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ON FATOU-TYPE THEOREMS FOR NON-RADIAL KERNELS

SADAHIRO SAEKI

In his 1906 paper [2], P. Fatou proved that if μ is a bounded Borel measure on the unit circle T and if μ is differentiable at some $z_0 \in T$, then the Poisson integral $P[\mu]$ of μ has a nontangential limit at z_0. The "standard" proof of this classical result heavily depends upon some special properties of the Poisson kernel and integration-by-parts, which renders the proof somewhat complicated; see A. Zygmund [9; p. 99–101] and, for a detailed proof, K. Hoffman [3; pp. 34–37]. (The proofs in [9] and in [3] must be modified because the function $g(t)$ in [9], that is, the function $G(t)$ in [3] is *not* integrable on any neighborhood of $t = \pi$ in general.)

Recently higher dimensional analogues of Fatou's theorem and some partial converses to them have been obtained by W. Ramey and D. Ullrich [6] and also by J. Brossard and L. Chevalier [1]. The harmonicity of the Poisson kernel is essential in [6] while the authors of [1] have proved Fatou-type theorems for some radially decreasing continuous kernels including the Gauss kernel. In the present paper, we shall obtain several analogues of Fatou's theorem for a wide class of kernels which are not necessarily radial or real-valued.

In the sequel, we shall choose and fix a norm $\|\cdot\|$ on R^n for which the unit open ball $B := \{x: \|x\| < 1\}$ has $|B| = 1$, where $|B|$ is the Lebesgue measure of B. A nonnegative function K on R^n is said to be radially decreasing if $\|x\| \leqq \|x'\|$ implies $K(x) \geqq K(x')$. We shall write λ for Lebesgue measure on R^n. In the following definitions, F is a Borel function on R^n, $x_0 \in R^n$, and μ is a Radon measure on R^n, that is, a countably additive complex-valued set-function defined on the ring of all bounded Borel subsets of R^n.

Define $F_y(x) = y^{-n}F(x/y)$ for $y > 0$ and $x \in R^n$. Let

$$F[\mu](x_0, y) = (F_y * \mu)(x_0) = \int F_y(x_0 - x)\, d\mu(x) \tag{0.1}$$

for each $y > 0$ for which the integral exists. The least radially decreasing majorant of F is the function F^* defined by

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(0.2) $$F^*(x) = \sup\{|F(x')|: x' \in \mathsf{R}^n \quad \text{and} \quad \|x'\| \geqq \|x\|\} \quad \forall x \in \mathsf{R}^n.$$

Let E be a bounded Borel subset of R^n with $|E| > 0$. We say that μ *is differentiable at* x_0 *with respect to* E if there exists $\gamma \in \mathsf{C}$ such that

(0.3) $$\lim_{r \downarrow 0} \mu(x_0 + rx + rE)/|rE| = \gamma \quad \forall x \in \mathsf{R}^n.$$

In this case, we call γ *the derivative of* μ *at* x_0 *with respect to* E and write $(D_E\mu)(x_0)$ for γ. If, in addition, $E = B$, then we shall drop "with respect to B" in the above definitions and write $(D\mu)(x_0)$ for γ. *The Hardy-Littlewood maximal function* of μ at x_0 is

(0.4) $$(M\mu)(x_0) = \sup\{|\mu|\,(x_0 + rB)/|rB|: r > 0\}.$$

If μ is real-valued, we define the *upper derivative* and the *symmetric upper derivative* of μ at x_0 by setting

(0.5) $$(\mathrm{UD}\,\mu)(x_0) = \sup\{\limsup_{r \downarrow 0} \mu(x_0 + rx + rB)/|rB|: x \in \mathsf{R}^n\}, \quad \text{and}$$

(0.6) $$(\mathrm{SUD}\,\mu)(x_0) = \limsup_{r \downarrow 0} \mu(x_0 + rB)/|rB|.$$

Now let H be a function defined on a strip $\mathsf{R}^n \times (0, t)$ for some $t > 0$. We say that H has a *sectorial limit* $\gamma \in \mathsf{C}$ at $(x_0, 0)$ if

(0.7) $$\lim_{y \downarrow 0} H(x_0 + yx, y) = \gamma \quad \forall x \in \mathsf{R}^n.$$

A *nontangential null sequence* is a sequence (x_k, y_k) in the open upper half-space $\mathsf{R}^n \times \mathsf{R}^+$ such that $\|x_k\| + y_k = o(1)$ and $\|x_k\|/y_k = O(1)$ as $k \to \infty$. H is said to have a *nontangential limit* γ at $(x_0, 0)$ if

(0.8) $$\lim_{k \to \infty} H(x_0 + x_k, y_k) = \gamma$$

for all such sequences. Finally, let K be a strictly positive Borel function on R^n. We say that K satisfies the *comparison condition* if

(0.9) $$\sup\{K_y(x)/K(x): 0 < y < 1 \quad \text{and} \quad \|x\| \geqq 1\} < \infty.$$

For some examples of such functions, see Examples 1.7 stated below.

If $T = \chi_B$ and $F = P$ (the Poisson kernel), then Fatou's theorem for the real line is the statement that if $|F|\,[|\mu|] < \infty$ everywhere and if $T[\mu]$ has a sectorial limit γ at $(x_0, 0)$, then $F[\mu]$ has nontangential limit γ at $(x_0, 0)$. It is natural to ask: *For which test functions* T *and kernels* F, *is this abstract form of Fatou's theorem valid?* (In our usage, a kernel simply means an L_1-function.) In section 1, we shall prove that if x_0 is a Lebessgue point of μ with $(M\mu)(x_0) < \infty$, then $F[\mu]$ has a nontangential limit at $(x_0, 0)$ whenever F is a bounded Borel kernel with $\|F^*\|_1 < \infty$. In the other three sections, x_0 is always chosen to be the origin of R^n. In section 2, the Lebesgue point condition at 0 is replaced by the weaker condition that $(\mathrm{SUD}\,|\mu|)(0) < \infty$. We shall prove that if $(M\mu)(0) < \infty$ and if there exists a bounded Borel kernel T on R^n such that $\int T\,dx = 1$, $\int T^*\,dx < \infty$ and $T[\mu]$ has a sectoril limit γ at $(0, 0)$, then $F[\mu]$ has nontangential limit $\gamma \int F\,dx$ at $(0, 0)$ for each bounded continuous kernel F with $\int F^*\,dx < \infty$. The major hypothesis in section 3 is that $(\mathrm{UD}\,|\mu|)(0) < \infty$. Results similar to those in section 2 will be proved for a wide class of bounded almost continuous kernels. (For the definition of almost continuity, see section 3.) The final section contains some generalizations of Fatou's theorem for the real line and the circle, each with a simple proof.

§1. Nontangential Convergence at Lebesgue Points.

Let μ be a Radon measure on R^n. Recall that $x_0 \in \mathsf{R}^n$ is called a Lebesgue point of μ if there exists $\gamma \in \mathsf{C}$ such that

$$|\mu - \gamma\lambda|\,(x_0 + rB)/|rB| = 0(1) \quad \text{as} \quad r \downarrow 0,$$

where λ is Lebesgue measure on R^n. In this case, $(D\mu)(x_0)$ exists and equals γ. The following result with $K = P$ (the Poisson kernel) is well-known and is proved in Stein [7; pp. 197–198]; see also Stein and Weiss [8; pp. 13–15].

THEOREM 1.1. *Let the Radon measure μ satisfy*

$$\text{(i)} \qquad |\mu|\,(rB)/|rB| = 0(1) \quad \text{as} \quad r \uparrow \infty,$$

and let K be a bounded Borel function on R^n such that $\int K^*\,dx < \infty$. Then $K^*[|\mu|] < \infty$ everywhere and $K[\mu]$ has nontangential limit $(D\mu)(x_0)\int K\,dx$ at $(x_0, 0)$ for each Lebesgue point x_0 of μ.

PROOF. We shall prove the first conclusion at the very end of the proof.

To prove the second conclusion, choose and fix a Lebesgue point x_0 of μ. We may and do suppose that $(D\mu)(x_0) = 0$ after replacing μ by $\mu - (D\mu)(x_0)\lambda$. It will suffice to show that $K^*[|\mu|]$ has nontangenrial limit 0 at $(x_0, 0)$. Therefore we may further assume that $K = K^*$ and $\mu \geqq 0$, and also $K(x) > 0$ for some $x \neq 0$ and $u := \sup\{K(x): x \neq 0\} = K(0)$.

First consider the case $x_0 = 0$, so $(M\mu)(0) < \infty$ by (i). For each $0 < t < u$, let $B_t = \{x: K(x) > t\}$. Then B_t is a ball centered at 0 because K is a radially decreasing L_1-function. Moreover, Fubini's Theorem ensures that $y > 0$ implies

$$(1)\qquad (K_y * \mu)(0) = y^{-n}\int K(x/y)\,d\mu(x) = y^{-n}\int\int_0^{K(x/y)} dt\,d\mu(x)$$
$$= y^{-n}\int_0^u \mu(\{x: K(x/y) > t\})\,dt$$
$$= y^{-n}\int_0^u \mu(yB_t)\,dt$$
$$= \int_0^u \frac{\mu(yB_t)}{|yB_t|}\cdot|B_t|\,dt.$$

This with $\mu = \lambda$ shows that $t \to (M\mu)(0)|B_t|$ is an L_1-function over $(0, u)$, and the integrand of the last integral in (1) is dominated by this L^1-function and converges pointwise to 0 as $y \downarrow 0$ since $(D\mu)(0) = 0$. It follows from Lebesgue's Convergence Theorem that $(K_y * \mu)(0) \to 0$ as $y \downarrow 0$. (Notice that this proof does *not* require the boundedness of K, so it gives a simple proof of Theorem 1.25 in [8; p. 13].)

Now let $\alpha > 0$ be given. By our additional assumption assumption on K, $K(x) > 0$ for some $x \neq 0$. Therefore Lemma 1.3 stated below yields $\beta = \beta(\alpha) > 0$ such that $\|x'\| < \alpha$ implies $K(x' + x) \leqq \beta K(x/\beta)$ for all $x \in \mathrm{R}^n$. Therefore, if $y > 0$ and $\|x'\|/y < \alpha$, then

$$(2)\qquad K[\mu](x', y) = y^{-n}\int K((x' - x)/y)\,d\mu(x)$$
$$\leqq y^{-n}\beta\int K(-x/\beta y)\,d\mu(x)$$
$$= \beta^{n+1}(K_{\beta y} * \mu)(0) = o(1) \quad \text{as} \quad y \downarrow 0.$$

Since $\alpha > 0$ was arbitrary, this confirms that $K[\mu]$ has nontangential limit 0 at $(0, 0)$.

In general, define a Radon measure μ_0 by setting $\mu_0(E) = \mu(E + x_0)$ for each bounded Borel subset E of R^n. Then μ_0 satisfies condition (i), 0 is a Lebesgue point of μ_0 with $(D\mu_0)(0) = 0$, and a simple calculation shows that

$$K[\mu](x, y) = K[\mu_0](x - x_0, y) \quad \forall x \in \mathrm{R}^n \quad \text{and} \quad y > 0.$$

Hence, by the above result applied to μ_0, $K[\mu]$ has nontangential limit 0 at $(x_0, 0)$.

Finally, pick any Lebesgue point x_0 of μ; such a point exists by Lebesgue's Differentiation Theorem. Define μ_0 as in the last paragraph. Then (1) and (2) applied to μ_0 show that $K[\mu_0] < \infty$ everywhere, which completes the proof.

