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L^p-IMPROVING PROPERTIES FOR SOME MEASURES SUPPORTED ON CURVES

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1. Introduction.

Let Γ be the curve in $\mathbf{R}^3$ defined by

$$\Gamma(t) = (t, t^2, t^3), \qquad t \geqq 0.$$

In [4] D. Oberlin showed that the Lebesgue measure σ on Γ has the following L^p-improving property: there are $p, q \in (1, \infty)$, $p < q$, such that

$$\sigma * L^p(\mathbf{R}^3) \subset L^q(\mathbf{R}^3).$$

More specifically, he proved the following $L^{3/2} \to L^2$ inequality:

THEOREM 1. *There is a constant C such that*

$$\|\sigma * f\|_{L^2(\mathbf{R}^3)} \leqq C \|f\|_{L^{3/2}(\mathbf{R}^3)},$$

for $f \in L^{3/2}(\mathbf{R}^3)$.

The cureve Γ is said to be nondegenerate because it has nonzero curvature and torsion. For more general curves, Drury and Marshall ([2]) pointed out that the natural measures to consider are the affine arclength measures on the given curves. Let $a = (a_1, a_2, a_3)$, $0 < a_1 < a_2 < a_3$, and $|a| = a_1 + a_2 + a_3$. Define $\Gamma_a: (0, \infty) \to \mathbf{R}^3$ by

$$(1) \qquad \Gamma_a(t) = (t^{a_1}, t^{a_2}, t^{a_3}).$$

Then the affine arclength measure σ_a on Γ_a is $t^{|a|/6 - 1}\,dt$ up to a multiplicative constant. In [1] Drury studied the curves Γ_a for $a = (1, 2, k)$ and proved the following:

THEOREM 2. *Suppose* $a = (1, 2, k)$ *and* $k \geqq 4$. *Then there exists a constant* $C > 0$ *such that*

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(2) $$\|\sigma_a * f\|_{L^2(\mathrm{R}^3)} \leqq C\,\|f\|_{L^{3/2}(\mathrm{R}^3)},$$

for $f \in L^2(\mathrm{R}^3)$.

Subsequently Pan [6] showed that (2) holds if $a = (1, 2, k)$ and $3 < k < 4$, thus filling the gap between Theorem 1 and Theorem 2.

In this paper we shall present a proof of inequality (2) for a large class of Γ_a's. Namely we have

THEOREM 3. *Let* Γ_a *be given as in* (1). *If* $a_2 + a_3 \geqq 5a_1$ *then there exists a constant* $C = C(a)$ *such that*

(3) $$\|\sigma_a * f\|_{L^2(\mathrm{R}^3)} \leqq C\,\|f\|_{L^{3/2}(\mathrm{R}^3)},$$

for $f \in L^{3/2}(\mathrm{R}^3)$.

When $|a| = 6$ we have $\sigma_a \approx dt$. Therefore (3) implies that

$$\left\| \int_0^\infty f(x - \Gamma_a(t))\,dt \right\|_{L^2(\mathrm{R}^3)} \leqq C\,\|f\|_{L^{3/2}(\mathrm{R}^3)}$$

holds if $\Gamma_a(t) = (t, t^2, t^3)$ (previously known), $\Gamma_a(t) = (t, t^{3/2}, y^{7/2})$, or $\Gamma_a(t) = (t, t^{7/3}, t^{8/3})$.

The paper is organized as follows. In the next section we establish some estimates which will be used in the proof of Theorem 3. In section 3 we shall use the method of "cut curves" and Stein's interpolation theorem to complete the proof of Theorem 3. The method of "cut curves" was first used by Oberlin for nondegenerate curves, and later by Drury for degenerate curves. Related results can be found in Ricci and Stein [7], Oberlin [5].

