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EXPLICIT FORMULAS AND ASYMPTOTIC EXPANSIONS FOR CERTAIN MEAN SQUARE OF HURWITZ ZETA-FUNCTIONS I

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1. Introduction.

Let $s = \sigma + it$ be a complex variable, $\alpha > 0$, and $\zeta(s,\alpha)$ the Hurwitz zeta-function defined by the analytic continuation of the Dirichlet series $\sum_{n=0}^{\infty}(n+\alpha)^{-s}$. Define $\zeta_1(s,\alpha) = \zeta(s,\alpha) - \alpha^{-s}$, and

$$I(s) = \int_0^1 |\zeta_1(s,\alpha)|^2 \, d\alpha.$$

The integral $I(\frac{1}{2}+it)$ was first studied by Koksma-Lekkerkerker [10], who proved the upper-bound $I(\frac{1}{2}+it) = O(\log t)$ for any $t \geqq 2$. Gallagher [4] applied this result to the study of Dirichlet L-functions. Balasubramanian [3] gave an improvement, that is the asymptotic formula

$$I(\tfrac{1}{2}+it) = \log t + O(\log\log t),$$

and the error term was refined to $O(1)$ by Rane [13]. A further improvement was obtained by Sitaramachandrarao [14], who proved

$$I(\tfrac{1}{2}+it) = \log(t/2\pi) + \gamma + O(t^{-3/16}(\log t)^{3/8}), \tag{1.1}$$

where γ is Euler's constant. Sitaramachandrarao's work was announced in p. 28 of Hardy-Ramanujan Journal, volume 10 (1987), but it seems that his paper [14] has not yet been published. Meanwhile, independent of Sitaramachandrarao, Zhang [17] obtained the essentially same result as (1.1), and then, he improved the error term to $O(t^{-7/36}(\log t)^{25/18})$ in [18]. In these papers, Zhang conjectured

$$I(\tfrac{1}{2}+it) = \log(t/2\pi) + \gamma + O(t^{-1/4}). \tag{1.2}$$

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Perhaps this conjecture had been well-known among Indian number theorists. When Professor K. Ramachandra visited Japan, he mentioned (1.2) as an interesting conjecture, in a private discussion with the authors.

The main tool in the aforementioned papers is the approximate functional equation of $\zeta(s,\alpha)$, but this is actually insufficient for the problem of evaluating $I(\frac{1}{2}+it)$, as has turned out in the following works. In December 1992, the authors found a proof of the conjecture (1.2), and wrote a brief sketch of the method in [7]. The author's method is a variant of Atkinson's device, and the basic idea is due to Atkinson [2], Motohashi [12] and the authors [6], concerning the mean values of the Riemann zeta and Dirichlet L-functions. However, in March 1993, the preprint of Zhang [19] arrived at the second-named author. In this article, by an ingenious simple argument based on the functional equation of $\zeta(s,\alpha)$ ((2.17.3) of Titchmarsh [15]), Zhang proved the stronger result

$$I(\tfrac{1}{2}+it)=\log(t/2\pi)+\gamma-2\,\mathrm{Re}\frac{\zeta(\frac{1}{2}+it)}{\frac{1}{2}+it}+O(t^{-1}), \tag{1.3}$$

where $\zeta(s)$ is the Riemann zeta-function. Later, the authors noticed the existence of Andersson's article [1], in which a different proof of (1.3) is given. Andersson's proof is based on Mikolás' idea [11] of using Parseval's identity.

By refining the argument of the authors sketched in [7], it is possible to give an alternative proof of (1.3). In fact, the authors' method can give more precise information. The main result in the present paper is the explicit formula (2.1), stated in the next section. As special limit cases of this main formula, we can deduce two kinds of refinements of (1.3) (Corollaries 2 and 5).

Here we list up several advantages of our method. Our method supplies a clear view of the structure of the whole theory (see "a little digression" in Section 6). Also we can treat the discrete mean square

$$\sum_{1\leqq a\leqq q}\left|\zeta\left(s,\frac{a}{q}\right)\right|^2, \tag{1.4}$$

where q is a positive integer, by the same method (see the remark in Section 4). Moreover, our main theorem is important as the starting point of further researches, which will be given in forthcoming papers.