REMARK 1.2. Let K be a radially decreasing bounded L_1-function on R^n with

$K(x) > 0$ for some $x \neq 0$, and let μ be a Radon measure on R^n. Then the last proof shows that

(a) $(K_y * |\mu|)(0) \leqq \|K\|_1 (M\mu)(0) \quad \forall y > 0,$

and that given $\alpha > 0$, there exists $\delta = \beta^{n+1} > 0$ such that

(b) $K[|\mu|](x', y) \leqq \delta \, \|K\|_1 (M\mu)(0)$

whenever $x' \in \mathsf{R}^n$, $y > 0$ and $\|x'\|/y < \alpha$.

LEMMA 1.3. *Let K be a bounded nonnegative radially decreasing function on R^n such that $K(x) > 0$ for some $x \neq 0$. Given $\alpha > 0$, there exists $\beta > 0$ such that*

$$K(x' + x) \leqq \beta K(x/\beta) \quad \forall x' \in \alpha B \quad \textit{and} \quad x \in \mathsf{R}^n.$$

PROOF. Fix $\alpha > 0$ and any unit vector x_0 in R^n. Choose $\beta > 2$ so large that $K(0) < \beta K(2\alpha x_0/\beta)$. If $\|x'\| < \alpha$ and $\|x\| \leqq 2\alpha$, then

$$K(x' + x) \leqq K(0) \leqq \beta K(2\alpha x_0/\beta) \leqq \beta K(x/\beta)$$

since K is radially decreasing. If $\|x'\| < \alpha$ and $\|x\| > 2\alpha$, then $\|x\|/2 \leqq \|x' + x\|$; hence

$$K(x' + x) \leqq K(x/2) \leqq \beta K(x/\beta),$$

as desired.

REMARK 1.4. Suppoose μ is a Radon measure on R^n, K is a bounded nonnegative radially decreasing function on R^n, and $(K_t * |\mu|)(0) < \infty$ for some $t > 0$. Then $K[|\mu|] < \infty$ on the strip $\mathsf{R}^n \times (0, t)$.

To see this, fix $\alpha > 1$. If $x', x \in \mathsf{R}^n$ and $\|x\| \geqq \alpha \|x'\|$, then $\|x' - x\| \geqq \|x\| - \|x'\| \geqq \alpha^{-1}(\alpha - 1)\|x\|$. Therefore $x' \in \mathsf{R}^n$ and $0 < y < \alpha^{-1}(\alpha - 1)t$ implies

$$\begin{aligned} y^n K[|\mu|](x', y) &\leqq K(0)\,|\mu|\,(\alpha \|x'\| B) + \int_{\|x\| \geqq \alpha \|x'\|} K(y^{-1}(x' - x))\, d\,|\mu|\,(x) \\ &\leqq K(0)\,|\mu|\,(\alpha \|x'\| B) + \int K(y^{-1}\alpha^{-1}(\alpha - 1)x)\, d\,|\mu|\,(x) \\ &\leqq K(0)\,|\mu|\,(\alpha \|x'\| B) + t^n (K_t * |\mu|)(0) < \infty. \end{aligned}$$

This confirms our assertion.

THEOREM 1.5. *Let K be a bounded strictly positive radially decreasing function in $L_1(\mathsf{R}^n)$ that satisfies the comparison condition (0, 9), and let μ be a Radon measure on R^n such that $(K_t * |\mu|)(0) < \infty$ for some $t > 0$. Then, for each Borel function F on R^n with $|F| \leqq K$ and each Lebesgue point x_0 of μ, $F[\mu]$ has nontangential limit $(D\mu)(x_0) \int F\, dx$ at $(x_0, 0)$.*

PROOF. We may and do suppose that $(D\mu)(x_0) = 0$ and $\mu \geqq 0$. It will suffice to show that $K[\mu]$ has nontangential limit 0 at x_0.

First consider the case $x_0 = 0$. Let $t > 0$ be as above, and define $\mu' = \chi_{tB}\mu$ and $\mu'' = \mu - \mu'$. Then μ' is a bounded positive measure with $(D\mu')(0) = 0$. Hence $K[\mu']$ has nontangential limit 0 at $(0, 0)$ by Theorem 1.1. To prove this conclusion for $K[\mu'']$, let $\alpha > 0$ and let $\beta = \beta(\alpha) > 0$ be as in Lemma 1.3. Then, if $x' \in \mathrm{R}^n$, $y > 0$ and $\|x'\|/y < \alpha$, we have

$$(1) \qquad \begin{aligned} K[\mu''](tx', ty) &= (ty)^{-n} \int K((tx' - x)/ty)\, d\mu''(x) \\ &\leqq (ty)^{-n}\beta \int K(x/\beta ty)\, d\mu''(x) \\ &= \beta^{n+1} \int K_{\beta ty}(x)\, d\mu''(x) \\ &= \beta^{n+1} \int Q(x, y) K_t(x)\, d\mu(x), \end{aligned}$$

where $Q(x, y) = 0$ for $\|x\| < t$ and

$$Q(x, y) = K_{\beta ty}(x)/K_t(x) = K_{\beta y}(t^{-1}x)/K(t^{-1}x)$$

for $\|x\| \geqq t$. Since $K^* = K \in L_1$, it is easy to check that $K(x) = o(1/\|x\|^n)$ as $\|x\| \to \infty$ (see [8; p. 14]); hence $Q(x, y) = o(1)$ as $y \downarrow 0$ for each fixed $x \in \mathrm{R}^n$. Moreover, Q is uniformly bounded on $\mathrm{R}^n \times (0, 1/\beta)$ because K satisfies (0.9). It follows from (1) and Lebesgue's Convergence Theorem that $K[\mu''](x_k, y_k) = o(1)$ as $k \to \infty$ for each nontangential null sequence (x_k, y_k) with $\sup_k \|x_k\|/y_k < \alpha$. As $\alpha > 0$ was arbitrary, we conclude that $K[\mu'']$ has nontangential limit 0 at $(0, 0)$. Hence $K[\mu] = K[\mu'] + K[\mu'']$ has the same property.

The proof for the general case ($x_0 \neq 0$) will be obvious from the corresponding part in the proof of Theorem 1.1 combined with Remark 1.4.

REMARK 1.6. The comparison condition on K is essential in Theorem 1.5. In fact, suppose that the function K in Theorem 1.5 does *not* satisfy that condition. Then there exists a sequence $(y_k)_1^\infty$ in $(0, 1)$ and a sequence $(x_k)_1^\infty$ in $\mathrm{R}^n \backslash B$ such that $K_{y_k}(x_k)/K(x_k) > k^3$ for all k. It is easy to check that $y_k \to 0$ and $\|x_k\| \to \infty$. Define $\mu = \sum_1^\infty \{k^2 K(x_k)\}^{-1} \delta_{x_k}$, where δ_x is the Dirac measure at x. Then 0 is a Lebesgue point of μ with $(D\mu)(0) = 0$ and $K[\mu](0, 1) = \int K\, d\mu < \infty$. But $K[\mu](0, y_k) = \int K_{y_k}\, d\mu > k$ for all k. Hence the conclusion of Theorem 1.5 does *not* hold.

EXAMPLES 1.7. Let $\alpha \geqq n$ and $\beta \geqq 0$ satisfy either (i) $\alpha > n$ or (ii) $\beta > 1$. Define

$$K(x) = K(x, \alpha, \beta) = 1/[(\|x\|^\alpha + 1)\{\log(2 + \|x\|)\}^\beta] \quad \forall x \in \mathrm{R}^n.$$

Then K is in $L_1(\mathrm{R}^n)$ and satisfies condition (0.9). Similarly, if $\alpha > 0$ and $\beta > 0$, then the function $\exp(-\alpha\|x\|^\beta)$ satisfies (0.9). In particular, both the Poisson kernel and the Gauss kernel satisfy (0.9).

The proof is straightforward.

Now consider the class of all $\phi \in L_2(\mathsf{R})$ having the property that the functions $g_{j,k}(x) := \phi(2^j x - k)$ with $j, k \in \mathsf{Z}$ span a dense subspace of $L_2(\mathsf{R})$. This class of functions is of vital importance in the study of wavelets. The following result gives a sufficient condition in order that ϕ possess the above property.

THEOREM 1.8. *Suppose that ϕ is a bounded measurable function on R^n such that $\int \phi^* \, dx < \infty$ and $\int \phi \, dx = 1$. Let $(\alpha_j)_1^\infty$ be a sequence of positive reals such that $\alpha_j \to \infty$, and let E be a nonempty subset of R^n such that*

(i) $$\liminf_{j\to\infty}\{\text{distance}(\alpha_j x, E)\} < \infty \quad \textit{for} \quad \text{a.a. } x \in \mathsf{R}^n.$$

Then the functions

(ii) $$g_{j,x'}(x) := \phi(\alpha_j x - x') \quad \textit{with} \quad j \in \mathsf{N} \quad \textit{and} \quad x' \in E$$

span a dense subspace of $L_p \cap L_1(\mathsf{R}^n)$ with respect to the norm $\|f\|_{p,1} := \|f\|_p + \|f\|_1$ whenever $1 \leqq p < \infty$.

PROOF. Suppose this is false for some $1 \leqq p < \infty$. Then the Hahn-Banach Theorem yields a nonzero element $f \in (L_{p'} + L_\infty)(\mathsf{R}^n)$ such that

(1) $$\int \phi(\alpha_j x - x') f(x)\, dx = 0 \quad \forall j \in \mathsf{N} \quad \text{and} \quad x' \in E.$$

Write $y_j = 1/\alpha_j$ and $\psi(x) = \phi(-x)$. Then $j \in \mathsf{N}$ and $x' \in E$ implies

(2) $$\psi[f\lambda](y_j x', y_j) = \int \psi_{y_j}(y_j x' - x) f(x)\, dx$$
$$= \alpha_j^n \int \phi(\alpha_j x - x') f(x)\, dx = 0.$$

Now let $x_0 \in \mathsf{R}^n$ be a Lebesgue point of f satisfying (i). So there exists a subsequence $(\alpha_j')_1^\infty$ of $(\alpha_j)_1^\infty$ and a sequence $(x_j)_1^\infty$ of elements of E such that $\sup_j \|\alpha_j' x_0 - x_j\| < \infty$. Since $\alpha_j \to \infty$, it follows that the sequene $(y_j' x_j, y_j')$, where $y_j' = 1/\alpha_j'$, converges nontangentially to $(x_0, 0)$. Since ψ^* is a bounded radially decreasing L_1-function and $\int \psi \, dx = 1$, it follows from Theorem 1.1 that $\psi[f\lambda](y_j' x_j, y_j') \to f(x_0)$ as $j \to \infty$; hence $f(x_0) = 0$ by (2). Since this is true for all x_0 as above, we conclude from Lebesgue's Differentiation Theorem that $f = 0$ almost everywhere, which contradicts our choice of f.

§2. Fatou's Theorem for Continuous Kernels.

Throughout this section, we choose and fix a strictly positive radially decreasing bounded L_1-function K on R^n that satisfies the comparison condition (0.9), and also a Radon measure μ on R^n such that

$$(2.0) \qquad (\mathrm{SUD}\,|\mu|)(0) < \infty \quad \text{and} \quad (K_t * |\mu|)(0) < \infty \quad \text{for some} \quad t > 0.$$

Thus Remark 1.4 ensures that if F is a Borel function on R^n with $|F| \leqq K$, then $F[\mu]$ is defined on the strip $\mathsf{R}^n \times (0, t)$.