2. Some preliminary estimates.

Let $\alpha, \beta \in \mathrm{R}$. Define ϕ and ψ by

$$\phi(t) = (t+1)^\alpha - t^\alpha, \qquad \psi(t) = (t+1)^\beta - t^\beta,$$

for $t \in (0, \infty)$. *For* $b > a > 0$, we defines $S^{\alpha\beta}$ on R^2 by

(4) $$S^{\alpha\beta} g(x, y) = \int_a^b g(x - \phi(t), y - \psi(t))(t+1)^{(\alpha+\beta-5)/3}\,dt.$$

We have the following uniform $L^{3/2} \to L^3$ boundedness result:

THEOREM 4. *Suppose* $\alpha > 1$, $\beta > 1$, $\alpha \neq \beta$. *Then there exists a constant* $C = C(\alpha, \beta)$ *which is independent of a and b such that*

(5) $$\|S^{\alpha\beta} g\|_{L^3(\mathrm{R}^2)} \leqq C\,\|g\|_{L^{3/2}(\mathrm{R}^2)},$$

for $g \in L^{3/2}(\mathrm{R}^2)$.

In order to prove Theorem 4, we first list and prove a few lemmas.

LEMMA 1. *Suppose* $v > 0$. *Then there are constants* $c_v, C_v > 0$ *such that*

$$c_v(t+1)^{v-1} \leqq (t+1)^v - t^v \leqq C_v(t+1)^{v-1}, \tag{6}$$

for $t > 0$.

LEMMA 2. *Suppose* $\beta > \alpha > 1$. *Then there exists a constant* $C = C(\alpha, \beta) > 0$ *such that*

$$\psi''(t)\phi'(t) - \phi''(t)\psi'(t) \geqq C(t+1)^{\alpha+\beta-5}, \tag{7}$$

for $t > 0$.

The proof of Lemma 1 involves a simple use of the mean value theorem and we omit the details. To prove Lemma 2, we observe that

$$\begin{aligned}
\frac{[\psi''(t)\phi'(t) - \phi''(t)\psi'(t)]}{\alpha\beta} &= (\beta-1)[(t+1)^{\beta-2} - t^{\beta-2}][(t+1)^{\alpha-1} - t^{\alpha-1}] \\
&\quad -(\alpha-1)[(t+1)^{\beta-1} - t^{\beta-1}][(t+1)^{\alpha-2} - t^{\alpha-2}] \\
&= (\beta-\alpha)(t+1)^{-1}[(t+1)^{\beta-1} - t^{\beta-1}] \\
&\quad \times [(t+1)^{\alpha-1} - t^{\alpha-1}] \\
&\quad +(\alpha-1)(\beta-1)(t+1)^{-1}t^{\alpha+\beta-3} \\
&\quad \times \left[\frac{\left(\frac{t+1}{t}\right)^{\beta-1} - 1}{\beta-1} - \frac{\left(\frac{t+1}{t}\right)^{\alpha-1} - 1}{\alpha-1}\right] \\
&\geqq C(t+1)^{\alpha+\beta-5},
\end{aligned}$$

where we used Lemma 1 and the inequality

$$\frac{s^x - 1}{x} > \frac{s^y - 1}{y}$$

which holds when $x > y > 0$, $s > 1$. The proof of Lemma 2 is complete.

We shall need the following lemma, due to van der Corput ([10]), in order to establish a key estimate for certain oscillatory integrals (Lemma 4).

LEMMA 3. *Suppose* φ *and* h *are smooth on* $[a, b]$ *and* φ *is real-valued. If* $|\varphi'(x)| \geqq \lambda$, *and* φ' *is monotone on* $[a, b]$, *then*

$$\left|\int_a^b e^{i\varphi(x)}h(x)\,dx\right| \leqq 4\lambda^{-1}\left(|h(b)| + \int_a^b |h'(x)|\,dx\right).$$

LEMMA 4. *Suppose $\beta > \alpha > 1$, $b > a > 0$, $\xi, \eta, \zeta \in \mathrm{R}$. Define*

$$(8)\qquad I(\xi,\eta,\zeta) = \int_a^b e^{i(\xi\phi(t)+\eta\psi(t))}(t+1)^{(\beta-3)/2+i\zeta}\,dt.$$

Then there exists a constant $C = C(\alpha,\beta)$ which is independent of a, b, ζ, ξ and η such that

$$(9)\qquad |I(\xi,\eta,\zeta)| \leqq C|\eta|^{-1/2}(1+|\zeta|)^{1/2},$$

for $\xi,\eta,\zeta \in \mathrm{R}$, and $a, b > 0$.

