}\zeta|} dV(\eta) + \left| c_s \int_B f(\eta) \frac{\chi(\eta)(1-|\eta|^2)^s}{(1-\bar{\eta}z)^{n+1+s}} dV(\eta) \right|$$
$$+ c \int_{B\setminus\Omega_1'} |f(\eta)| \frac{(1-|\eta|^2)^s |\bar{\eta}(z-\zeta)|}{|1-\bar{\eta}z|^{n+1+s}|1-\bar{\eta}\zeta|} dV(\eta) =: J_1 + J_2 + J_3.$$

By lemma 3.3 we have $|1-\bar{\eta}z| \leqq c|1-\bar{\zeta}z|$ in Ω_1' and thus for $0 < \varepsilon < 1$ we have

$$J_1 \leqq c\|f\|_\infty \int_B \frac{|1-\bar{\zeta}z|^\varepsilon (1-|\eta|^2)^s}{|1-\bar{\eta}z|^{n+s+\varepsilon}|1-\bar{\eta}\zeta|} dV(\eta).$$

Hence, by lemma 3.4 with $k = 0$, we obtain $J_1 \leqq c\|f\|_\infty$.

We can use theorem 3.6 to obtain

$$J_2 = \left| f(z)\chi(z) - \int_B K(\eta, z) f(\eta) \wedge \bar{\partial}\chi(\eta)\, dV(\eta) \right|$$
$$\leqq \|f\|_\infty \left(1 + \int_{\Omega_1\setminus\Omega_1'} |K_0(\eta, z)|\, |\bar{\partial}\chi(\eta)|\, dV(\eta) \right.$$
$$\left. + \int_{\Omega_1\setminus\Omega_1'} |K_1(\eta, z)|\, |\bar{\partial}(1-|\eta|^2) \wedge \bar{\partial}\chi(\eta)|\, dV(\eta) \right).$$

Then using theorem 3.6 and properties c) and d) of the function χ, the result follows.

To prove the estimate for J_3 observe that in $B\setminus\Omega_1'$ we have $d(\zeta, z) \leqq c\, d(\eta, z)$. Therefore, we obtain

$$|\bar{\eta}(z-\zeta)| \leqq |\eta - z|\, |z-\zeta| + |\bar{z}(z-\zeta)| \leqq c d(\zeta, z)^{\frac{1}{2}} d(\eta, z)^{\frac{1}{2}}$$

and

$$J_3 \leqq c\|f\|_\infty d(\zeta, z)^{\frac{1}{2}} \int_B \frac{(1-|\eta|^2)^s}{|1-\bar{\eta}z|^{n+\frac{1}{2}+s}|1-\bar{\eta}\zeta|} dV(\eta).$$

Thus, the result follows from lemma 3.4 with $k = 0$.

To prove ii) note that

$$\frac{1}{1-\bar{\eta}\zeta} = \frac{1}{1-\bar{z}\zeta} + \frac{(\zeta - z)(\bar{\eta} - \bar{z}) + 1 - |z|^2 - (1-\bar{\eta}z)}{(1-\bar{\eta}\zeta)(1-\bar{z}\zeta)}$$

and therefore

$$I_2 \leqq \frac{|f(z)|}{|1-\bar{z}\zeta|} + c\frac{\|f\|_\infty}{|1-\bar{z}\zeta|^{\frac{1}{2}}} \int_B \frac{(1-|\eta|^2)^s}{|1-\bar{\eta}z|^{n+\frac{1}{2}+s}|1-\bar{\eta}\zeta|} + \frac{1}{|1-\bar{z}\zeta|} I_1.$$

Finally we apply lemma 3.4 and estimate i) to end the proof.

In the next lemma we will obtain some estimates of the forms defined in (3.3) and (3.4). Similar estimates were obtained by M. Andersson and M. Carlsson in Proposition 3.2 of [AN-CA2] for an analogous Hefer decomposition.