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2. Statement of results.

Let u, v be complex variables, and E the set of (u, v) at which some factor in (2.1) below has a singularity. Let $\Gamma(s)$ be the gamma-function, and $\psi(s) = (\Gamma'/\Gamma)(s)$. We use the Pochhammer symbol $(s)_n = \Gamma(s+n)/\Gamma(s)$ for any integer n. In what follows, the empty sum is to be considered as zero. The main general result of the present paper can be stated as follows.

THEOREM. *Let $N \geqq 1$ be an integer, $-N+1 < \operatorname{Re} u < N+1$, $-N+1 < \operatorname{Re} v < N+1$, and $(u, v) \notin E$. Then it holds*

$$\begin{aligned}(2.1)\quad &\int_0^1 \zeta_1(u,\alpha)\zeta_1(v,\alpha)\,d\alpha \\ &= \frac{1}{u+v-1} + \Gamma(u+v-1)\zeta(u+v-1)\left(\frac{\Gamma(1-v)}{\Gamma(u)} + \frac{\Gamma(1-u)}{\Gamma(v)}\right) \\ &\quad - S_N(u,v) - S_N(v,u) - T_N(u,v) - T_N(v,u),\end{aligned}$$

where

$$S_N(u,v) = \sum_{n=0}^{N-1} \frac{(u)_n}{(1-v)_{n+1}}(\zeta(u+n)-1),$$

$$T_N(u,v) = \frac{(u)_N}{(1-v)_N}\sum_{l=1}^{\infty} l^{1-u-v}\int_l^{\infty} \beta^{u+v-2}(1+\beta)^{-u-N}\,d\beta.$$

Moreover, $T_N(u,v)$ has the expression

$$\begin{aligned}(2.2)\quad T_N(u,v) &= \sum_{k=1}^{K}(-1)^{k-1}\frac{(2-u-v)_{k-1}(u)_{N-k}}{(1-v)_N}\sum_{l=1}^{\infty} l^{-k}(l+1)^{-u-N+k} \\ &\quad + (-1)^K\frac{(2-u-v)_K(u)_{N-K}}{(1-v)_N}\sum_{l=1}^{\infty} l^{1-u-v}\int_l^{\infty}\beta^{u+v-K-2}(1+\beta)^{-u-N+K}\,d\beta\end{aligned}$$

for any integer $K \geqq 0$.

It is easy to see that (u, v) belongs to E if and only if $u+v$ is an integer ($\leqq 2$). or u is an integer, or v is an integer. The above theorem in particular shows that our method works not only on the critical line (or in the critical strip), but on the whole plane. In the case of Dirichlet L-functions such a phenomenon has been observed in Katsurada [5] and the authors [9].

Taking $u = \sigma + it$ and $v = \sigma - it$ in the theorem, we have

COROLLARY 1. *Let N, K be integers with $N \geqq 1$ and $K \geqq 0$. Then, for any σ satisfying $-N+1 < \sigma < N+1$, $2\sigma - 1 \notin \{1, 0, -1, -2, \cdots\}$ and any $t \geqq 1$, we have*

(2.3)
$$
\begin{aligned}
I(\sigma + it) &= \frac{1}{2\sigma - 1} + 2\Gamma(2\sigma - 1)\zeta(2\sigma - 1)\,\mathrm{Re}\,\frac{\Gamma(1 - \sigma + it)}{\Gamma(\sigma + it)} \\
&\quad - 2\,\mathrm{Re}\sum_{n=0}^{N-1} \frac{(\sigma + it)_n}{(1 - \sigma + it)_{n+1}}(\zeta(\sigma + it + n) - 1) \\
&\quad - 2\,\mathrm{Re}\sum_{k=1}^{K} (-1)^{k-1} \frac{(2 - 2\sigma)_{k-1}(\sigma + it)_{N-k}}{(1 - \sigma + it)_N} \sum_{l=1}^{\infty} l^{-k}(l + 1)^{-\sigma - N + k - it} \\
&\quad + O(t^{-K-1}),
\end{aligned}
$$

where the O-constant depends only on N, K and σ.