PROOF OF LEMMA 4. Let $x = \phi(t)$, $\Phi(x) = \psi(\phi^{-1}(x))$. Define $p(x)$ by

$$p(x) = \frac{(t+1)^{(\beta-3)/2+i\zeta}}{\phi'(t)},$$

where $t = \phi^{-1}(x)$. Without loss of generality we may assume that $\eta > 0$. By Lemma 1 and Lemma 2 we have

(i) $C_1(t+1)^{\alpha-1} \leqq x \leqq C_2(t+1)^{\alpha-1}$,

(ii) $C_3(t+1)^{\beta-\alpha} \leqq \Phi'(x) \leqq C_4(t+1)^{\beta-\alpha}$;

(iii) $\Phi''(x) \geqq C_5(t+1)^{\beta-2\alpha+1}$;

(iv) $\displaystyle\int_{\phi^{-1}([z,w])} (t+1)^{(\beta-3)/2}\left|\left(\frac{1}{\phi'(t)}\right)_t'\right| dt \leqq C\max\{z^{(\beta-2\alpha+1)/2(\alpha-1)}\}$,

if $1 < z < w$ and $\beta - 2\alpha + 1 \neq 0$.

Let $A = \phi(a)$, $B = \phi(b)$, and $\rho = -\xi/\eta$. Then $B > A > 1$ and

$$(10)\qquad I(\xi,\eta,\zeta) = \int_A^B e^{i(\xi x+\eta\Phi(x))}p(x)\,dx.$$

For $\Phi'(A) < \rho < \Phi'(B)$, let $\gamma = (\Phi')^{-1}(\rho)$; for $\rho \leqq \Phi'(A)$, let $\gamma = A$; for $\gamma \geqq \Phi'(B)$, let $\gamma = B$.

Let $\theta = \beta - 2\alpha + 1$. We shall examine the following three cases separately: (I) $\theta > 0$; (II) $\theta < 0$; (III) $\theta = 0$.

Case I: $\theta > 0$.

We pick $\delta > 0$ such that

$$(11)\qquad (\gamma+\delta)^{\theta/2(\alpha-1)}\delta = \eta^{-1/2}(1+|\zeta|)^{1/2}.$$

Let $d = \min\{B, \gamma+\delta\}$, $J_k = [2^k(\gamma+\delta), 2^{k+1}(\gamma+\delta)] \cap [\gamma, B]$, $k \geqq 0$. For $x \in [\gamma, B]$ we have

$$\frac{d}{dx}(\xi x + \eta\Phi(x)) \geqq \eta(\Phi'(x) - \Phi'(\gamma)). \tag{12}$$

By (iii) we also have

$$\Phi'(x) - \Phi'(\gamma) \geqq Cx^{\theta/(\alpha-1)}\delta, \tag{13}$$

for $x \geqq \gamma + \delta$. By (iv) we have

$$\begin{aligned} \int_{J_k} |p'(x)|\, dx &\leqq C(1 + |\zeta|) \int_{J_k} x^{\theta/2(\alpha-1)-1}\, dx \\ &\quad + C \int_{\phi^{-1}(J_k)} (t+1)^{(\beta-3)/2} \left| \left(\frac{1}{\phi'(t)} \right)'_t \right| dt \\ &\leqq C(1 + |\zeta|)(\gamma + \delta)^{\theta/2(\alpha-1)} 2^{\theta k/2(\alpha-1)}. \end{aligned} \tag{14}$$