LEMMA 3.8. *Let g_j be a bounded function on B and $\tilde{h}_j^0$, $h_j^1, \omega_j^0, \omega_j^1$ the forms defined in (3.3) and (3.4). Then we have*

$$\text{i)}\ |\tilde{h}_j^0(\zeta, z)| \leqq \frac{c}{|1-\bar{\zeta}z|^{\frac{1}{2}}}$$

$$\text{ii)}\ |h_j^1(\zeta, z) \leqq \frac{c}{|1-\bar{\zeta}z|}$$

$$\text{iii)}\ |\omega_j^0(\zeta)| \leqq \frac{c}{(1-|\zeta|^2)^{\frac{1}{2}}}$$

$$\text{iv)}\ |\omega_j^1(\zeta)| \leqq \frac{c}{1-|\zeta|^2}$$

Moreover, if g_j is a multiplier of the Bloch space we have

$$\text{v)}\ |\omega_j^0(\zeta)| \leqq \frac{c}{(1-|\zeta|^2)^{\frac{1}{2}}} \log^{-1} \frac{2}{1-|\zeta|^2}$$

$$\text{vi)}\ |\omega_j^1(\zeta)| \leqq \frac{c}{1-|\zeta|^2} \log^{-1} \frac{2}{1-|\zeta|^2}$$

PROOF. From the definition of $h_{j,i}^0$, (definition 3.1) we obtain

$$|h_{j,i}^0(\zeta, z)| \leqq c \left| \int_B g_j(\eta) \frac{(1-|\eta|^2)^s(\bar{z}_i - \bar{\zeta}_i)}{(1-\bar{\eta}z)^{n+1+s}(1-\bar{\eta}\zeta)} dV(\eta) \right|$$
$$+ c\|g_j\|_\infty \int_B \frac{(1-|\eta|^2)^s}{|1-\bar{\eta}z|^{n+\frac{1}{2}+s}|1-\bar{\eta}\zeta|} dV(\eta)$$
$$+ c\|g_j\|_\infty \int_B \frac{(1-|\eta|^2)^s}{|1-\bar{\eta}z||1-\bar{\eta}\zeta|^{n+\frac{1}{2}+s}} dV(\eta).$$

Hence i) follows from lemmas 3.4 with $k = 0$ and 3.7. Part ii) follows in the same way.

The estimates iii) and iv) are well known.

On the other hand the representation formula of theorem 3.1 and the integration by parts formula of lemma 3.3 give

$$\left|\left(\frac{\partial}{\partial\zeta_i} - \bar\zeta_i R\right) g_j(\zeta)\right| = \left|c_{s+1}\left(\frac{\partial}{\partial\zeta_i} - \bar\zeta_i R\right)\int_B R_s g_j(\eta)\frac{(1-|\eta|^2)^{s+1}}{(1-\bar\eta\zeta)^{n+1+s}}\,dV(\eta)\right|$$
$$\leqq c\int_B \frac{(1-|\eta|^2)^s}{|1-\bar\eta\zeta|^{n+\frac12+s}}\log^{-1}\frac{2}{1-|\eta|^2}\,dV(\eta).$$

Thus, v) follows from lemma 3.4 with $k = -1$ and $\zeta = 0$.

Finally, vi) follows trivially from the characterization of the multipliers of the Bloch space.

LEMMA 3.9. *If the function g_j is a multiplier of the Bloch space, then the derivatives of the Hefer functions defined in (3.1) satisfy:*

i) $|\partial_z \tilde h_j^0(\zeta,z)| \leqq \dfrac{c}{(1-|z|^2)|1-\bar\zeta z|^{\frac12}}\log^{-1}\dfrac{2}{1-|z|^2}$

ii) $|\partial_z h_j^1(\zeta,z)| \leqq \dfrac{c}{(1-|z|^2)|1-\bar\zeta z|}\log^{-1}\dfrac{2}{1-|z|^2} + \dfrac{c}{(1-|\zeta|^2)|1-\bar\zeta z|}\log^{-1}\dfrac{2}{1-|\zeta|^2}$