LEMMA 2.1. *Let H be a radially decreasing bounded L_1-function on R^n. Then there exists a radially decreasing function $S \in C(\mathsf{R}^n) \cap L_1(\mathsf{R}^n)$ such that $H < S$ pointwise and*

$$\sup\{H(x' - t^{-1}x)/S(x) : \|x'\| < \alpha \quad and \quad 0 < t < \alpha\} \to 0 \quad as \quad \|x\| \to \infty$$

for each fixed $\alpha > 0$.

PROOF. It is easy to construct a radially decreasing function $F \in C(\mathsf{R}^n) \cap L_1(\mathsf{R}^n)$ such that $H < F$ pointwise. Regard F as a function on $[0, \infty)$ in the obvious fashion. Then we can choose a sequence $r_0 = 0 < r_1 < r_2 < \cdots$ such that

$$(1) \qquad F(r_{j-1}/2^{j-1}) > (j+1)F(r_j/2^j) \quad \forall j \in \mathsf{N}, \quad \text{and}$$

$$(2) \qquad \sum_{j=1}^{\infty} (j+2)! \int_{R(r_{j-1}, r_j)} F(x/2^j)\, dx < \infty,$$

where $R(r, s) = \{x : r \leqq \|x\| < s\}$ for $r < s$. Define

$$(3) \qquad S(x) = j! \cdot \max\{F(x/2^{j-1}), (j+1)F(r_j/2^j)\} \quad \forall x \in R(r_{j-1}, r_j) \;\&\; j \in \mathsf{N}.$$

Since F is a radially decreasing continuous function on R^n, it follows from (1) and (3) that S is also such a function on R^n. Moreover, $S \geqq F > H$ pointwise and

$$(4) \qquad \int_{R(r_{j-1}, r_j)} S(x)\, dx \leqq j! \int_{R(r_{j-1}, r_j)} \{F(x/2^{j-1}) + (j+1)F(r_j/2^j)\}\, dx$$
$$\leqq (j+2)! \int_{R(r_{j-1}, r_j)} F(x/2^j)\, dx.$$

Hence $\int S(x)\, dx < \infty$ by (2).

Finally, let $\alpha > 0$ be given. By Lemma 1.3, there exists $k \in \mathsf{N}$ such that $k > \alpha$ and $H(x' - x) \leqq kF(2x/k)$ whenever $x' \in \alpha B$ and $x \in \mathsf{R}^n$. Therefore, if $\|x'\| < \alpha$, $0 < t < \alpha$ and $\|x\| > r_k$, then

$$H(x' - t^{-1}x)/S(x) \leqq kF(2x/k^2)/\{(k+1)!F(x/2^k)\} \leqq 1/k!$$

by (3). Since k can be chosen as large as one wishes, this completes the proof.

LEMMA 2.2. *Suppose* $(M\mu)(0) < \infty$, $F \in C(\mathsf{R}^n)$, $\int F^* \, dx < \infty$, *and* (x_k, y_k) *is a sequence in* $\mathsf{R}^n \times \mathsf{R}^+$ *such that* $x_k/y_k \to x_0$ *for some* $x_0 \in \mathsf{R}^n$. *Then*

$$\lim_{k \to \infty} \{F[\mu](x_k, y_k) - F[\mu](y_k x_0, y_k)\} = 0.$$

PROOF. Let S be as in Lemma 2.1 with $H = F^*$, and let $\varepsilon > 0$ be given. Choose $\alpha > 0$ so that $\|x_k\|/y_k < \alpha$ for every k. Then the conclusion of Lemma 2.1 yields $\delta > 0$ such that

$$\text{(1)} \qquad \sup\{F^*(x' - x) : \|x'\| \leqq \alpha\} < \varepsilon S(x)/2 \quad \forall x \in \mathsf{R}^n \backslash (\delta B).$$

Since F is continuous and $\inf\{S(x) : \|x\| < \delta\} > 0$, we also have

$$\text{(2)} \qquad |F(y_k^{-1} x_k - x) - F(x_0 - x)| < \varepsilon S(x) \quad \forall x \in \delta B$$

whenever k is large enough. For each such k, the inequality in (2) holds for all $x \in \mathsf{R}^n$ by (1); hence

$$\text{(3)} \qquad \begin{aligned} |F[\mu](x_k, y_k) - F[\mu](y_k x_0, y_k)| &\leqq y_k^{-n} \int |F(y_k^{-1}(x_k - x)) \\ &\quad - F(x_0 - y_k^{-1} x)| \, d|\mu|(x) \\ &\leqq \varepsilon y_k^{-n} \int S(y_k^{-1} x) \, d|\mu|(x) \leqq \varepsilon \|S\|_1 (M\mu)(0), \end{aligned}$$

where the last inequality follows from Remark 1.2. As $\varepsilon > 0$ was arbitrary, this completes the proof.

We are now ready to prove our main result in the present section.

THEOREM 2.3. *Given a complex number* γ, *the following assertions are equivalent:*

(a) *Given* $F \in C(\mathsf{R}^n)$ *with* $|F| \leqq K$, $F[\mu]$ *has nontangential limit* $\gamma \int F \, dx$ *at* $(0, 0)$.

(b) *There exists* $T \in C(\mathsf{R}^n)$ *with* $|T| \leqq K$ *and a dense subset* D *of* R^n *such that* $\int T \, dx = 1$ *and* $\lim_{y \downarrow 0} T[\mu](yx, y) = \gamma$ *for every* $x \in D$.

(c) *There exists a Borel function* T *on* R^n *with* $|T| \leqq K$ *such that* $\int T \, dx = 1$ *and* $\lim_{y \downarrow 0} T[\mu](yx, y) = \gamma$ *for* a.a. $x \in \mathsf{R}^n$.

(d) *There exists a Borel function* T *on* R^n *with* $|T| \leqq K$ *and a function* $h \in L_1(\mathsf{R}^n)$ *such that* supp h *is compact*, $\int T \, dx = \int h \, dx = 1$ *and*

$$\lim_{y \downarrow 0} \int T[\mu](yx, y) h(x' - x) \, dx = \gamma \quad \textit{for } \text{a.a.} \quad x' \in \mathsf{R}^n.$$

PROOF. Without loss of generality, assume $\gamma = 0$. Let $\mu' = \chi_B \mu$ and $\mu'' = \mu - \mu'$. Then 0 is a Lebesgue point of μ'' with $(D\mu'')(0) = 0$. Since K satisfies condition (0.9) by our standing hypothesis, it follows from Theorem 1.5 combined with the second condition in (2.0) that $K[|\mu''|]$ has nontangential limit 0 at $(0,0)$. Therefore, upon replacing μ by μ' if necessary and taking (2.0) into account, we may and do suppose that $(M\mu)(0) < \infty$.

(a) $\Rightarrow$ (b). Trivial.

(b) $\Rightarrow$ (c). Let T and D be as in (b). To prove (c), it will suffice to show that $T[\mu]$ has sectorial limit 0 at $(0,0)$. To this end, fix any $x_0 \in \mathrm{R}$ and $\varepsilon > 0$. Let S and $\delta > 0$ be as in the proof of Lemma 2.2 with $F = T$ and $\alpha = \|x_0\| + 1$. Since D is dense in R^n and T is continuous, we can find an $x' \in D$ such that $\|x' - x_0\| < 1$ and

$$|T(x' - x) - T(x_0 - x)| < \varepsilon S(x) \quad \forall x \in \delta B.$$

Then the proof of Lemma 2.2 with $F = T$ shows that

$$|T[\mu](yx', y) - T[\mu](yx_0, y)| \leqq \varepsilon \, \|S\|_1 (M\mu)(0) \quad \forall y > 0.$$

Since $T[\mu](yx', y) \to \gamma = 0$ as $y \downarrow 0$ by (b) and since $\varepsilon > 0$ was arbitrary, it follows that $T[\mu](yx_0, y) \to 0$ as $y \downarrow 0$ for each $x_0 \in \mathrm{R}^n$.

(c) $\Rightarrow$ (d). Let T be as in (c), and let $h \in L_1(\mathrm{R}^n)$ have compact support. Since $(M\mu)(0) < \infty$ and $T^* \in L_1 \cap L_\infty$, the function $x \to \sup_{y>0} T^*[|\mu|](yx, y)$ is bounded on each compact set by Remark 1.2. Hence the conclusion of (d) follows from (c) combined with Lebesgue's Convergence Theorem.

(d) $\Rightarrow$ (a). Let T and h be as in (d). Plainly $h * T \in C_0(\mathrm{R}^n)$ and $\int h * T \, dx = 1$. To show that $\|(h * T)^*\|_1 < \infty$, choose $\alpha > 0$ such that $\operatorname{supp} h \subset \alpha B$. By Lemma 1.3, there exists $\beta > 0$ such that

$$\text{(1)} \qquad T^*(x' - x) \leqq \beta T^*(x'/\beta) \quad \forall x' \in \mathrm{R}^n \quad \text{and} \quad x \in \alpha B.$$

So $x' \in \mathrm{R}^n$ implies

$$\text{(2)} \qquad \begin{aligned} |(h * T)(x')| &\leqq \int_{\alpha B} T^*(x' - x)\, |h(x)|\, dx \\ &\leqq \beta T^*(x'/\beta) \int |h(x)|\, dx = \beta \, \|h\|_1 T^*(x'/\beta). \end{aligned}$$

Hence $\|(h * T)^*\|_1 \leqq \beta^{n+1} \|h\|_1 \|T^*\|_1 < \infty$. Moreover, an easy application of Fubini's Theorem shows that $x' \in \mathrm{R}^n$ and $y > 0$ implies

$$(3)\qquad (h * T)[\mu](yx', y) = (h_y * T_y * \mu)(yx')$$
$$= \int (T_y * \mu)(yx' - x)h_y(x)\,dx$$
$$= \int (T_y * \mu)(yx' - yx)h(x)\,dx$$
$$= \int (T_y * \mu)(yx)h(x' - x)\,dx$$
$$= \int T[\mu](yx, y)h(x' - x)\,dx.$$

Hence $(h * T)[\mu]$ has sectorial limit 0 at $(0, 0)$ by the conclusion of (d) and the proof of (b) $\Rightarrow$ (c). Accordingly, after replacing T by $h * T$, we may and do suppose that $T \in C(\mathsf{R}^n)$, $\|T^*\|_1 < \infty$, $\int T\,dx = 1$ and $T[\mu]$ has sectorial limit 0 at $(0, 0)$.