By (11) we have

$$\begin{aligned} \left| \int_\gamma^d e^{i(\xi x + \eta\Phi(x))} p(x)\, dx \right| &\leqq C \int_\gamma^d x^{\theta/2(\alpha-1)}\, dx \\ &\leqq C(\gamma + \delta)^{\theta/2(\alpha-1)}\delta \\ &= C\eta^{-1/2}(1 + |\zeta|)^{1/2}. \end{aligned} \tag{15}$$

By (12)–(14) and van der Corput's lemma, we find

$$\begin{aligned} \left| \int_{J_k} e^{i(\xi x + \eta\Phi(x))} p(x)\, dx \right| &\leqq C(\eta\delta)^{-1}[2^k(\gamma + \delta)]^{-\theta/(\alpha-1)} \\ &\quad \times \left([2^{k+1}(\gamma + \delta)]^{\theta/2(\alpha-1)} + \int_{J_k} |p'(x)|\, dx \right) \\ &\leqq C(\eta\delta)^{-1}(1 + |\zeta|)(\gamma + \delta)^{-\theta/2(\alpha-1)} 2^{-\theta k/2(\alpha-1)}. \end{aligned} \tag{16}$$

Therefore we have

$$\begin{aligned} \left| \int_d^B e^{i(\xi + \eta\Phi(x))} p(x)\, dx \right| &\leqq \sum_{k \geqq 0} \left| \int_{J_k} e^{i(\xi x + \eta\Phi(x))} p(x)\, dx \right| \\ &\leqq C\eta^{-1/2}(1 + |\zeta|)^{1/2}. \end{aligned} \tag{17}$$

Next we treat the integral over the interval $[A, \gamma]$. Let

$$\tau = \eta^{-1/2}(1 + |\zeta|)^{1/2}\gamma^{-\theta/2(\alpha-1)}$$

and $D = \max\{A, \gamma - \tau\}$. Then we have

(18) $$\left|\int_D^\gamma e^{i(\xi x+\eta\Phi(x))}p(x)\,dx\right| \leqq C\gamma^{\theta/2(\alpha-1)}\tau$$
$$=C\eta^{-1/2}(1+|\zeta|)^{1/2}.$$

For $x\in[A,D]$ we have

(19) $$\left|\frac{d}{dx}(\xi x+\eta\Phi(x))\right| \geqq \eta(\Phi'(\gamma)-\Phi'(x))$$
$$\geqq C\eta\gamma^{\theta/(\alpha-1)}\tau.$$

By van der Corput's lemma we obain

(20) $$\left|\int_{\tilde{J}_k} e^{i(\xi x+\eta\Phi(x))}\,p(x)\,dx\right| \leqq C(\eta\tau)^{-1}\gamma^{-\theta/(\alpha-1)}(1+|\zeta|)(2^{-k}D)^{\theta/2(\alpha-1)},$$

where $\tilde{J}_k=[2^{-(k+1)}D,2^{-k}D]\cap[A,D]$, $k\geqq 0$. We have

(21) $$\left|\int_A^D e^{i(\xi x+\eta\Phi(x))}p(x)\,dx\right| \leqq C(\eta\tau)^{-1}(1+|\zeta|)\gamma^{-\theta/2(\alpha-1)}\left(\sum_{k\geqq 0}2^{-k\theta/2(\alpha-1)}\right)$$
$$=C\eta^{-1/2}(1+|\zeta|)^{1/2}.$$

By combining (15) (17) (18) and (21) we find

(22) $$\left|\int_A^B e^{i(\xi x+\eta\Phi(x))}p(x)\,dx\right| \leqq C\eta^{-1/2}(1+|\zeta|)^{1/2}$$

if $\theta=\beta-2\alpha+1>0$.

Case II: $\theta<0$.