PROOF. As in the above lemma, we can use the definition 3.1 of $h_{j,i}^0$ and the integration by parts formula of lemma 2.3 to obtain

$$|\partial_z h_{j,i}^0(\zeta,z)| \leqq c\int_B |R_s g_j(\eta)|\frac{(1-|\eta|^2)^{s+1}}{|1-\bar\eta z|^{n+2+s}|1-\bar\eta\zeta|^{\frac12}}\,dV(\eta)$$
$$+\, c\int_B |R_s g_j(\eta)|\frac{(1-|\eta|^2)^{s+1}}{|1-\bar\eta z|^2\,|1-\bar\eta\zeta|^{n+s+\frac12}}\,dV(\eta).$$

Hence, from $|R_s g_j(\eta)| \leqq c\dfrac{\|g_j\|_{\mathscr B}}{1-|\eta|^2}\log^{-1}\dfrac{2}{1-|\eta|^2}$ and the estimates of lemmas 3.4 and 3.5 we obtain the result i).

To prove ii), note that

$$|\partial_z h_j^1(\zeta,z)| \leqq c\int_B |R_s g_j(\eta)|\frac{(1-|\eta|^2)^{s+1}}{|1-\bar\eta z|^{n+2+s}|1-\bar\eta\zeta|}\,dV(\eta)$$
$$+\, c\int_B |R_s g_j(\eta)|\frac{(1-|\eta|^2)^{s+1}}{|1-\bar\eta z|^2\,|1-\bar\eta\zeta|^{n+s+1}}\,dV(\eta)$$
$$\leqq c\int_B |R_s g_j(\eta)|\frac{(1-|\eta|^2)^{s+1}}{|1-\bar\eta z|^{n+2+s}|1-\bar\eta\zeta|}\,dV(\eta)$$
$$+\,\frac{c}{1-|\zeta|^2}\int_B |R_s g_j(\eta)|\frac{(1-|\eta|^2)^{s+1}}{|1-\bar\eta z|^2\,|1-\bar\eta\zeta|^{n+s}}\,dV(\eta).$$

Thus, the estimate follows from lemmas 3.4 and 3.5.

As corollary of these results we obtain:

COROLLARY 3.10. *Assume that the functions g_j are multipliers of the Bloch space. Then, with the same notations of formula (3.5) and* $0 < |K| + |I| = |K'| + |I'| = k$, $|I|, |I'| \leqq 1$, *we have*

i) $$|(\bar{G}^0_K \wedge \bar{G}^1_I \wedge H^0_{K'} \wedge H^1_{I'})(\zeta, z)| \leqq \frac{c}{(1-|\zeta|^2)^{\frac{k-1}{2}+1}|1-\bar{\zeta}z|^{\frac{k-1}{2}+1}} \log^{-k}\frac{2}{1-|\zeta|^2}$$

ii) $$|(\bar{G}^0_K \wedge H^0_{K'})(\zeta, z)| \leqq \frac{c}{(1-|\zeta|^2)^{\frac{k}{2}}|1-\bar{\zeta}z|^{\frac{k}{2}}} \log^{-k}\frac{2}{1-|\zeta|^2}$$

iii) $$|\partial_z(\bar{G}^0_K \wedge \bar{G}^1_I \wedge H^0_{K'} \wedge H^1_{I'})(\zeta, z)|$$
$$\leqq \frac{c}{(1-|\zeta|^2)^{\frac{k-1}{2}+1}|1-\bar{\zeta}z|^{\frac{k-1}{2}+1}(1-|z|^2)} \log^{-k}\frac{2}{1-|\zeta|^2}\log^{-1}\frac{2}{1-|z|^2}$$
$$+ \frac{c}{(1-|\zeta|^2)^{\frac{k-1}{2}+2}|1-\bar{\zeta}z|^{\frac{k-1}{2}+1}} \log^{-(k+1)}\frac{2}{1-|\zeta|^2}$$

iv) $$|\partial_z(\bar{G}^0_K \wedge H^0_{K'})(\zeta,z)| \leqq \frac{c}{(1-|\zeta|^2)^{\frac{k}{2}}|1-\bar{\zeta}z|^{\frac{k}{2}}(1-|z|^2)} \log^{-k}\frac{2}{1-|\zeta|^2}\log^{-1}\frac{2}{1-|z|^2}$$
$$+ \frac{c}{(1-|\zeta|^2)^{\frac{k}{2}+1}|1-\bar{\zeta}z|^{\frac{k}{2}}} \log^{-(k+1)}\frac{2}{1-|\zeta|^2}$$

The following results summarize some estimates that we will need to prove theorem B.