The error estimate $O(t^{-K-1})$ in (2.3) follows from the facts

$$
\frac{(2 - 2\sigma)_K(\sigma + it)_{N-K}}{(1 - \sigma + it)_N} = O(t^{-K})
$$

and

$$
\sum_{l=1}^{\infty} l^{1-2\sigma} \int_l^{\infty} \beta^{2\sigma - K - 2}(1 + \beta)^{-\sigma - it - N + K}\, d\beta = O(t^{-1}),
$$

where the last estimate can be proved by integration by parts.

Choose $N = 1$, and put $\sigma = \frac{1}{2} + \frac{\delta}{2}$ in (2.3), where δ is a small positive number. In view of the formulas $\Gamma(\delta) = \delta^{-1} - \gamma + O(\delta)$ and $\zeta(\delta) = -\frac{1}{2} - \frac{1}{2}(\log 2\pi)\delta + O(\delta^2)$, taking the limit $\delta \to 0$, we obtain the following refinement of (1.3).

COROLLARY 2. *For any integer $K \geqq 0$ and any real $t \geqq 1$, we have the asymptotic expansion*

$$
\begin{aligned}
I(\tfrac{1}{2} + it) &= \gamma - \log 2\pi + \mathrm{Re}\,\psi(\tfrac{1}{2} + it) - 2\,\mathrm{Re}\,\frac{\zeta(\frac{1}{2} + it) - 1}{\frac{1}{2} + it} \\
&\quad - 2\,\mathrm{Re}\sum_{k=1}^{K} \frac{(-1)^{k-1}(k - 1)!}{(\frac{3}{2} - k + it)(\frac{5}{2} - k + it)\cdots(\frac{1}{2} + it)} \sum_{l=1}^{\infty} l^{-k}(l + 1)^{-\frac{3}{2} + k - it} \\
&\quad + O(t^{-K-1}),
\end{aligned}
$$

where the O-constant depends only on K.

REMARK. Since $\mathrm{Re}\,\psi(\frac{1}{2} + it) = \log t + O(t^{-2})$, Corollary 2 implies (1.3).

Other exceptional cases can also be treated as the limit cases. For example, since $(2 - u - v)_{k-1} = 0$ for $k \geqq 2$ if $u + v = 2$, taking the limit $u \to 1 + it$ and $v \to 1 - it$ in the theorem, we have

COROLLARY 3. *For any integer $N \geqq 1$ and any real $t \geqq 1$, we have*

$$I(1+it) = 1 - t^{-2} - 2\,\mathrm{Re}\,\frac{\psi(1+it)}{it} - 2\,\mathrm{Re}\,\frac{1}{it}\sum_{n=0}^{N-1}(\zeta(1+n+it)-1) - 2\,\mathrm{Re}\,\frac{1}{it}\sum_{l=1}^{\infty}\frac{1}{l(l+1)^{N+it}}.$$

Next, let $N_0 \geqq 1$ be an integer, $-N_0+1 < \mathrm{Re}\,u < N_0+1$, $-N_0+1 < \mathrm{Re}\,v < N_0+1$, and $(u,v) \notin E$. Then the theorem holds for any $N \geqq N_0$. By using Stirling's formula we have

$$\frac{(u)_N}{(1-v)_N} = O(N^{\mathrm{Re}(u+v)-1}),$$

where the O-constant depends on u and v. Combining this estimate with the inequality

$$\left|\sum_{l=1}^{\infty} l^{1-u-v}\int_l^{\infty}\beta^{u+v-2}(1+\beta)^{-u-N}\,d\beta\right| \leqq 2^{-N+N_0}\sum_{l=1}^{\infty} l^{1-\mathrm{Re}(u+v)}\int_l^{\infty}\beta^{\mathrm{Re}(u+v)-2}(1+\beta)^{-\mathrm{Re}\,u-N_0}\,d\beta \ll 2^{-N},$$

with the implied constant depending on u, v and N_0, we have

$$T_N(u,v) \ll 2^{-N}N^{\mathrm{Re}(u+v)-1},$$

hence $T_N(u,v) \to 0$ as $N \to \infty$. Therefore, letting $N \to \infty$ in the theorem, we obtain the following explicit formula.