First we prove that

(23) $$\left|\int_\gamma^B e^{i(\xi x+\eta\Phi(x))}p(x)\,dx\right| \leqq C\eta^{-1/2}(1+|\zeta|)^{1/2}.$$

To this end we pick $s>0$ such that

(24) $$(\gamma+s)^{(\beta-1)/2(\alpha-1)}-\gamma^{(\beta-1)/2(\alpha-1)}=\eta^{-1/2}(1+|\zeta|)^{1/2}.$$

Since $\theta/2(\alpha-1)+1=(\beta-1)/2(\alpha-1)>0$, by Lemma 1 we have

(25) $$(\gamma+s)^{\theta/2(\alpha-1)}s \geqq C[(1+\gamma/s)^{(\beta-1)/2(\alpha-1)}-(\gamma/s)^{(\beta-1)/2(\alpha-1)}]s^{(\beta-1)/2(\alpha-1)}$$
$$=C\eta^{-1/2}(1+|\zeta|)^{1/2}.$$

By (i) and (ii) there exists a constant $L>1$ such that $\Phi'(x)>2\Phi'(y)$ whenever $x\geqq Ly$, $y>1$. Let $\Omega_1=[\gamma,B]\cap[\gamma,\gamma+s]$, $\Omega_2=[\gamma,B]\cap[\gamma+s,L(\gamma+s)]$, and $\Omega_3=[\gamma,B]\cap[L(\gamma+s),\infty)$. For $x\in\Omega_3$ we have

$$\frac{d}{dx}(\xi x + \eta\Phi(x)) \geqq (\eta/2)\Phi'(x), \tag{26}$$

By van der Corput's lemma and (26) we have

$$\begin{aligned} \left|\int_{\Omega_3} e^{i(\xi x + \eta\Phi(x))} p(x)\,dx\right| &\leqq C\eta^{-1}(\gamma + s)^{-(\beta-\alpha)/(\alpha-1)}(1 + |\zeta|)(\gamma + s)^{\theta/2(\alpha-1)} \\ &= C\eta^{-1}(1 + |\zeta|)(\gamma + s)^{-(\beta-1)/2(\alpha-1)} \\ &\leqq C\eta^{-1/2}(1 + |\zeta|)^{1/2}. \end{aligned} \tag{27}$$

Let $k \geqq 0$, $\omega_k = \Omega_2 \cap [\gamma + 2^k s, \gamma + 2^{k+1}s]$. For $x \in \omega_k$ we have

$$\frac{d}{dx}(\xi x + \eta\Phi(x)) \geqq C\eta(\gamma + s)^{\theta/(\alpha-1)}2^k s. \tag{28}$$

By van der Corput's lemma, (28) and (25), we have

$$\begin{aligned} \left|\int_{\Omega_3} e^{i(\xi x + \eta\Phi(x))} p(x)\,dx\right| &\leqq \sum_{k\geqq 0}\left|\int_{\omega_k} e^{i(\xi x + \eta\Phi(x))} p(x)\,dx\right| \\ &\leqq C\sum_{k\geqq 0}(\eta s)^{-1}(\gamma + s)^{-\theta/(\alpha-1)} \\ &\qquad \times 2^{-k}(1 + |\zeta|)(\gamma + s)^{\theta/2(\alpha-1)} \\ &\leqq C\eta^{-1/2}(1 + |\zeta|)^{1/2}. \end{aligned} \tag{29}$$

By (24) we also have

$$\left|\int_{\Omega_1} e^{i(\xi x + \eta\Phi(x))} p(x)\,dx\right| \leqq C\eta^{-1/2}(1 + |\zeta|)^{1/2}. \tag{30}$$

By combining (27) (29) and (30) we see that (23) holds.

It remains for us to show that

$$\left|\int_A^\gamma e^{i(\xi x + \eta\Phi(x))} p(x)\,dx\right| \leqq C\eta^{-1/2}(1 + |\zeta|)^{1/2}. \tag{31}$$

If $\gamma^{(\beta-1)/2(\alpha-1)} \leqq \eta^{-1/2}(1 + |\zeta|)^{1/2}$, then we have

$$\begin{aligned} \left|\int_A^\gamma e^{i[\xi x + \eta\Phi(x))} p(x)\,dx\right| &\leqq \int_0^\gamma x^{\theta/2(\alpha-1)}\,dx \\ &= C\eta^{-1/2}(1 + |\zeta|)^{1/2}. \end{aligned}$$