LEMMA 3.11. *Let g_j be a bounded holomorphic function on B. Then we have*

i) $$|\partial_z \tilde{h}^0_j(\zeta, z)| \leqq c\frac{|\partial g(z)|}{|1-\bar{\zeta}z|^{\frac{1}{2}}} + c\frac{1}{(1-|z|^2)^{\frac{1}{2}}|1-\bar{\zeta}z|}$$

ii) $$|\partial_z h^1_j(\zeta, z)| \leqq c\frac{|\partial g(z)|}{|1-\bar{\zeta}z|} + c\frac{1}{(1-|z|^2)^{\frac{1}{2}}|1-|\zeta|^2)^{\frac{1}{2}}|1-\bar{\zeta}z|}.$$

PROOF. By definition 3.1 we have

$$h^0_{j,i}(\zeta, z) = c_s\int_B g_j(\eta)(1-|\eta|^2)^s \sum_{l=0}^{n+s} \frac{\bar{\eta}_i - \bar{\zeta}_i}{(1-\bar{\eta}z)^{n+1+s-l}(1-\bar{\eta}\zeta)^{l+1}}\,dV(\eta)$$
$$= \frac{\bar{z}_j - \bar{\zeta}_j}{1-\bar{z}\zeta} g_j(z) + c_s\int_B g_j(\eta)\frac{(1-|\eta|^2)^s}{(1-\bar{\eta}z)^{n+1+s}}\left(\frac{\bar{\eta}_i - \bar{\zeta}_i}{(1-\bar{\eta}\zeta)} - \frac{\bar{z}_i - \bar{\zeta}_i}{(1-\bar{z}\zeta)}\right)dV(\eta)$$
$$+ c_s\int_B g_j(\eta)(1-|\eta|^2)^s \sum_{l=1}^{n+s} \frac{\bar{\eta}_i - \bar{\zeta}_i}{(1-\bar{\eta}z)^{n+1+s-l}(1-\bar{\eta}\zeta)^{l+1}}\,dV(\eta).$$

We can now differentiate under the integral sign and use the estimates of lemmas 3.4 and 3.5 to obtain the estimates of i).

Part ii) follows in the same way.

COROLLARY 3.12. *Assume that g_j are bounded holomorphic functions on B. With the same notations of formula (3.5) and* $|K| + |I| = |K'| + |I'| = k, |I|, |I'| \leqq 1$, *we have*

i) $$|(\bar{G}_K^0 \wedge \bar{G}_I^1 \wedge H_{K'}^0 \wedge H_{I'}^1)(\zeta, z)| \leqq \frac{|\partial g(\zeta)|}{(1-|\zeta|^2)^{\frac{k-1}{2}}|1-\bar{\zeta}z|^{\frac{k-1}{2}+1}}$$

ii) $$|(\bar{G}_K^0 \wedge H_{K'}^0)(\zeta, z)| \leqq c\frac{|\partial g(\zeta) \wedge \partial(1-|\zeta|^2)|}{(1-|\zeta|^2)^{\frac{k-1}{2}}|1-\bar{\zeta}z|^{\frac{k}{2}}}$$

iii) $$|\partial_z(\bar{G}_K^0 \wedge \bar{G}_I^1 \wedge H_{K'}^0 \wedge H_{I'}^1)(\zeta, z)| \leqq \frac{|\partial g(z)|\,|\partial g(\zeta)|}{(1-|\zeta|^2)^{\frac{k-1}{2}}|1-\bar{\zeta}z|^{\frac{k-1}{2}+1}} + c\frac{|\partial g(\zeta)|}{(1-|z|^2)^{\frac{1}{2}}(1-|\zeta|^2)^{\frac{k}{2}}|1-\bar{\zeta}z|^{\frac{k-1}{2}+1}}$$