Now let $F \in C_c(\mathsf{R}^n)$ and $x' \in \mathsf{R}^n$ be given. If $t, y > 0$, then (3) with $h * T$ replaced by $F * T_t$ shows that

$$(4)\qquad (F * T_t)[\mu](yx', y) = \int T[\mu](yx, ty)F(x' - x)\,dx.$$

It follows from Lebesgue's Convergence Theorem that each $(F * T_t)[\mu]$ has sectorial limit 0 at $(0, 0)$. To prove that $F[\mu]$ satisfies this conclusion, let S be as in Lemma 2.1 with $H = T^*$ and let $\varepsilon > 0$ be given. Choose $\alpha > 0$ so large that $\{x'\} \cup (\operatorname{supp} F) \subset \alpha B$ and

$$(5)\qquad T^*(x/2) < \varepsilon S(x)/\|F\|_1 \quad \forall x \in \mathsf{R}^n \backslash \alpha B.$$

Since $\int T\,dx = 1$ and $F \in C_c(\mathsf{R}^n)$, we can find $0 < t < 1$ such that $\|F - F * T_t\|_\infty < \varepsilon/(4\alpha)^n$. If $\|x\| \geqq 4\alpha$, then $F(x' - x) = 0$ and so

$$(6)\qquad |F(x' - x) - (F * T_t)(x' - x)| \leqq \int_{\alpha B} T_t^*(x' - x - u)\,|F(u)|\,du$$
$$\leqq \int T_t^*(x/2)\,|F(u)|\,du$$
$$\leqq \varepsilon S_t(x)$$

by (5). Hence

$$(7)\qquad |F(x' - x) - (F * T_t)(x' - x)| \leqq \varepsilon\{(4\alpha)^{-n}\chi_{4\alpha B}(x) + S_t(x)\} \quad \forall x \in \mathsf{R}^n.$$

It follows that $y > 0$ implies

$$(8)\quad |F[\mu](yx',y) - (F * T_t)[\mu](yx',y)| \leqq (|F_y - (F * T_t)_y| * |\mu|)(yx')$$
$$= y^{-n}\int |F(x' - y^{-1}x) - (F * T_t)(x' - y^{-1}x)|\, d|\mu|(x)$$
$$\leqq \varepsilon y^{-n}\int \{(4\alpha)^{-n}\chi_{4\alpha B}(y^{-1}x) + S_t(y^{-1}x)\}\, d|\mu|(x)$$
$$= \varepsilon\left\{(4y\alpha)^{-n}|\mu|(4y\alpha B) + \int S_{ty}(x)\, d|\mu|(x)\right\}$$
$$\leqq \varepsilon\{(M\mu)(0) + (M\mu)(0)\|S\|_1\}$$

by (7) and Remark 1.2. Since $(F * T_t)[\mu]$ has sectorial limit 0 at $(0, 0)$, we infer from (8) that

$$(9)\qquad \limsup_{y\downarrow 0} |F[\mu](yx',y)| \leqq \varepsilon(M\mu)(0)\{1 + \|S\|_1\}.$$

As $\varepsilon > 0$ was arbitrary, this confirms that $F[\mu]$ has sectorial limit 0 at $(0,0)$ whenever $F \in C_c(\mathsf{R}^n)$.

Now let $F \in C(\mathsf{R}^n)$ be an arbitrary function with $\|F^*\|_1 < \infty$. To prove that $F[\mu]$ has nontangential limit 0 at $(0, 0)$, we may and do suppose $F \geqq 0$. Let $x' \in \mathsf{R}^n$ and $\varepsilon > 0$ be given. Choose $\beta = \beta(x') > 1$ so that $x \in \mathsf{R}^n$ implies $F^*(x' - x) \leqq \beta F^*(x/\beta)$. For each $\delta > 0$, let $F^* \wedge \delta = \min\{F^*, \delta\}$ pointwise. Then

$$(F^* \wedge \delta)(x' - x) \leqq \beta \cdot \min\{F^*(x/\beta), \delta\} = \beta^{n+1}(F^* \wedge \delta)_\beta(x)$$

for all $x \in \mathsf{R}^n$. Hence, as in the proof of Theorem 1.1, we have

$$(F^* \wedge \delta)[|\mu|](yx',y) \leqq \beta^{n+1}\int (F^* \wedge \delta)_{\beta y}\, d|\mu| \leqq \beta^{n+1}(M\mu)(0)\,\|F^* \wedge \delta\|_1$$

for all $y > 0$. Therefore, if we choose $\delta > 0$ sufficiently small, then

$$(10)\qquad |(F \wedge \delta)[\mu](yx',y)| < \varepsilon \quad \forall y > 0.$$

On the other hand, $(F - \delta)^+ \in C_c(\mathsf{R}^n)$ and so $(F - \delta)^+[\mu]$ has sectorial limit 0 at $(0,0)$. Since $F = (F - \delta)^+ + F \wedge \delta$, it follows from (10) that

$$(11)\qquad \limsup_{y\downarrow 0} |F[\mu](yx',y)| \leqq \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, this confirms that $F[\mu]$ has sectorial limit 0 at $(0, 0)$.

Finally, suppose that (x_k, y_k) is an arbitrary nontangential null sequence in

$\mathsf{R}^n \times \mathsf{R}^+$. To prove that $F[\mu](x_k, y_k) \to 0$, we may suppose that $x_k/y_k \to x_0$ for some $x_0 \in \mathsf{R}^n$. But then

$$|F[\mu](x_k, y_k) - F[\mu](y_k x_0, y_k)| \to 0 \quad \text{as} \quad k \to \infty$$

by Lemma 2.2. Hence $F[\mu](x_k, y_k) \to 0$ as $k \to \infty$, which completes the proof.

REMARK 2.4. If $(M\mu)(0) < \infty$, then the above proof shows that the condition $|F| \leqq K$ (or $|T| \leqq K$) in Theorem 2.3 may be replaced by the weaker condition that $F^* \in L_1 \cap L_\infty$ (or $T^* \in L_1 \cap L_\infty$). *This comment applies to all of the results in the present section and in section* 3.

THEOREM 2.5. *Suppose that there exist two bounded Borel functions T and ϕ on R^n and a function $h \in L_1(\mathsf{R}^n)$ such that* supp h *is compact,* $|T| \leqq K$, $\int T\,dx = \int h\,dx = 1$, *and*

(i) $$\lim_{y \downarrow 0} \int T[\mu](yx, y) h(x' - x)\,dx = (\phi * h)(x') \quad \textit{for a.a.} \quad x' \in \mathsf{R}^n.$$

Then $\gamma := \lim_{r \uparrow \infty} r^n \phi_r$ exists in the weak- topology of $L_\infty(\mathsf{R}^n)$ and*

(ii) $$\lim_{k \to \infty} F[\mu](x_k, y_k) = (F * \gamma)(x_0)$$

whenever $F \in C(\mathsf{R}^n)$, $|F| \leqq K$, and (x_k, y_k) is a nontangential null sequence in $\mathsf{R}^n \times \mathsf{R}^+$ such that $x_k/y_k \to x_0 \in \mathsf{R}^n$.

We shall not prove this generalization of Theorem 2.3. Instead we shall give a simple result of this type in section 3 (Corollary 3.5) and another in section 4 (Corollary 4.2).

COROLLARY 2.6. *Let E_1 and E_2 be two bounded Borel subsets of R^n with $|E_1| = |E_2| = 1$, and let γ_1 and γ_2 be complex numbers. If*

(i) $$\lim_{y \downarrow 0} \mu(yx + yE_k)/|yE_k| = \gamma_k \quad \textit{for a.a.} \quad x \in \mathsf{R}^n (k = 1, 2).$$

then $\gamma_1 = \gamma_2$.

PROOF. Fix $k \in \{1, 2\}$ and set $E = E_k$, $\gamma = \gamma_k$ and $T(x) = \chi_E(-x)$. Then $x' \in \mathsf{R}^n$ and $y > 0$ implies

$$T[\mu](yx', y) = y^{-n} \int \chi_{yE}(x - yx')\,dx = \mu(yx' + yE)/|yE|.$$

It follows from (i) that $T[\mu](yx', y) \to \gamma$ as $y \downarrow 0$ for a.a. $x' \in \mathsf{R}^n$. Therefore Theorem 2.3 ensures that if $F \in C_c(\mathsf{R}^n)$ and $\int F\,dx = 1$, then $F[\mu](0, y) \to \gamma$ as $y \downarrow 0$. Hence $\gamma_1 = \gamma_2$.

The special case of the following result with $E = B$ and $F = K$ is due to Ramey and Ullrich [6] if $K = P$ (the Poisson kernel) and $\mu \geqq 0$ or $\mu = f\lambda$ with $f \in \mathrm{BMO}$, and to Brossard and Chevalier [1] if K belongs to a certain class of radially decreasing kernels including the Gauss kernel.

COROLLARY 2.7. *Suppose that μ is differentiable at 0 with respect to some bounded Borel subset E of R^n with $|E| = 1$. Then, for each $F \in C(\mathsf{R}^n)$ with $|F| \leqq K$, $F[\mu]$ has nontangential limit $(D_E\mu)(0) \int F\,dx$ at $(0,0)$.*

PROOF. Take $T(x) = \chi_E(-x)$ in Theorem 2.3 (c).

THEOREM 2.8. *Suppose that there exists a function $T \in C(\mathsf{R}^n)$ with $\|T^*\|_1 < \infty$, a set $E \subset \mathsf{R}^n$ and a complex number γ such that $\int T\,dx = 1$;*

(a) *the zero measure is the only Radon measure ν on R^n such that $(K + |T^*|)[|\nu|] < \infty$ everywhere and $T[\nu](yx', y) = 0$ for every $y > 0$ and $x' \in E$;*

(b) $\lim_{y \downarrow 0} T[\mu](yx', y) = \gamma$ *for each $x' \in E$.*

Then, for each $F \in C(\mathsf{R}^n)$ with $|F| \leqq K$, $F[\mu]$ has nontangential limit $\gamma \int F\,dx$ at $(0,0)$.

PROOF. As in the proof of Theorem 2.3, we may and do suppose that $\gamma = 0$ and $(M\mu)(0) < \infty$.

Choose and fix an arbitrary $F \in C(\mathsf{R}^n)$ with $\|F^*\|_1 < \infty$. By Lemma 2.1, there exists a radially decreasing function $S \in C(\mathsf{R}^n) \cap L_1(\mathsf{R}^n)$ such that $F^* + K + T^* < S$ everywhere and

$$(1) \quad \{F^*(x' - t^{-1}x) + K(x' - t^{-1}x) + T^*(x' - t^{-1}x)\}/S(x) \to 0 \quad \text{as} \quad \|x\| \to \infty$$

for each fixed $x' \in \mathsf{R}^n$ and $t > 0$. Thus, for each such x' and t, the continuous functions $F(x' - t^{-1}x)/S(x)$ and $T(x' - t^{-1}x)/S(x)$ vanish at infinity.

Now fix $x_0 \in \mathsf{R}^n$ and $\varepsilon > 0$. We then claim that there exist $\alpha_1, \ldots, \alpha_m \in \mathsf{C}$, $t_1, \ldots, t_m > 0$ and $x_1, \ldots, x_m \in E$ such that

$$(2) \qquad \left| F(x_0 - x) - \sum_{j=1}^{m} \alpha_j T_{t_j}(t_j x_j - x) \right| \Big/ S(x) < \varepsilon \quad \forall x \in \mathsf{R}^n.$$

In fact, suppose this is false. Then the Hahn-Banach Theorem combined with the Riesz Representation Theorem (and the above remark) yields a nonzero bounded measure σ on R^n such that

$$(3) \qquad \int T_t(tx' - x)\frac{d\sigma(x)}{S(x)} = 0 \quad \forall t > 0 \text{ and } x' \in E.$$

Define $d\nu(x) = d\sigma(x)/S(x)$. Then $(K + |T^*|)[|\nu|] < \infty$ everywhere by (1), and (3) simply says that $T[\nu](tx', t) = 0$ for every $t > 0$ and $x' \in E$. Hence $\sigma = S\nu = 0$ by (a), which contradicts our choice of σ. This *reductio ad absurdum* establishes our claim.