COROLLARY 4. *Let u, v be complex numbers with $(u,v) \notin E$. Then*

$$\int_0^1 \zeta_1(u,\alpha)\zeta_1(v,\alpha)\,d\alpha = \frac{1}{u+v-1} + \Gamma(u+v-1)\zeta(u+v-1)\left(\frac{\Gamma(1-v)}{\Gamma(u)} + \frac{\Gamma(1-u)}{\Gamma(v)}\right) - \sum_{n=0}^{\infty}\frac{(u)_n}{(1-v)_{n+1}}(\zeta(u+n)-1) - \sum_{n=0}^{\infty}\frac{(v)_n}{(1-u)_{n+1}}(\zeta(v+n)-1).$$

Note that Andersson [1] proved this result as well as the next Corollary 5 in a different way. By taking the limit $u \to \frac{1}{2}+it$ and $v \to \frac{1}{2}-it$ in Corollary 4, we obtain another refinement of (1.3).

COROLLARY 5. *For any real t we have*

$$I(\tfrac{1}{2}+it)=\gamma-\log 2\pi+\operatorname{Re}\psi(\tfrac{1}{2}+it)-2\operatorname{Re}\sum_{n=0}^{\infty}\frac{\zeta(\frac{1}{2}+n+it)-1}{\frac{1}{2}+n+it}.$$

We note that the special case $t=0$ in Corollary 5 is given in Zhang [19].

Once the statement of Corollary 2 has been known, it is actually possible to find a simple argument by which one can deduce Corollary 2 from Corollary 5. In fact, expanding $\left(1-\dfrac{1}{l+1}\right)^{-k}$ $(l\geqq 1)$ by the binomial theorem we find

$$\sum_{k=1}^{K}\frac{(-1)^{k-1}(k-1)!}{(\frac{3}{2}-k+it)(\frac{5}{2}-k+it)\cdots(\frac{1}{2}+it)}l^{-k}(l+1)^{-\frac{3}{2}+k-it}$$
$$=\sum_{p=0}^{\infty}(l+1)^{-p-\frac{3}{2}-it}A_K(p),$$

where

$$A_K(p)=\sum_{k=1}^{K}\frac{(-1)^{k-1}(p+1)\cdots(p+k-1)}{(\frac{3}{2}-k+it)\cdots(\frac{1}{2}+it)}.$$

Induction on K shows that

$$A_K(p)=\frac{1}{p+\frac{3}{2}+it}\left(1+\frac{(-1)^{k-1}(p+1)\cdots(p+K)}{(\frac{3}{2}-K+it)\cdots(\frac{1}{2}+it)}\right)$$
$$=\frac{1}{p+\frac{3}{2}+it}+O_K(p^Kt^{-K-1}).$$

Since $\sum_{l\geqq 1}\sum_{p\geqq 0}p^K(l+1)^{-p-\frac{3}{2}}$ is convergent, the double sum in Corollary 2 is equal to

$$\sum_{p=0}^{\infty}\frac{\zeta(p+\frac{3}{2}+it)-1}{p+\frac{3}{2}+it}+O_K(t^{-K-1}).$$

Therefore, Corollary 2 is a consequence of Corollary 5. However, this argument does not give a way of finding the formula in Corollary 2.

The above mentioned results except Corollary 3 have been announced in [8]. The rest of this paper is devoted to the proof of the theorem.

3. Atkinson's dissection.

Let u, v be complex variables, $\alpha>0$, and at first assume $\operatorname{Re}u>1$, $\operatorname{Re}v>1$. Then

$$\zeta(u,\alpha)\zeta(v,\alpha)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(m+\alpha)^{-u}(n+\alpha)^{-v},$$

and dividing this sum into three parts corresponding to the conditions $m = n$, $m < n$ and $m > n$, we have

$$\zeta(u,\alpha)\zeta(v,\alpha) = \zeta(u+v,\alpha) + f(u,v;\alpha) + f(v,u;\alpha), \tag{3.1}$$

where

$$f(u,v;\alpha) = \sum_{m=0}^{\infty} (m+\alpha)^{-u} \sum_{n=1}^{\infty} (m+n+\alpha)^{-v}.$$