iv) $$|\partial_z(\bar{G}_K^0 \wedge H_{K'}^0)(\zeta, z)| \leqq c\frac{|\partial g(z)|\,|\partial g(\zeta) \wedge \partial(1-|\zeta|^2)}{(1-|\zeta|^2)^{\frac{k-1}{2}}|1-\bar{\zeta}z|^{\frac{k}{2}}} + c\frac{|\partial g(\zeta) \wedge \partial(1-|\zeta|^2)|}{(1-|z|^2)^{\frac{1}{2}}(1-|\zeta|^2)^{\frac{k-1}{2}}|1-\bar{\zeta}z|^{\frac{k-1}{2}+1}}$$

LEMMA 3.13. *If g is a multiplier of the space $A_{1,1}^1$, then $|\partial g(z) \wedge \partial(1-|z|^2)|(1-|z|^2)^{-\frac{1}{2}}\,dV(z)$ is a $A_{1,1}^1$-Carleson measure.*

PROOF. First recall that if $h \in A_{1,1}^1$, then the coefficients of $(\partial h \wedge \partial(1-|\zeta|^2))$ belong to the weighted L^1-space $L^1_{\frac{1}{2},0}(B) := L^1(B, (1-|\zeta|^2)^{-\frac{1}{2}}\,dV(\zeta))$ and

$$\int_B |(\partial h \wedge \partial(1-|\zeta|^2)|(1-|\zeta|^2)^{-\frac{1}{2}}\,dV(\zeta) \leqq c\|h\|_{1,1,1}.$$

Thus, if g is a multiplier of $A_{1,1}^1$, we have

$$\begin{aligned}\int_B &|f(\zeta)\partial g(\zeta)) \wedge \partial(1-|\zeta|^2)|(1-|\zeta|^2)^{-\frac{1}{2}}\,dV(\zeta)\\ &\leqq \int_B |\partial(gf)(\zeta) \wedge \partial(1-|\zeta|^2)|(1-|\zeta|^2)^{-\frac{1}{2}}\,dV(\zeta)\\ &\quad + \|g\|_\infty \int_B |\partial f(\zeta) \wedge \partial(1-|\zeta|^2)|(1-|\zeta|^2)^{-\frac{1}{2}}\,dV(\zeta)\\ &\leqq c\|gf\|_{1,1,1} + c\|f\|_{1,1,1} \leqq c\|f\|_{1,1,1}.\end{aligned}$$

Hence the lemma is proved.

4. Sufficient conditions.

SUFFICIENT CONDITIONS IN THEOREM A. We will prove that the opeators T_j defined in (3.5) maps $\mathscr{B}$ into $\mathscr{B}$. We need to show that

$$\sup_{z \in B} \{(1 - |z|^2)|\partial_z T_j(f)(z)|\} \leqq c \|f\|_{\mathscr{B}}.$$

By differentiation under the integral sign and using the estimates of corollary 3.10, we obtain

$$|\partial_z T_j(f)(z)| \leqq c \left| \partial_z \int_B f(\zeta) \frac{\bar{g}_j(\zeta)}{|g(\zeta)|^2} \frac{(1 - |\zeta|)^s}{(1 - \bar{\zeta} z)^{n+1+s}} \left(\frac{g(z)\bar{g}(\zeta)}{|g(\zeta)|^2} \right)^{k_0} dV(\zeta) \right|$$
$$+ \frac{c}{1 - |z|^2} \log^{-1} \frac{2}{1 - |z|^2} \sum_{k=1}^{k_0} \int_B |f(\zeta)| \frac{(1 - |\zeta|^2)^s}{|1 - \bar{\zeta} z|^{n+1+s}} \log^{-k} \frac{2}{1 - |\zeta|^2} dV(\zeta)$$
$$+ c \sum_{k=1}^{k_0} \int_B |f(\zeta)| \frac{(1 - |\zeta|^2)^s}{|1 - \bar{\zeta} z|^{n+2+s}} \log^{-k} \frac{2}{1 - |\zeta|^2} dV(\zeta).$$