Now let $y > 0$ be given. Then (2) is equivalent to

$$(4) \qquad \left| F_y(yx_0 - x) - \sum_{j=1}^{m} \alpha_j T_{t_j y}(t_j y x_j - x) \right| < \varepsilon S_y(x) \quad \forall x \in \mathbf{R}^n.$$

We integrate each side of (4) with respect to $d\,|\mu|\,(x)$ to get

$$(5) \qquad \left| F[\mu](yx_0, y) - \sum_{j=1}^{m} \alpha_j T[\mu](t_j y x_j, t_j y) \right| \leqq \varepsilon \int S_y(x)\, d\,|\mu|\,(x)$$
$$\leqq \varepsilon (M\mu)(0) \|S\|_1.$$

Upon letting $y \downarrow 0$ in (5) and utilizing condition (b) with $\gamma = 0$, we obtain

$$(6) \qquad \limsup_{y \downarrow 0} |F[\mu](yx_0, y)| \leqq \varepsilon (M\mu)(0)\, \|S\|_1.$$

Since $\varepsilon > 0$ was arbitrary, (6) shows that $F[\mu](yx_0, y) \to 0$ as $y \downarrow 0$. Since $x_0 \in \mathbf{R}^n$ was arbitrary, we conclude from Lemma 2.2 that $F[\mu]$ has nontangential limit 0 at $(0, 0)$, as desired.

In order to give nontrivial examples of test functions T and sets E that satisfy condition (a) of the last theorem, let P and G be the Poisson kernel and the Gauss kernel for $\mathbf{R}^n$, respectively. Thus

$$P(x) = \frac{\alpha_n}{(x^2 + 1)^{\frac{n+1}{2}}} \quad \text{and} \quad G(x) = \pi^{-n/2} \exp(-x^2),$$

where x^2 is the inner product of $x \in \mathbf{R}^n$ with itself and the number α_n is chosen so that $\int P(x)\, dx = 1$.

The following result appears to be new even for $n = 1$.

THEOREM 2.9. *Let T be either P or G, let V be an open nonempty subset of $\mathbf{R}^n \times \mathbf{R}^+$, and let ν be a real Radon measure on $\mathbf{R}^n$ such that $T[|\nu|] < \infty$ on $(\mathbf{R}^n \times \mathbf{R}^+) \cap \bar{V}$ and $T[\nu](x, y) \leqq My$ for all $(x, y) \in (\mathbf{R}^n \times \mathbf{R}^+) \cap \partial V$ and for some $0 \leqq M < \infty$. If the set $C := \{x \in \mathbf{R}^n : (x, 0) \in \partial V\}$ has $|\nu|$-measure 0, then $T[\nu](x, y) \leqq My$ whenever $(x, y) \in V$. If, in addition, $T[\nu] = 0$ on $(\mathbf{R}^n \times \mathbf{R}^+) \cap \partial V$, then $\nu = 0$.*

PROOF. (I) Choose $\alpha > 0$ so large that $T(x) \leqq \alpha / \|x\|^n$ for all nonzero $x \in \mathbf{R}^n$. Given $\varepsilon > 0$, define $A = C_\varepsilon = \{x \in \mathbf{R}^n : \operatorname{dist}(x, C) < \varepsilon\}$ and

$$(1) \quad F(x, y) = T[\nu - \chi_A|\nu| - \chi_{\alpha B'}|\nu|](x, y) - (\varepsilon + M)y \quad \forall (x, y) \in (\mathbf{R}^n \times \mathbf{R}^+) \cap \bar{V},$$

where B' is the complement of the unit ball B in $\mathbf{R}^n$.

Since $T[v] \leqq My$ on $(\mathsf{R}^n \times \mathsf{R}^+) \cap \partial V$, F is negative on $(\mathsf{R}^n \times \mathsf{R}^+) \cap \partial V$. We claim that if $x_0 \in C$, then the limit supremum of $F(x, y)$ as $(x, y) \in V$ approaches $(x_0, 0)$ is nonpositive. In fact, if $x', x \in \mathsf{R}^n$, $x \notin A$ and $\|x' - x_0\| < \varepsilon/2$, then $\|x' - x\| \geqq \|x_0 - x\| - \|x' - x_0\| \geqq \|x_0 - x\|/2$. Therefore $(x', y) \in V$ and $\|x' - x_0\| < \varepsilon/2$ implies

(2)
$$\begin{aligned} F(x', y) &\leqq T[v - \chi_A|v|](x', y) \leqq T[\chi_{A'}|v|](x', y) \\ &= y^{-n} \int_{A'} T(y^{-1}(x' - x))\, d|v|(x) \\ &\leqq y^{-n} \int_{A'} T(y^{-1}(x_0 - x)/2)\, d|v|(x) \\ &= 2^n T[\chi_{A'}|v|](x_0, 2y) = o(1) \quad \text{as} \quad y \downarrow 0 \end{aligned}$$

by Theorem 1.5, which establishes our claim.

Next suppose that V is unbounded. We claim that the limit supremum of $F(x, y)$ as $(x, y) \in V$ recedes to the point at infinity is nonpositive. In fact, the definition of F shows that

(3)
$$\begin{aligned} F(x', y) &\leqq T[v - \chi_{\alpha B'}|v|](x', y) \leqq T[\chi_{\alpha B}|v|](x', y) \\ &= y^{-n} \int_{\alpha B} T(y^{-1}(x' - x))\, d|v|(x) \end{aligned}$$

for each $(x', y) \in V$. In particular,

(4)
$$F(x', y) \leqq T(0)\,|v|(\alpha B)/y^n \quad \forall (x', y) \in V.$$

On the other hand, (3) and our choice of α ensure that $(x', y) \in V$ and $\|x'\| > 2\alpha$ implies

(5)
$$\begin{aligned} F(x', y) &\leqq y^{-n} \int_{\alpha B} T(y^{-1}x'/2)\, d|v|(x) \\ &\leqq y^{-n} \cdot \alpha(2y/\|x'\|)^n\, |v|(\alpha B) = 2^n \alpha \cdot |v|(\alpha B)/\|x'\|^n. \end{aligned}$$

Combine (4) and (5) to conclude that the limit supremum of $F(x', y)$ as $(x', y) \in V$ recedes to the point at infinity is nonpositive.

(II) We shall show that $T[v] \leqq T[\chi_C|v|] + My$ on V. To this end, let α, ε, A and F be as in (I). First suppose that $T = P$. Then F is defined and harmonic on $\mathsf{R}^n \times \mathsf{R}^+$. Therefore, from (I) and the maximum modulus principle for harmonic functions, we infer that $F \leqq 0$ on V, or equivalently

(6)
$$T[v](x, y) \leqq T[\chi_A|v|](x, y) + T[\chi_{\alpha B'}|v|](x, y) + (\varepsilon + M)y \quad \forall (x, y) \in V.$$

Next suppose $T = G$. A direct calculation shows that the function $Q(x, y) := G_y(x)$ satisfies a modified heat equation, namely,

$$\left(\frac{2}{y}\frac{\partial}{\partial y} - \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}\right) Q(x, y) = 0 \quad \forall (x, y) \in \mathsf{R}^n \times \mathsf{R}^+. \tag{7}$$

Therefore it is easy to check that $F(x, y) + (\varepsilon + M)y$ also satisfies (7) with $\mathsf{R}^n \times \mathsf{R}^+$ replaced by V. Hence

$$\left(\frac{2}{y}\frac{\partial}{\partial y} - \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}\right) F(x, y) = -2(\varepsilon + M)/y < 0 \quad \forall (x, y) \in V, \tag{8}$$

so F has no local maximum in V (cf. F. John [4; p. 215]). Therefore it must be the case that $F \leqq 0$ on V by (I). Thus we have confirmed that (6) holds in either case.

Now it is obvious that C is a closed set. Moreover, $A = \{x\colon \operatorname{dist}(x, C) < \varepsilon\}$ and (6) holds for each sufficiently large $\alpha > 0$ and each $\varepsilon > 0$. Therefore $T[\nu] \leqq T[\chi_C|\nu|] + My$ on V.

(III) Suppose $|\nu|(C) = 0$. Then $T[\nu] \leqq My$ on V by (II). If we further assume that $T[\nu] = 0$ on $(\mathsf{R}^n \times \mathsf{R}^+) \cap \partial V$, then we may apply the above conclusion to $-\nu$ to get $T[\nu] = 0$ on V. But it is obvious that $T[\nu]$ is real-analytic in each variable. Hence $T[\nu] = 0$ identically, or equivalently, $T_y * \nu = 0$ identically for each $y > 0$ small enough.

Finally, let F be an arbitrary element of $C_c(\mathsf{R}^n)$. Then $T_y * (\nu * F) = (T_y * \nu) * F = 0$ identically for each $y > 0$ small enough. Hence $\nu * F = 0$ identically by Theorem 1.5 with $\mu = (\nu * F)\lambda$. Therefore $\nu = 0$, which completes the proof.

LEMMA 2.10. *Let T be a strictly positive continuous function on R^n with $T \leqq K$, let ν be a Radon measure on R^n such that $(K_t * |\nu|) < \infty$ for some $t > 0$, and let $x_0 \in \mathsf{R}^n$ be given. If there exists a nontangential null sequence (x_k, y_k) in $\mathsf{R}^n \times \mathsf{R}^+$ such that $T[\nu](x_0 + x_k, y_k) = o(y_k^{-n})$ as $k \to \infty$, then $\nu(\{x_0\}) = 0$.*

PROOF. Let $\gamma = \nu(\{x_0\})$ and $\sigma = \nu - \gamma\delta_{x_0}$. Given $\varepsilon > 0$, choose $\alpha > 0$ so small that $|\sigma|(x_0 + \alpha B) < \varepsilon/\|T\|_\infty$. Write $\sigma' = \chi_{x_0 + \alpha B}\sigma$ and $\sigma'' = \sigma - \sigma'$. Then $T[\sigma'']$ has nontangential limit 0 at $(x_0, 0)$ by Theorem 1.5. Moreover, $x' \in \mathsf{R}^n$ and $0 < y < t$ implies

$$y^n \cdot |T[\sigma'](x', y)| \leqq \|T\|_\infty \cdot |\sigma|(x_0 + \alpha B) < \varepsilon.$$

Since $T[\sigma] = T[\sigma'] + T[\sigma'']$ and since $\varepsilon > 0$ was arbitrary, it follows that $y^n T[\sigma](x, y) \to 0$ nontangentially at $(x_0, 0)$. Therefore, if (x_k, y_k) is a nontangential null sequence as above, then

$$\gamma T(x_k/y_k) = y_k^n T[\gamma\delta_{x_0}](x_0 + x_k, y_k) \tag{1}$$
$$= y_k^n T[\nu - \sigma](x_0 + x_k, y_k) = o(1).$$

But T is a strictly positive continuous function and $(x_k/y_k)_1^\infty$ is a bounded sequence in $\mathbb{R}^n$. Hence (1) is possible if and only if $\gamma = 0$.