Therefore we may assume that $\gamma^{(\beta-1)/2(\alpha-1)} > \eta^{-1/2}(1 + |\zeta|)^{1/2}$. Pick $\varepsilon, \kappa \in (0, \gamma)$ such that

(32) $\kappa^{(\beta-1)/2(\alpha-1)} = \eta^{-1/2}(1+|\zeta|)^{1/2} = \gamma^{(\beta-1)/2(\alpha-1)} - (\gamma-\varepsilon)^{(\beta-1)/2(\alpha-1)}.$

Thus we have

$$\gamma^{\theta/2(\alpha-1)}\varepsilon \geqq C\eta^{-1/2}(1+|\zeta|)^{1/2}. \tag{33}$$

We then write $[A,\gamma] = \bigcup_{j=1}^{4} \tilde{\Omega}_j$, where $\tilde{\Omega}_1 = [0,\kappa] \cap [A,\gamma]$, $\tilde{\Omega}_2 = ([\gamma-\varepsilon,\gamma] \cap [A,\gamma])\backslash\tilde{\Omega}_1$, $\tilde{\Omega}_3 = ([0,\gamma/2] \cap [\kappa,\gamma] \cap [A,\gamma])\backslash\tilde{\Omega}_2$ and $\tilde{\Omega}_4 = \bigcup_{k\geqq 0} \tilde{\omega}_k$, with

$$\tilde{\omega}_k = ([\gamma - 2^{k+1}\varepsilon, \gamma - 2^k\varepsilon] \cap [\gamma/2,\gamma] \cap [A,\gamma])\backslash\tilde{\Omega}_1.$$

To prove (31), it suffices to show that

$$\left|\int_{\tilde{\Omega}_j} e^{i[\xi x+\eta\Phi(x))}p(x)\,dx\right| \leqq C\eta^{-1/2}(1+|\zeta|)^{1/2}, \tag{34}$$

for $j = 1, 2, 3, 4$.

By (32) we see that (34) holds for $j = 1,2$. For $x \in \tilde{\Omega}_3$ we have

$$\left|\frac{d}{dx}(\xi x + \eta\Phi(x))\right| \geqq C\eta\gamma^{\theta/(\alpha-1)}\gamma,$$

and

$$\left|\int_{\tilde{\Omega}_3} e^{i[\xi x+\eta\Phi(x))}p(x)\,dx\right| \leqq C\eta^{-1}\gamma^{-\theta/(\alpha-1)-1}(1+|\zeta|)\kappa^{\theta/2(\alpha-1)}$$
$$\leqq C\eta^{-1/2}(1+|\zeta|)^{1/2},$$

which proves (34) for $j = 3$. Finally, for $x \in \tilde{\omega}_k$ we have $\gamma/2 \leqq x \leqq \gamma$ and

$$\left|\frac{d}{dx}(\xi x + \eta\Phi(x))\right| \geqq C\eta\gamma^{\theta/(\alpha-1)}2^k\varepsilon.$$

Therefore we have

$$\left|\int_{\tilde{\Omega}_4} e^{i[\xi x+\eta\Phi(x))}p(x)\,dx\right| \leqq \sum_{k\geqq 0}\left|\int_{\tilde{\omega}_k} e^{i[\xi x+\eta\Phi(x))}p(x)\,dx\right|$$
$$\leqq C(1+|\zeta|)\sum_{k\geqq 0}(\eta\varepsilon\gamma^{\theta/(\alpha-1)})^{-1}2^{-k}\gamma^{\theta/2(\alpha-1)}$$
$$\leqq C\eta^{-1/2}(1+|\zeta|)^{1/2}.$$

This completes the proof of case (II).

Case III: $\theta = 0$.