We can now apply the integration by parts formula of lemma 2.3 to the first term and use

$$\sup_{\zeta \in B} \left\{ |f(\zeta)| \log^{-1} \frac{2}{1 - |\zeta|^2} \right\} \leqq c \|f\|_{\mathscr{B}},$$

to obtain

$$|\partial_z T_j(f)(z)| \leqq c \|f\|_{\mathscr{B}} \int_B \frac{(1 - |\zeta|^2)^s}{|1 - \bar{\zeta} z|^{n+2+s}} dV(\zeta)$$
$$+ \|f\|_{\mathscr{B}} \frac{1}{1 - |z|^2} \log^{-1} \frac{2}{1 - |z|^2} \int_B \frac{(1 - |\zeta|^2)^s}{|1 - \bar{\zeta} z|^{n+1+s}} dV(\zeta)$$
$$\leqq c \frac{\|f\|_{\mathscr{B}}}{1 - |z|^2},$$

which proves the theorem.

SUFFICIENT CONDITIONS IN THEOREM B. The estimates of corollary 3.12 give

$$|T_j(f)(z)| \leqq c\left|\int_B f(\zeta)\frac{\bar{g}_j(\zeta)}{|g(\zeta)|^2}\frac{(1-|\zeta|)^s}{(1-\bar{\zeta}z)^{n+1+s}}\left(\frac{g(z)\bar{g}(\zeta)}{|g(\zeta)|^2}\right)^{k_0}dV(\zeta)\right|$$
$$+\int_B |f(\zeta)|\,|\partial g(\zeta)|\frac{(1-|\zeta|^2)^s}{|1-\bar{\zeta}z|^{n+s}}dV(\zeta)$$
$$+\int_B |f(\zeta)|\,|\partial g(\zeta)\wedge\partial(1-|\zeta|^2)|\frac{(1-|\zeta|^2)^s}{|1-\bar{\zeta}z|^{n+\frac{1}{2}+s}}dV(\zeta)=:I_1+I_2+I_3.$$

and

$$|\partial_z T_j(f)(z)| \leqq c\left|\partial_z\int_B f(\zeta)\frac{\bar{g}_j(\zeta)}{|g(\zeta)|^2}\frac{(1-|\zeta|)^s}{(1-\bar{\zeta}z)^{n+1+s}}\left(\frac{g(z)\bar{g}(\zeta)}{|g(\zeta)|^2}\right)^{k_0}dV(\zeta)\right|$$
$$+c|\partial g(z)|\int_B |f(\zeta)|\,|\partial g(\zeta)|\frac{(1-|\zeta|^2)^s}{|1-\bar{\zeta}z|^{n+s}}dV(\zeta)$$
$$+c|\partial g(z)|\int_B |f(\zeta)|\,|\partial g(\zeta)\wedge\partial(1-|\zeta|^2)|\frac{(1-|\zeta|^2)^s}{|1-\bar{\zeta}z|^{n+\frac{1}{2}+s}}dV(\zeta)$$
$$+\frac{c}{(1-|z|^2)^{\frac{1}{2}}}\int_B |f(\zeta)|\,|\partial g(\zeta)|\frac{(1-|\zeta|^2)^s}{|1-\bar{\zeta}z|^{n+\frac{1}{2}+s}}dV(\zeta)$$
$$+\frac{c}{(1-|z|^2)^{\frac{1}{2}}}\int_B |f(\zeta)|\,|\partial g(\zeta)\wedge\partial(1-|\zeta|^2)|\frac{(1-|\zeta|^2)^s}{|1-\bar{\zeta}z|^{n+1+s}}dV(\zeta)$$
$$=:J_1+J_2+J_3+J_4+J_5.$$

To prove the theorem we need to show that each of the terms I_i, J_j is in $L^1(B)$.