COROLLARY 2.11. *Let T be either P or G, let E be a dense subset of ∂U for some bounded open nonempty subset U of $\mathbb{R}^n$, and let ν be a Radon measure on $\mathbb{R}^n$ such that $(T_t * |\nu|)(0) < \infty$ for some $t > 0$. If for each $x \in E$, $\{y \in (0, t)\colon T[\nu](yx, y) = 0\}$ contains an infinite compact set, then $\nu = 0$. In particular, T and E satisfy condition* (a) *of Theorem* 2.8 *with $K = T$.*

PROOF. Without loss of generality, assume that ν is real-valued. Let $V_t = \{(yx, y)\colon 0 < y < t \text{ and } x \in U\}$. Then V_t is open in $\mathbb{R}^n \times \mathbb{R}^+$, $\bar{V}_t \cap (\mathbb{R}^n \times \{0\}) = \{(0,0)\}$, and

$$\partial V_t = \{(yx, y)\colon 0 \leqq y \leqq t \quad \text{and} \quad x \in \partial U\} \cup (t\bar{U} \times \{t\}).$$

For each $x \in E$, the real-analytic function $y \to T[\nu](yx, y)$ vanishes on an infinite compact set by the hypotheses. Therefore $T[\nu](yx, y) = 0$ for all $0 < y < t$ and all $x \in E$. But $T[\nu]$ is continuous on $\mathbb{R}^n \times (0, t)$ and E is dense in ∂U. Hence

$$T[\nu](yx, y) = 0 \quad \forall x \in \partial U \quad \text{and} \quad 0 < y < t. \tag{1}$$

Since $\partial U \neq \emptyset$, we have $\nu(\{0\}) = 0$ by (1) and Lemma 2.10.

Now suppose $T = P$. Then it is obvious that we may take $t = \infty$ in (1). Hence $\nu = 0$ by Theorem 2.9. So suppose $T = G$. Given a sufficiently large $\alpha > 0$ and a sufficiently small $\varepsilon > 0$, let $F = F_\varepsilon$ be as in the proof of Theorem 2.9 with $V = V_t$ and $M = 0$. Then (8) in that proof shows that F has no local maximum in V_t, and that the restriction of F to $(\mathbb{R}^n \times \mathbb{R}^+) \cap \overline{V_{t/2}}$ has no maximum in $(t/2)\cdot(\bar{U} \times \{1\})$. Hence $F \leqq 0$ on $V_{t/2}$ by (II) in that proof. Since $\nu(\{0\}) = 0$ and since $\varepsilon > 0$ was arbitrary, it follows that $G[\nu] \leqq 0$ on $V_{t/2}$. Therefore $G[\nu] = 0$ on $V_{t/2}$; hence $\nu = 0$ by Theorem 2.9.

COROLLARY 2.12. *Let T and $E \subset \partial U$ be as in Corollary 2.11, and let $\{A_x\colon x \in E\}$ be a collection of infinite compact subsets of $(0, \infty)$. Then the linear span of the functions*

$$T(x' - y^{-1}x) \quad \text{with} \quad x' \in E \quad \text{and} \quad y \in A_{x'}$$

is uniformly dense in $C_0(\mathbb{R}^n)$, norm-dense in $L_p \cap L_1(\mathbb{R}^n)$ with respect to $\|f\|_{p,1} := \|f\|_p + \|f\|_1$ if $1 \leqq p < \infty$, and weak-$*$ dense in $L_\infty(\mathbb{R}^n)$.

PROOF. Combine Corollary 2.11 with the Hahn-Banach Theorem.

COROLLARY 2.13. *Let $T = P$ or G, let V be an open subset of* $\mathsf{R}^n \times \mathsf{R}^+$ *such that* $(\mathsf{R}^n \times \{0\}) \cap \bar{V} = \{(0,0)\}$ *and* $v(\{0\}) = 0$. *If the limit supremum of $T[v](x,y)$ as $(x,y) \in \partial V$ approaches $(0,0)$ is nonpositive, then the limit supremum of $T[v](x,y)$ as $(x,y) \in V$ approaches $(0,0)$ is nonpositive. Therefore, if $\gamma \in [-\infty, \infty]$ and if $T[v](x,y) \to \gamma$ as $(x,y) \in \partial V$ approaches $(0,0)$, then $T[v](x,y) \to \gamma$ as $(x,y) \in V$ approaches $(0,0)$.*

PROOF. Suppose that the limit supremum of $T[v](x,y)$ as $(x,y) \in \partial V$ approaches $(0,0)$ is nonpositive. Given $\varepsilon > 0$, choose $t > 0$ so small that $T[v] < \varepsilon$ on $(tB \times (0,t]) \cap \partial V$ and $T[|v|]$ is bounded on $V \cap (tB \times \{t\})$. Thus there exists $M > 0$ such that $T[v](x,y) - My < \varepsilon$ on $(\mathsf{R}^n \times \mathsf{R}^+) \cap \partial V_t$, where $V_t = (tB \times (0,t)) \cap V$. Therefore Theorem 2.9 ensures that $T[v](x,y) \leqq My + \varepsilon$ whenever $(x,y) \in V_t$. As $\varepsilon > 0$ was arbitrary, this proves that the limit supremum of $T[v]$ as $(x,y) \in V$ approaches $(0,0)$ is nonpositive.

Now suppose $\gamma \in [-\infty, \infty]$ and $T[v](x,y) \to \gamma$ as $(x,y) \in \partial V$ approaches $(0,0)$. We may suppose $\gamma < \infty$ (if necessary, replace v by $-v$). Given a real number $\gamma' > \gamma$, we then have that $T[v - \gamma'\lambda](x,y) \to \gamma - \gamma'$ as $(x,y) \in \partial V$ approaches $(0,0)$. Hence the limit supremum of $T[v](x,y) - \gamma'$ as $(x,y) \in V$ approaches $(0,0)$ is nonpositive. As $\gamma' > \gamma$ was arbitrary, it follows that the limit supremum of $T[v](x,y)$ as $(x,y) \in V$ approaches $(0,0)$ is $\leqq \gamma$. If $\gamma = -\infty$, then we are done. If $\gamma > -\infty$, then we may apply the above argument to $-v$ to see that the limit supremum of $T[-v](x,y)$ as $(x,y) \in V$ approaches $(0,0)$ is $\leqq -\gamma$. Hence $T[v](x,y) \to \gamma$ as $(x,y) \in V$ approaches $(0,0)$.

§3. Fatou's Theorem for Almost Continuous Kernels.

Throughout this section, let K be a strictly positive radially decreasing bounded L_1-function on R^n that satisfies the comparison condition, and let μ be a Radon measure on R^n such that

$$(3.0) \qquad (\mathrm{UD}\,|\mu|)(0) < \infty \quad \text{and} \quad (K_t * |\mu|)(0) < \infty \quad \text{for some} \quad t > 0.$$

A function on R^n is said to be *almost continuous* if it is continuous on $\mathsf{R}^n \backslash C$ for some closed subset C of R^n with $|C| = 0$. Thus, if A is a subset of R^n, then χ_A is almost continuous if and only if $|\partial A| = 0$.

LEMMA 3.1. *Suppose that v is a real Radon measure on* R^n *such that*

$$\text{(i)} \qquad (\mathrm{UD}\,|v|)(0) < \infty \quad \text{and} \quad (Mv)(0) < \infty,$$

that F is an almost continuous nonnegative bounded Borel function on R^n *with $\|F^*\|_1 < \infty$, and that (x_k, y_k) is a nontangential null sequence in* $\mathsf{R}^n \times \mathsf{R}^+$. *Then*

$$\text{(ii)} \qquad \limsup_{k \to \infty} F[v](x_k, y_k) \leqq (\mathrm{UD}\,v)(0)\,\|F\|_1.$$

PROOF. Without loss of generality, suppose $x_k/y_k \to x_0$ for some $x_0 \in \mathrm{R}^n$. Let $T = \chi_B$. If $F \in C_c^+(\mathrm{R}^n)$ and $t > 0$, then

(1) $$\limsup_{y \downarrow 0} (F * T_t)[v](yx_0, y) \leqq (\mathrm{UD}\, v)(0)\, \|F\|_1.$$

In fact, (4) in the proof of Theorem 2.3 shows that

(2) $$\begin{aligned}(F * T_t)[v](yx_0, y) &= \int T[v](yx, ty)F(x_0 - x)\,dx \\ &= (ty)^{-n} \int v(yx + tyB)F(x_0 - x)\,dx \\ &= \int Q(x, y)F(x_0 - x)\,dx,\end{aligned}$$

where $Q(x, y) = v(yx + tyB)/|tyB|$. Since $(Mv)(0) < \infty$ and the support C_{x_0} of $F(x_0 - x)$ is compact, $\sup_{y>0} |Q(x, y)|$ is bounded on C_{x_0}. Moreover, $\limsup_{y\downarrow 0} Q(x, y) \leqq (\mathrm{UD}\, v)(0)$ by the definition of $\mathrm{UD}\, v$. Hence (1) follows from (2) combined with Fatou's Lemma.

Now suppose $F \in C_c^+(\mathrm{R}^n)$ and $\varepsilon > 0$. Then (8) with $S = P$ in the proof of Theorem 2.3 shows that there exists $0 < t < 1$ such that $y > 0$ implies

(3) $$|F[v](yx_0, y) - (F * T_t)[v](yx_0, y)| \leqq \varepsilon(Mv)(0)\{1 + \|S\|_1\}.$$

Since $\varepsilon > 0$ was arbitrary, it follows from (3) and (1) that

(4) $$\limsup_{y \downarrow 0} F[v](yx_0, y) \leqq (\mathrm{UD}\, v)(0)\, \|F\|_1.$$

This, combined with Lemma 2.2, ensures that (ii) holds if $F \in C_c^+(\mathrm{R}^n)$.

Next suppose that the function F has compact support. Since F is almost continuous, F is continuous on $\mathrm{R}^n \backslash C$ for some closed set C with $|C| = 0$. Upon replacing C by $C \cap (\operatorname{supp} F)$, we may suppose C is compact. Given $\varepsilon > 0$ choose $g \in C_c^+(\mathrm{R}^n)$ such that $g \leqq 1$ *on* R^n, $g = 1$ on C and $\|g\|_1 + \|gF\|_1 < \varepsilon$. Then $(1 - g)F \in C_c^+(\mathrm{R}^n)$, and so

(5) $$\limsup_{k \to \infty} ((1 - g)F)[v](x_k, y_k) \leqq (\mathrm{UD}\, v)(0)\, \|(1 - g)F\|_1.$$

Moreover, the result proved in the last paragraph holds with v replaced by $|v|$. Hence

(6) $$\begin{aligned}\limsup_{k \to \infty} (gF)[v](x_k, y_k) &\leqq \limsup_{k \to \infty} \|F\|_\infty g[|v|](x_k, y_k) \\ &\leqq \|F\|_\infty(\mathrm{UD}\, |v|)(0)\varepsilon.\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, (5) and (6) establish (ii).

In general, we use the decomposition $F = (F - \delta)^+ + F \wedge \delta$ for a sufficiently small $\delta > 0$. Upon arguing as in the paragraph before last in the proof of Theorem 2.3, we can prove that F satisfies (ii).

THEOREM 3.2. *Suppose that there exists an almost continuous Borel function T on R^n with $|T| \leqq K$ and $\int T\,dx = 1$, a set $E \subset \mathrm{R}^n$ and a complex number γ such that:*

(a) *If $\phi \in L_\infty(\mathrm{R}^n)$ and $(T_y * \phi)(yx') = 0$ for all $y > 0$ and $x' \in E$, then $\phi = 0$ a.e.*

(b) *For each $x' \in E$, $\lim_{y \downarrow 0} T[\mu](yx', y) = \gamma$.*

Then, for each almost continuous Borel function F on R^n with $|F| \leqq K$, $F[\mu]$ has nontangential limit $\gamma \int F\,dx$ at $(0,0)$. In particular, $(D\mu)(0) = \gamma$.