For $x > 1$ we have $\phi'(\phi^{-1}(x)) \approx x^{(\alpha-2)/(\alpha-1)}$ and $|p(x)| \leqq C$. Suppose $z, w > 1$ and $w > z$. Then

$$\int_z^w |p'(x)|\,dx \leqq C(1+|\zeta|)\left[\int_z^w \frac{dx}{x} + \int_{\phi^{-1}(z)}^{\phi^{-1}(w)} \frac{|\phi''(t)|}{\phi'(t)}\,dt\right] \tag{35}$$
$$\leqq C(1+|\zeta|)(1+\ln(w/z)).$$

Let $v = \eta^{-1/2}(1+|\zeta|)^{1/2}$, $\Delta_k = [\gamma + 2^k v, \gamma + 2^{k+1} v]$, $k \geqq 0$. Since

$$\left|\frac{d}{dx}(\xi x + \eta\Phi(x))\right| \geqq C\eta(2^k v)$$

for $x \in [\gamma, B] \cap \Delta_k$, we have

$$\left|\int_\gamma^B e^{i[\xi x + \eta\Phi(x))} p(x)\,dx\right| \leqq C\left[v + (1+|\zeta|)\sum_{k\geqq 0}(\eta 2^k v)^{-1}\ln 2\right] \tag{36}$$
$$\leqq C\eta^{-1/2}(1+|\zeta|)^{1/2}.$$

It remains for us to show that

$$\left|\int_A^\gamma e^{i[\xi x + \eta\Phi(x))} p(x)\,dx\right| \leqq C\eta^{-1/2}(1+|\zeta|)^{1/2}. \tag{37}$$

To prove (37) it suffices to show that

$$\left|\int_E e^{i[\xi x + \eta\Phi(x))} p(x)\,dx\right| \leqq C\eta^{-1/2}(1+|\zeta|)^{1/2}, \tag{38}$$

where $E = [A,\gamma] \cap [0, \gamma - v] \cap [v, \infty)$. Clearly we may assume that $\gamma > 4v$. Let $k_0 \in \mathbf{N}$ such that $2^{k_0+1} v < \gamma \leqq 2^{k_0+2} v$. Let $\tilde{\Delta}_k = [\gamma - 2^{k+1} v,\ \gamma - 2^k v]$ for $k = 0, 1, \ldots, k_0 - 1$, and $\tilde{\Delta}_{k_0} = [v, \gamma - 2^{k_0} v]$. Then we have

$$\int_{\tilde{\Delta}_k} |p'(x)|\,dx \leqq C(1+|\zeta|)\ln 2 \tag{39}$$

for $k = 0, 1, \ldots, k_0 - 1$, and

$$\int_{\tilde{\Delta}_{k_0}} |p'(x)|\,dx \leqq C(1+|\zeta|)k_0. \tag{40}$$

Therefore we have

$$\left|\int_E e^{i[\xi x + \eta\Phi(x))} p(x)\,dx\right| \leqq C(\eta v)^{-1}(1+|\zeta|)\left(\sum_{k=0}^{k_0-1}\frac{\ln 2}{2^k} + \frac{k_0}{2^{k_0}}\right) \tag{41}$$
$$\leqq C\eta^{-1/2}(1+|\zeta|)^{1/2}.$$

This completes the proof of case (III). Lemma 4 is proved.

We are now ready to prove Theorem 4.

Proof of Theorem 4. Without loss of generality we may assume that $\beta > \alpha > 1$. For $z \in \mathbb{C}$, define T_z by

(42) $$\widehat{T_z g}(x, y) = \frac{1}{\Gamma\left(\frac{z+1}{2}\right)} \int_{-\infty}^{\infty} \int_a^b g(x - \phi(t),$$

$$y - \psi(t) - s)\,|s|^z\,(t+1)^{\alpha - 2 - z\theta/3}\,dt\,ds,$$

where $\theta = \beta - 2\alpha + 1$. By the calculation in [3] we find

(43) $$T_z g(\xi, \eta) = m_z(\xi, \eta)\hat{g}(\xi, \eta),$$

where

(44) $$m_z(\xi, \eta) = \frac{2^{z+1}\pi^{1/2}|\eta|^{-z-1}}{\Gamma(-z/2)} \int_a^b e^{-i(\xi\phi(t) + \eta\psi(t))}(t+1)^{\alpha - 2 - z\theta/3}\,dt.$$