Fubini Theorem gives

$$\int_B |I_1(z)|\,dV(z) \leqq c\int_B\int_B |f(\zeta)|\frac{(1-|\zeta|^2)^s}{|1-\bar{\zeta}z|^{n+1+s}}dV(\zeta)\,dV(z) \leqq c\,\|f\|_{1,1,1}.$$

The estimates of I_2 and I_3 follow from Fubini Theorem, the fact that $|\partial g_j(\zeta)|\,dV(\zeta)$ is a $A^1_{1,1}$-Carleson measure and lemma 3.13.

By integration by parts and differentiation under the integral sign we have

$$|J_1(z)| \leqq c|\partial g(z)|\int_B (|R_s f(\zeta)|+|f(\zeta)R_s g(\zeta)|)\frac{(1-|\zeta|^2)^{s+1}}{|1-\bar{\zeta}z|^{n+1+s}}dV(\zeta)$$
$$+\int_B (|R_s f(\zeta)|+|f(\zeta)R_s g(\zeta)|)\frac{(1-|\zeta|^2)^{s+1}}{|1-\bar{\zeta}z|^{n++2+s}}dV(\zeta).$$

Therefore we have

$$\int_B |J_1(z)|\,dV(z) \leqq c\int_B (|R_s f(\zeta)| + |f(\zeta)\,R_s g(\zeta)|)$$
$$\times\,(1-|\zeta|^2)^{s+1}\int_B \frac{|\partial g(z)|}{|1-\bar{\zeta}z|^{n+1+s}}\,dV(z)\,dV(\zeta)$$
$$+\,c\int_B (|R_s f(\zeta) + |f(\zeta)\,R_s g(\zeta)|)\int_B \frac{(1-|\zeta|^2)^{s+1}}{|1-\bar{\zeta}z|^{n+2+s}}\,dV(z)\,dV(\zeta)$$
$$\leqq c\int_B (|R_s f(\zeta)| + |f(\zeta)\,R_s g(\zeta)|)\,dV(\zeta) \leqq c\,\|f\|_{1,1,1}.$$

Furthermore, the same reasoning gives

$$\int_B |J_2(z)|dV(z) \leqq c\int_B |f(\zeta)\partial g(\zeta)|(1-|\zeta|^2)^s\int_B \frac{|\partial g(z)|}{|1-\bar{\zeta}z|^{n+s}}\,dV(z)\,dV(\zeta)$$
$$\leqq c\int_B |f(\zeta)\partial g(\zeta)|\,dV(\zeta) \leqq c\,\|f\|_{1,1,1}$$

and

$$\int_B |J_3(z)|dV(z) \leqq c\int_B |f(\zeta)\partial g(\zeta) \wedge \partial(1-|\zeta|^2)|\,(1-|\zeta|^2)^{-\frac{1}{2}}dV(\zeta) \leqq c\,\|f\|_{1,1,1}.$$

Analogously, we obtain the estimates of J_4 and J_5.

Hence the lemma is proved.

FINAL REMARKS. If we consider the problem for the intermediate cases $1 < p < \infty$, the same reasoning used in section 2 shows that the functions g_j are multipliers of the Besov space $A^p_{p,1}$ and that $|g|^2 \geqq \delta > 0$ are needed to solve the problem.

For $p = 2$ these conditions are sufficient according to [AN-CA2].

Then we will finish with the following question:

With the above hypothesis, do the operators T_j map $A^p_{p,1}$ into itself for $1 < p < \infty$?

Note that the same reasoning of section 4 and Hölder inequalities give

$$\|I_1 + I_2 + I_3 + J_1 + J_4 + J_5\|_{p,p,1} \leqq c\,\|f\|_{p,p,1}$$

where I_i, J_i are the terms defined in the above proof. However, using this method we can not obtain these estimates for the terms J_2 and J_3.

Probably, as happen for $p = 2$ [*AN-CA*2], it is necessary to obtain more information about Hefer functions and they derivatives in terms of $A^p_{p,1}$-Carleson measures in addition to the pointwise estimates already obtained.

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