PROOF. Once again, we may and do suppose that $\gamma = 0$ and $(M\mu)(0) < \infty$.

Choose and fix an arbitrary almost continuous bounded Borel function F on R^n with $\|F^*\|_1 < \infty$ and $x_0 \in \mathrm{R}^n$. In light of the Hahn-Banach Theorem, (a) is equivalent to the condition that the function $T_t(tx' + x)$ with $t > 0$ and $x' \in E$ span a dense subspace of $L_1(\mathrm{R}^n)$. So there exist $\alpha_1, \ldots, \alpha_m \in \mathrm{C}$, $t_1, \ldots, t_m > 0$ and $x_1, \ldots, x_m \in E$ such that $\|H\|_1 < \varepsilon$, where

$$H(x) = F(x_0 + x) - \sum_{j=1}^{m} \alpha_j T_{tj}(t_j x_j + x) \quad \forall x \in \mathrm{R}^n. \tag{1}$$

Notice that H is an almost continuous bounded Borel function on R^n, and that $\|H^*\|_1 < \infty$ by an application of Lemma 1.3. (Note $F^* + T^* \in L_1 \cap L_\infty$.) Moreover, $\nu = |\mu|$ fulfills the assumption (i) of Lemma 3.1 by (3.0) and $(M\mu)(0) < \infty$. Furthermore, (1) shows that $y > 0$ implies

$$\left| F[\mu](yx_0, y) - \sum_{j=1}^{m} \alpha_j T[\mu](t_j y x_j, t_j y) \right| = |H[\mu](0, y)| \leqq |H|\,[\nu](0, y). \tag{2}$$

Let $y \downarrow 0$ in (2), and utilize condition (b) with $\gamma = 0$ and Lemma 3.1. The result is

$$\limsup_{y \downarrow 0} |F[\mu](yx_0, y)| \leqq (\mathrm{UD}\,\nu)(0)\,\|H\|_1 \leqq (\mathrm{UD}\,|\mu|)(0)\varepsilon. \tag{3}$$

As $\varepsilon > 0$ was arbitrary, this shows that $F[\mu]$ has sectorial limit 0 at $(0,0)$. Therefore Theorem 2.3 ensures that if $J \in C_c(\mathrm{R}^n)$, then $J[\mu]$ has nontangential limit 0 at $(0,0)$.

Now let (x_k, y_k) be an arbitrary nontangential null sequence in $\mathrm{R}^n \times \mathrm{R}^+$. Given $\varepsilon > 0$, choose $J \in C_c(\mathrm{R}^n)$ such that $\|F - J\|_1 < \varepsilon$. Then Lemma 3.1 with $\nu = |\mu|$ ensures that

$$\begin{aligned} \limsup_{k \to \infty} |F[\mu](x_k, y_k) - J[\mu](x_k, y_k)| &\leqq \limsup_{k \to \infty} |F - J|\,[\nu](x_k, y_k) \\ &\leqq (\mathrm{UD}\,\nu)(0)\,\|F - J\|_1 \leqq (\mathrm{UD}\,\nu)(0)\varepsilon. \end{aligned} \tag{4}$$

Since $J[\mu](x_k, y_k) \to 0$ and since $\varepsilon > 0$ was arbitrary, it follows from (4) that $F[\mu](x_k, y_k) \to 0$, as desired.

The special case of the following result with $K = T = P$ and $F = \chi_B$ is due to L.H. Loomis [5] if $n = 1$ and $\mu \geqq 0$; to Ramey and Ullrich [6] if $\mu \geqq 0$ or $\mu = f\lambda$ with $f \in \mathrm{BMO}$; and to Brossard and Chevalier [1] under their own assumptions which are stronger than our assumptions.

COROLLARY 3.3. *Suppose that there exists an almost continuous Borel function T on R^n with $|T| \leqq K$ and $\int T\,dx = 1$, a dense subset E of R^n, and a complex number γ such that*

(i) $$\lim_{y \downarrow 0} T[\mu](yx', y) = \gamma \quad \forall x' \in E.$$

Then, for each almost continuous Borel function F on R^n with $|F| \leqq K$, $F[\mu]$ has nontangential limit $\gamma \int F\,dx$ at $(0, 0)$. In particular, $(D\mu)(0) = \gamma$.

PROOF. Pick any $\phi \in L_\infty(\mathsf{R}^n)$ with $(T_y * \phi)(yx') = 0$ for all $y > 0$ and $x' \in E$. Then, since $T_y * \phi$ is continuous on R^n and since E is dense in R^n, $T_y * \phi = 0$ identically on R^n. Moreover, $\int T\,dx = 1$, and so $T_y * \phi \to \phi$ in the weak-$*$ topology of $L_\infty(\mathsf{R}^n)$ as $y \downarrow 0$; hence $\phi = 0$ a.e. Therefore the desired result is an immediate consequence of Therem 3.2.

The following result with $K = P, F = \chi_B$, and $\mu \geqq 0$ or $\mu = f\lambda$ with $f \in \mathrm{BMO}$ is due to Ramey and Ullrich [6].

COROLLARY 3.4. *Let E be a dense subset of ∂U for some bounded open nonempty subset U of R^n, let K be either the Poisson kernel or the Gauss kernel, and let the measure μ satisfy (3.0). If there exists a complex number γ such that*

(i) $$\lim_{y \downarrow 0} K[\mu](yx', y) = \gamma \quad \forall x' \in E,$$

then, for each almost continuous Borel function F on R^n with $|F| \leqq K$, $F[\mu]$ has nontangential limit $\gamma \int F\,dx$ at $(0, 0)$.

PROOF. By Corollary 2.11, both P and G satisfy condition (a) of Theorem 3.2.

COROLLARY 3.5. *Let $n = 1$, let K be either $P(x) := \pi^{-1}(x^2 + 1)^{-1}$ or $G(x) := \pi^{-1/2}\exp(-x^2)$, and let μ be a Radon measure on R that satisfies (3.0). Then the following assertions are equivalent:*

(I) *Both $\lim_{y \downarrow 0} \mu([0, y])/y$ and $\lim_{y \downarrow 0} \mu([-y, 0])/y$ exist.*

(II) *There exist real numbers $x_1 < x_2$ such that for each $j = 1, 2,$ $K[\mu](yx_j, y)$ has a limit as $y \downarrow 0$.*

(III) *Whenever F is an almost continuous Borel function on R^n with $|F| \leqq K$ and*

(x_k, y_k) is a nontangential null sequence in $\mathsf{R} \times \mathsf{R}^+$ such that $x_k/y_k \to x_0$ for some $x_0 \in \mathsf{R}$, then

$$\lim_{k\to\infty} F[\mu](x_k, y_k) = \beta \int_{-\infty}^{x_0} F\,dz + \alpha \int_{x_0}^{\infty} F\,dx,$$

where α and β are complex numbers independent of F and (x_k, y_k).

PROOF. (II) $\Rightarrow$ (I) and (III). Let $x_1 < x_2$ be as in (II). Define

(1) $$\ell_j = \lim_{y\downarrow 0} K[\mu](yx_j, y) \quad \text{for} \quad j = 1, 2,$$

(2) $$\alpha = \ell_1 + (\ell_1 - \ell_2)\left\{\int_{-\infty}^{x_1} K\,dx\right\}\Big/\int_{x_1}^{x_2} K\,dx,$$

(3) $$\beta = \ell_1 - (\ell_1 - \ell_2)\left\{\int_{x_1}^{\infty} K\,dx\right\}\Big/\int_{x_1}^{x_2} K\,dx,$$

and $\phi = \alpha\chi_{(-\infty,0)} + \beta\chi_{(0,\infty)}$. Then $F \in L_1(\mathsf{R})$, $x' \in \mathsf{R}$ and $y > 0$ implies

(4) $$\begin{aligned} F[\phi\lambda](x', y) &= \alpha \int_{-\infty}^{0} F_y(x' - x)\,dx + \beta \int_{0}^{\infty} F_y(x' - x)\,dx \\ &= \alpha \int_{x'}^{\infty} F_y(x)\,dx + \beta \int_{-\infty}^{x'} F_y(x)\,dx \\ &= \alpha \int_{x'/y}^{\infty} F(x)\,dx + \beta \int_{-\infty}^{x'/y} F(x)\,dx. \end{aligned}$$

The reader checks that we have defined α and β in such a way that

(5) $$\lim_{y\downarrow 0} K[\mu - \phi\lambda](yx_j, y) = 0 \quad \text{for} \quad j = 1, 2.$$

It follows from Corollary 3.4 with $n = 1$ and $E = \{x_1, x_2\}$ that $\mu - \phi\lambda$ has derivative 0 at 0. This combined with the definition of ϕ shows that

(6) $$\lim_{y\downarrow 0} \mu([-y, 0])/y = \alpha \quad \text{and} \quad \lim_{y\downarrow +0} \mu([0, y])/y = \beta.$$

To prove (III), let F, (x_k, y_k) and x_0 be as in the hypotheses of (III). Since $\mu - \phi\lambda$ has derivative 0 at 0, Corollary 3.3 with $T = 2^{-1}\chi_{[-1,1]}$ ensures that $F[\mu - \phi\lambda](x_k, y_k) \to 0$ as $k \to \infty$. This combined with (4) establishes the conclusion of (III).

(III) $\Rightarrow$ (II). Trivial.

(I) $\Rightarrow$ (II). Suppose (I) holds, and define α and β by (6). Then $\mu - \phi\lambda$ has derivative 0 at 0, where ϕ is defined as above. Hence, by Corollary 3.3,

$K[\mu - \phi\lambda]$ has nontangential limit 0 at $(0,0)$. Thus (II) is an immediate consequence of (4).

REMARK 3.6. In Corollary 3.5, replace the assumption $(\mathrm{UD}\,|\mu|)(0) < \infty$ by the weaker assumption that $(\mathrm{SUD}\,|\mu|)(0) < \infty$. Then Theorem 2.3 combined with the above proof shows that each of (I) and (II) implies (III) with "almost continuous" replaced by "continuous". Notice also that the special case of (III) with $F = P$ becomes

$$\lim_{y\downarrow 0} P[\mu](yx', y) = \frac{\beta}{\pi}\left\{\operatorname{Arctan} x' + \frac{\pi}{2}\right\} + \frac{\alpha}{\pi}\left\{\frac{\pi}{2} - \operatorname{Arctan} x'\right\}$$
$$= \frac{\alpha+\beta}{2} + \frac{\beta-\alpha}{\pi}\operatorname{Arctan} x',$$

which is nothing but the familiar formula of F. Prym.

PROPOSITION 3.7. *Suppose, in addition to the hypotheses of Theorem* 3.2, *that* $\mu = \psi\lambda$ *for some* $\psi \in L_\infty(\mathsf{R}^n)$. *Then the conclusion of Theorem* 3.2 *holds for each* $F \in L_1(\mathsf{R}^n)$.

PROOF. Let $F \in L_1(\mathsf{R}^n)$ and let (x_k, y_k) be a nontangential null sequence in $\mathsf{R}^n \times \mathsf{R}^+$. Given $\varepsilon > 0$, choose $H \in C_c(\mathsf{R}^n)$ such that $\|F - H\|_1 < \varepsilon$. Then

$$|F[\mu] - H[\mu]| \leqq |F - H|\,[|\mu|] \leqq \|F - H\|_1\,\|\psi\|_\infty \leqq \|\psi\|_\infty\varepsilon$$

everywhere. Moreover, $H[\mu]$ has nontangential limit $\gamma\int H\,dx$ at $(0,0)$ by Theorem 3.2. As $\varepsilon > 0$ was arbitrary, it follows that $F[\mu](x_k, y_k) \to \gamma\int F\,dx$ as $k \to \infty$.