Let $\eta \geqq 0$, and now assume $\operatorname{Re} u > 0$, $\operatorname{Re} v > 1$ and $\operatorname{Re}(u+v) > 2$. Then the infinite series

$$\sum_{m=0}^{\infty} (\alpha+m)^{-u} \int_{\eta}^{\infty} \frac{e^{-(\alpha+m)y} y^{v-1}}{e^y - 1} dy$$

is convergent absolutely, uniformly with respect to η, because the above integral is bounded by

$$\begin{aligned}
&\ll \int_{\eta}^{1} e^{-(\alpha+m)y} y^{\operatorname{Re} v - 2} dy + \int_{1}^{\infty} e^{-(\alpha+m+1)y} y^{\operatorname{Re} v - 1} dy \\
&\ll (\alpha+m)^{-\operatorname{Re} v + 1} \Gamma(\operatorname{Re} v - 1) + (\alpha+m+1)^{-\operatorname{Re} v} \Gamma(\operatorname{Re} v) \\
&\ll (\alpha+m)^{-\operatorname{Re} v + 1},
\end{aligned}$$

where the implied constants are independent of η. Therefore,

$$\begin{aligned}
&\int_0^{\infty} \frac{y^{v-1}}{e^y - 1} \int_0^{\infty} \frac{e^{(1-\alpha)(x+y)} x^{u-1}}{e^{x+y} - 1} dx\, dy \\
&= \lim_{\eta \downarrow 0} \int_{\eta}^{\infty} \frac{y^{v-1}}{e^y - 1} \sum_{m=0}^{\infty} \int_0^{\infty} e^{-(\alpha+m)(x+y)} x^{u-1} dx\, dy \\
&= \lim_{\eta \downarrow 0} \Gamma(u) \sum_{m=0}^{\infty} (\alpha+m)^{-u} \int_{\eta}^{\infty} \frac{e^{-(\alpha+m)y} y^{v-1}}{e^y - 1} dy \\
&= \Gamma(u) \sum_{m=0}^{\infty} (\alpha+m)^{-u} \int_0^{\infty} \left(\sum_{n=1}^{\infty} e^{-(\alpha+m+n)y} \right) y^{v-1} dy \\
&= \Gamma(u)\Gamma(v) \sum_{m=0}^{\infty} (\alpha+m)^{-u} \sum_{n=1}^{\infty} (\alpha+m+n)^{-v},
\end{aligned} \tag{3.2}$$

which gives the analytic continuation of $f(u,v;\alpha)$ to $\operatorname{Re} u > 0$, $\operatorname{Re} v > 1$ and $\operatorname{Re}(u+v) > 2$.

For any complex z, put

$$h(z;\alpha)=\frac{e^{(1-\alpha)z}}{e^z-1}-\frac{1}{z},$$

and by $h^{(N)}(z;\alpha)$ we mean the Nth derivative of $h(z;\alpha)$ with respect to z. Let $\mathscr{C}$ be the contour which consists of the half-line on the positive real axis from infinity to a small positive constant δ, a circle of radius δ counterclockwise round the origin, and the other half-line on the positive real axis from δ to infinity. Note that $h(z;\alpha)$ is holomorphic at $z=0$. We now prepare

LEMMA 1. *For any integer $N \geqq 0$, we have*

$$h^{(N)}(x+\tau y;\alpha)=O(\alpha^N e^{-A\alpha|x|}+(|x|+1)^{-N-1})$$

with a positive absolute constant A, uniformly for any $x, y\in\mathscr{C}\cup[0,+\infty)$, $\tau\in[0,1]$ and any $\alpha\geqq 0$.

PROOF. We have

$$\begin{aligned}
&h^{(N)}(z;\alpha)\\
&=\sum_{k=0}^{N}\binom{N}{k}(-\alpha)^{N-k}e^{-\alpha z}\frac{d^k}{dz^k}\left(\frac{1}{e^z-1}+1\right)-\frac{(-1)^N N!}{z^{N+1}}\\
&=(-\alpha)^N e^{-\alpha z}\left(\frac{1}{e^z-1}+1\right)+\sum_{k=1}^{N}\binom{N}{k}(-\alpha)^{N-k}e^{-\alpha z}\frac{P_k(e^z)}{(e^z-1)^{k+1}}-\frac{(-1)^N N!}{z^{N+1}},
\end{aligned}$$