Therefore $T_{-1} = S^{\alpha\beta}$. Let $y \in \mathbb{R}$. Since $\phi'(t) \approx (t+1)^{\alpha-2}$, we have

$$\|T_{iy} g\|_\infty \leqq \frac{C}{\left|\Gamma\left(\frac{1+iy}{2}\right)\right|}\|g\|_1.$$

By (44), Lemma 4, and Plancherel's theorem, we have

$$\|T_{-3/2+iy} g\|_2 \leqq \frac{C(1+|y|)^{1/2}}{\left|\Gamma\left(\frac{3-2iy}{4}\right)\right|}\|g\|_2.$$

By invoking Stein's interpolation theorem ([8]), we find

$$\|T_{-1} g\|_3 \leqq C\,\|g\|_{3/2}$$

for some constant C which is independent of a, b. The proof of Theorem 4 is complete.

3. Proof of the main theorem.

We now prove Theorem 3. By using a change of variable if necessary, we may assume that $a_1 = 1$. Let $\alpha = a_2$, $\beta = a_3$. Then $1 < \alpha < \beta$ and $\alpha + \beta$ and $\alpha + \beta \geqq 5$. Let

$$Tf(x, y, z) = \int_0^\infty f(x - t, y - t^\alpha, z - t^\beta)\,t^{(\alpha+\beta-5)/6}\,dt.$$

We need to show that

(45) $$\|Tf\|_{L^2(\mathbb{R}^3)} \leqq C\|f\|_{L^{3/2}(\mathbb{R}^3)}.$$

Define I^+f and I^-f by

$$(I^+f)(x,y,z) = \int_0^\infty \int_0^\infty f(x-u, y-u^\alpha\phi(s),$$
$$z-u^\beta\psi(s))[(s+1)s]^{(\alpha+\beta-5)/6}u^{(\alpha+\beta-2)/3}\,ds\,du,$$
$$(I^-f)(x,y,z) = \int_0^\infty \int_0^\infty f(x+u, y+u^\alpha\phi(s),$$
$$z+u^\beta\psi(s))[(s+1)s]^{(\alpha+\beta-5)/6}u^{(\alpha+\beta-2)/3}\,ds\,du.$$

By using $(s+1)s \leqq (s+1)^2$ and Theorem 4 we obtain

(46) $$\|I^+f\|_{L^3(\mathbb{R}^3)} \leqq \left\| \int_0^\infty \left\| \int_0^\infty |f(x-u, y-u^\alpha\phi(s), \right.\right.$$
$$\left.\times\, z-u^\beta\psi(s))|\,|(s+1)^{(\alpha+\beta-5)/3}\,ds\right\|_{3,dydz}$$
$$\left.\times\, u^{(\alpha+\beta-2)/3}\,du\right\|_{3,dx}$$
$$\leqq C\left\| \int_0^\infty \left\| f(x-u,y,z)\right\|_{3/2,dydz} u^{-2/3}du\right\|_{3,dx}$$
$$\leqq C\|f\|_{L^{3/2}(\mathbb{R}^3)}.$$

Similarly we have

(47) $$\|I^-f\|_{L^3(\mathbb{R}^3)} \leqq C\|f\|_{L^{3/2}(\mathbb{R}^3)}.$$

Therefore we have

$$\|Tf\|^2_{L^2(\mathbb{R}^3)} \leqq \|T^*Tf\|_{L^3(\mathbb{R}^3)}\|f\|_{L^{3/2}(\mathbb{R}^3)}$$
$$= \|I^+f + I^-f\|_{L^3(\mathbb{R}^3)}\|f\|_{L^{3/2}(\mathbb{R}^3)}$$
$$\leqq C\|f\|^2_{L^{3/2}(\mathbb{R}^3)},$$

which proves (45). The proof of Theorem 3 is now complete.

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