§4. Fatou's Theorem for the Real Line and the Circle.

Choose and fix a Radon measure μ on R. Let $F = F_\mu$ be a distribution function of μ, i.e., a function on R such that

$$F(y) - F(x) = \mu((x, y]) \quad \text{whenever} \quad y \geqq x \quad \text{in} \quad \mathsf{R}.$$

Thus μ is differentiable at 0 if and only if F is differentiable at 0, in which case $(D\mu)(0) = F'(0)$. We say that μ is *semi-differentiable* at 0 if both

$$(D_\ell F)(0) := \lim_{x\uparrow 0}\frac{F(x) - F(0)}{x} \quad \text{and} \quad (D_r F)(0) := \lim_{x\downarrow 0}\frac{F(x) - F(0)}{x}$$

exist in C.

THEOREM 4.1. *Let K be a bounded nonnegative measurable function on* R, *with* $\int K\,dx = 1$, *such that K is increasing on* $(-\infty, 0)$ *and decreasing on* $(0, \infty)$. *Suppose also that* $(K_t * |\mu|)(0) < \infty$ *for some* $t > 0$ *and*

(i) $$|\mu([-r,0])| + |\mu([0,r])| = O(r) \quad as \quad r \uparrow \infty.$$

If μ is differentiable at 0, *then $K[\mu]$ has nontangential limit $F'(0)$ at* (0, 0).

PROOF. Without loss of generality, assume $F'(0) = 0$.

Case 1: $K = 0$ on $(-\infty, 0)$. For notational simplicity, we shall assume that K is lower semicontinuous on $(0, \infty)$ and $u := K(0) = \sup\{K(x): x > 0\}$. Thus, for each $0 < t < u$, $\{x: K(x) > t\} = [0, r(t))$ for some $r(t) > 0$. Note that

(1) $$\int_0^u r(t)\,dt = \int K(x)\,dx < \infty.$$

Now define $q(x) = \{F(x) - F(0)\}/x$ for each real $x \neq 0$ and $q(0) = 0$. Then

(2) $$F(x) - F(x') = xq(x) - x'q(x') \quad \forall x, x' \in \mathrm{R},$$

q is continuous at 0 since $F'(0) = 0$, and $\|q\|_\infty < \infty$ by (i). Moreover, if $x' \in \mathrm{R}$ and if $y > 0$ is small enough, then

(3) $$\begin{aligned} K[\mu](x', y) &= y^{-1} \int K((x' - x)/y)\,d\mu(x) \\ &= y^{-1} \int_0^u \mu(\{x: K((x' - x)/y) > t\})\,dt \\ &= y^{-1} \int_0^u \mu(x' - y[0, r(t)))\,dt \\ &= y^{-1} \int_0^u \{F(x') - F(x' - yr(t))\}\,dt = \int_0^u Q(t, x', y)\,dt, \end{aligned}$$

where

(4) $$Q(t, x', y) = y^{-1}x'q(x') - \{y^{-1}x' - r(t)\}q(x' - yr(t)).$$

If $\alpha > 0$ and $|x'/y| < \alpha$, then

$$|Q(t, x', y)| \leqq \|q\|_\infty (2\alpha + r(t)) \in L_1((0, u))$$

by (1). Moreover, if $0 < t < u$ is fixed, then $Q(t, x', y') \to 0$ as $(x', y) \to (0, 0)$ nontangentially because q is continuous at 0 and $q(0) = 0$. It follows from Lebesgue's Convergence Theorem and (3) that $K[\mu]$ has nontangential limit 0 at (0, 0).

Case 2: $K = 0$ on $(0, \infty)$. Similar to Case 1.

In general, write $H = \chi_{[0,\infty)}K$ and $J = K - H$. Then $K = H + J$, and so $K[\mu]$ has nontangential limit 0 at (0, 0) by Cases 1 and 2.

COROLLARY 4.2. *Let K and μ be as in the hypotheses of Theorem* 4.1. *If μ is semi-differentiable at* 0, *then*

$$\lim_{k\to\infty} K[\mu](x_k, y_k) = (D_\ell F)(0)\int_{x_0}^{\infty} K(x)\,dx + (D_r F)(0)\int_{-\infty}^{x_0} K(x)\,dx$$

whenever $x_0 \in \mathsf{R}$ *and* (x_k, y_k) *is a nontangential null sequence in* $\mathsf{R} \times \mathsf{R}^+$ *such that* $x_k/y_k \to x_0$.

PROOF. Let $\alpha = (D_\ell F)(0)$, $\beta = (D_r F)(0)$ and $\phi = \alpha\chi_{(-\infty,0)} + \beta\chi_{(0,\infty)}$. Then $\mu - \phi\lambda$ has derivative 0 at 0. Therefore $K[\mu - \phi\lambda]$ has nontangential limit 0 at $(0,0)$. Moreover, $x' \in \mathsf{R}$ and $y > 0$ implies

$$K[\phi\lambda](x', y) = \alpha\int_{x'/y}^{\infty} K(x)\,dx + \beta\int_{-\infty}^{x'/y} K(x)\,dx.$$

Since $K[\mu] = K[\phi\lambda] + K[\mu - \phi\lambda]$, this completes the proof.

REMARKS. If K is strictly positive and satisfies the comparison condition (0, 9), then condition (i) in the last two results may be omitted.

For each $\sigma > 1/2$, determine $C(\sigma) > 0$ by the requirement that

$$\int_{-\infty}^{\infty} \frac{C(\sigma)}{(x^2+1)^\sigma}\,dx = 1.$$

We define *the Poisson kernel (for the circle) of order* σ to be

$$P_\sigma(x, r) = \frac{C(\sigma)(1-r)^{2\sigma-1}}{\{1 - 2r\cos x + r^2\}^\sigma} \quad (x \in \mathsf{R} \quad \text{and} \quad 0 \leqq r < 1).$$

For each complex Borel measure μ concentrated on $[-\pi, \pi]$, let

$$P_\sigma[\mu](x', r) = \int P_\sigma(x' - x, r)\,d\mu(x) \quad (x' \in \mathsf{R} \quad \text{and} \quad 0 \leqq r < 1).$$

Thus, if $\sigma = 1$, then $P_\sigma[\mu]$ is essentially the usual Poisson integral of μ, in which case part (a) of the following result is due to P. Fatou [2].

COROLLARY 4.3. *Let* $\sigma > 1/2$, *and let* μ *be a complex Borel measure on* R *that is concentrated on* $[-\pi, \pi]$.

(a) *If* μ *is differentiable at* 0, *then* $P_\sigma[\mu]$ *has nontangential limit* $(D\mu)(0)$ *at* $(0, 1)$.

(b) *If* μ *is semi-differentiable at* 0 *and if* $x_0 \in \mathsf{R}$, *then*

$$\lim_{k\to\infty} P_\sigma[\mu](x_k, r_k) = (D_\ell F)(0) + \{(D_r F)(0) - (D_\ell F)(0)\}\int_{-\infty}^{x_0} \frac{C(\sigma)\,dx}{(x^2+1)^\sigma}$$

whenever (x_k, r_k) *is a sequence in* $[-\pi, \pi] \times (0, 1)$ *such that* $r_k \uparrow 1$ *and* $x_k/(1 - r_k) \to x_0$.

PROOF. Let $0 < r < 1$ and $y > 0$ be related via $y = (1 - r)/\sqrt{r}$. Define

(1) $$A = A(x,r) = 1/\{2(1 - \cos x) + y^2\},$$

(2) $$B = B(x,y) = 1/(x^2 + y^2) \quad \text{and} \quad K(x) = C(\sigma)/(x^2 + 1)^\sigma.$$

Since $1 - 2r\cos x + r^2 = 2r(1 - \cos x) + (1 - r)^2$, it is easy to check that

(3) $$\sqrt{r}P_\sigma(x,r) = C(\sigma)y^{2\sigma-1}A^\sigma, \quad \text{and}$$

(4) $$K_y(x) = y^{-1}K(x/y) = C(\sigma)y^{2\alpha-1}B^\sigma.$$

Note that $x^2 \geqq 2(1 - \cos x)$ for each real x, $\{x^2 - 2(1 - \cos x)\}/x^4 \leqq 1/12$ for each real x (by four applications of the generalized mean-value theorem), and $x^2/(1 - \cos x) \leqq \pi^2$ for $|x| \leqq \pi$. Hence $|x| \leqq \pi$ implies

(5) $$\begin{aligned} 0 \leqq A - B &= AB\{x^2 - 2(1 - \cos x)\} \\ &\leqq A \cdot x^2/12 \leqq 1. \end{aligned}$$

It follows from the Mean-Value Theorem that $\sigma \geqq 1$ and $|x| \leqq \pi$ implies

(6) $$\begin{aligned} 0 \leqq \sqrt{r}P_\sigma(x,r) - K_y(x) &= C(\sigma)y^{2\sigma-1}(A^\sigma - B^\sigma) \\ &\leqq \sigma C(\sigma)y^{2\sigma-1}A^{\sigma-1}(A - B) \\ &\leqq \sigma C(\sigma)y^{2\sigma-1} \cdot y^{-2(\sigma-1)} \cdot 1 = \sigma C(\sigma)y. \end{aligned}$$

Similarly, if $1/2 < \sigma < 1$, $0 < y < 1$ and $|x| \leqq \pi$, then

(7) $$\begin{aligned} 0 \leqq \sqrt{r}P_\sigma(x,r) - K_y(x) &\leqq \sigma C(\sigma)y^{2\sigma-1}B^{\sigma-1}(A - B) \\ &\leqq \sigma C(\sigma)y^{2\sigma-1}(\pi^2 + 1)^{1-\sigma} \leqq 4\sigma C(\sigma)y^{2\sigma-1}. \end{aligned}$$

Now it is obvious for each $\delta > 0$, $P_\sigma(x,r) \to 0$ uniformly in $x \in [-\pi,\pi]\backslash[-\delta,\delta]$ as $r \uparrow 1$. So, in order to prove the desired results, we may suppose that μ is concentrated on $[-1,1]$. Therefore, if $\sigma \geqq 1$, then (6) ensures that

(8) $$|\sqrt{r}P_\sigma[\mu](x',r) - K[\mu](x',y)| \leqq \sigma C(\sigma)\,\|\mu\|\,y$$

whenever $0 < r < 1$ and $|x'| < \pi - 1$; hence (a) and (b) follow from Theorem 4.1 and Corollary 4.2. In light of (7), the proof for $1/2 < \sigma < 1$ is similar.

Corollary 4.4. *Suppose that μ is concentrated on $[-\pi,\pi]$ and $(\mathrm{UD}\,|\mu|)(0) < \infty$. If there exist two distinct real numbers in x_1, x_2 in $(-\pi,\pi)$ such that for each k, $\lim_{r\uparrow 1} P_1[\mu]((1 - r)x_k, r)$ exists, then μ is semi-differentiable at 0.*

Proof. Combine the above proof with Corollary 3.5.

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DEPARTMENT OF MATHEMATICS
KANSAS STATE UNIVERSITY
MANHATTAN, KS 66506
USA