where $P_k(x)$ is, as we can see inductively, a polynomial of degree k in x (see the proof of Lemma 1 in [6]). Hence the assertion of the lemma follows when $|x+\tau y|>1$. Next, let $C=\{z\,|\,|z|=1\}$ and $D=\{z\,|\,|z|\leqq 1\}$. For any $\alpha\geqq 0$, $h^{(N)}(z,\alpha)$ is holomorphic on D, hence we can find a point $z_\alpha\in C$ such that $|h^{(N)}(z;\alpha)|\leqq|h^{(N)}(z_\alpha;\alpha)|$ for any $z\in D$. Since $h^{(N)}(z;\alpha)$ is continuous on $C\times[0,1]$ as a function in (z,α), we conclude that $h^{(N)}(z;\alpha)$ is uniformly bounded on $D\times[0,1]$. This implies Lemma 1 in case $|x+\tau y|\leqq 1$.

Using Lemma 1 and the fact

$$\int_0^\infty\frac{x^{u-1}}{x+y}dx=y^{u-1}\Gamma(u)\Gamma(1-u)\qquad(y>0,\ 0<\operatorname{Re}u<1),$$

we obtain from (3.2) that

$$\begin{aligned}
(3.3)\qquad f(u,v;\alpha)&=\frac{1}{\Gamma(u)\Gamma(v)}\int_0^\infty\frac{y^{v-1}}{e^y-1}\int_0^\infty h(x+y;\alpha)x^{u-1}\,dx\,dy\\
&\quad+\frac{1}{\Gamma(u)\Gamma(v)}\int_0^\infty\frac{y^{v-1}}{e^y-1}\int_0^\infty\frac{x^{u-1}}{x+y}\,dx\,dy\\
&=g(u,v;\alpha)+\Gamma(u+v-1)\zeta(u+v-1)\Gamma(1-u)\Gamma(v)^{-1}
\end{aligned}$$

for $\alpha > 0$, $0 < \operatorname{Re} u < 1$ and $\operatorname{Re}(u + v) > 2$, where

$$g(u, v; \alpha) = \frac{1}{\Gamma(u)\Gamma(v)(e^{2\pi iu} - 1)(e^{2\pi iv} - 1)} \times \int_{\mathscr{C}} \frac{y^{v-1}}{e^y - 1} \int_{\mathscr{C}} h(x + y; \alpha) x^{u-1} \, dx \, dy,$$

$x^u = e^{u \log x}$, $y^v = e^{v \log y}$, and $\operatorname{Im}(\log x)$, $\operatorname{Im}(\log y)$ vary from 0 to 2π round $\mathscr{C}$. Since $g(u, v; \alpha)$ is convergent absolutely for $\operatorname{Re} u < 1$ and any complex v, from (3.1) and (3.3) we now obtain

LEMMA 2. *Let* $\alpha > 0$, $\operatorname{Re} u < 1$ *and* $\operatorname{Re} v < 1$. *Then*

$$(3.4) \quad \zeta(u, \alpha)\zeta(v, \alpha) = \zeta(u + v, \alpha) + \Gamma(u + v - 1)\zeta(u + v - 1)\left\{\frac{\Gamma(1 - u)}{\Gamma(v)} + \frac{\Gamma(1 - v)}{\Gamma(u)}\right\} + g(u, v; \alpha) + g(v, u; \alpha).$$

REMARK. This is an analogue of [6, (2.2)], which is originally due to Motohashi [12].

Since $\zeta_1(s, \alpha) = \zeta(s, \alpha) - \alpha^{-s}$, it immediately follows from Lemma 2 that

$$(3.5) \quad \zeta_1(u, \alpha)\zeta_1(v, \alpha) = \zeta_1(u + v, \alpha) + \Gamma(u + v - 1)\zeta(u + v - 1)\left\{\frac{\Gamma(1 - u)}{\Gamma(v)} + \frac{\Gamma(1 - v)}{\Gamma(u)}\right\} + g(u, v; \alpha) + g(v, u; \alpha) - \alpha^{-u}\zeta_1(v, \alpha) - \alpha^{-v}\zeta_1(u, \alpha).$$

This identity has already been stated in [7, (3.1)], and plays the fundamental role in our method.

4. The analytic continuation of $g(u, v; \alpha)$.

In view of Lemma 1, we can divide

$$g(u, v; \alpha) = \frac{1}{\Gamma(u)\Gamma(v)(e^{2\pi iu} - 1)(e^{2\pi iv} - 1)} \times \int_{\mathscr{C}} \frac{y^{v-1}}{e^y - 1}(g_1(u) + g_2(u) + g_3(u)) \, dy$$

for $\alpha > 0$, $\operatorname{Re} u < 1$ and any v, where

$$g_1(u) = \int_{\mathscr{C}} (h(x+y;\alpha) - h(x;\alpha))x^{u-1}\,dx,$$

$$g_2(u) = \int_{\mathscr{C}} \frac{e^{(1-\alpha)x}}{e^x - 1} x^{u-1}\,dx$$

and

$$g_3(u) = -\int_{\mathscr{C}} x^{u-2}\,dx.$$

It is obvious that $g_3(u) = 0$ for $\operatorname{Re} u < 1$. Also it is well-known that

$$g_2(u) = (e^{2\pi i u} - 1)\Gamma(u)\zeta(u,\alpha) \tag{4.1}$$

(see Whittaker-Watson [16], Chap. 13). Therefore we have

$$g(u,v;\alpha) = \frac{1}{\Gamma(u)\Gamma(v)(e^{2\pi i u} - 1)(e^{2\pi i v} - 1)} \int_{\mathscr{C}} \frac{y^{v-1} g_1(u)}{e^y - 1}\,dy + \zeta(u,\alpha)\zeta(v) \tag{4.2}$$

for $\operatorname{Re} u < 1$ and any v.

Let N be a positive integer. Integrating by parts $(N-1)$-times, we have

$$\begin{aligned}
&h(x+y;\alpha) - h(x;\alpha)\\
&= \int_x^{x+y} h'(\xi;\alpha)\,d\xi\\
&= -\int_x^{x+y} \frac{\partial}{\partial\xi}(x+y-\xi)h'(\xi;\alpha)\,\mathrm{d}\xi\\
&= \sum_{n=1}^{N-1} h^{(n)}(x;\alpha)\frac{y^n}{n!} + \int_x^{x+y} \frac{(x+y-\xi)^{N-1}}{(N-1)!} h^{(N)}(\xi;\alpha)\,d\xi\\
&= \sum_{n=1}^{N-1} h^{(n)}(x;\alpha)\frac{y^n}{n!} + y^N \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} h^{(N)}(x+\tau y;\alpha)\,d\tau.
\end{aligned}$$

Hence,

$$\begin{aligned}
&g_1(u)\\
&= \sum_{n=1}^{N-1} \frac{y^n}{n!} \int_{\mathscr{C}} h^{(n)}(x;\alpha)x^{u-1}\,dx + y^N \int_{\mathscr{C}} \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} h^{(N)}(x+\tau y;\alpha)x^{u-1}\,d\tau\,dx.
\end{aligned}$$

Integrating by parts repeatedly and using Lemma 1 and (4.1), we have

$$\begin{aligned}\int_{\mathscr{C}} h^{(n)}(x;\alpha)x^{u-1}\,dx &= (-1)^n(u-1)\cdots(u-n)\int_{\mathscr{C}} h(x;\alpha)x^{u-n-1}\,dx \\ &= (-1)^n(u-1)\cdots(u-n)\int_{\mathscr{C}} \frac{e^{(1-\alpha)x}}{e^x-1}x^{u-n-1}\,dx \\ &= (-1)^n(e^{2\pi i u}-1)\Gamma(u)\zeta(u-n,\alpha)\end{aligned}$$

for $\operatorname{Re} u < 1$. Substituting these results into (4.2), we have

$$\begin{aligned}(4.3)\quad g(u,v;\alpha) &= \frac{1}{\Gamma(v)(e^{2\pi i v}-1)}\int_{\mathscr{C}} \frac{1}{e^y-1}\sum_{n=1}^{N-1}\frac{(-1)^n}{n!}y^{v+n-1}\zeta(u-n,\alpha)\,dy \\ &\quad + \zeta(u,\alpha)\zeta(v) + r_N(u,v;\alpha) \\ &= \sum_{n=0}^{N-1}\binom{-v}{n}\zeta(u-n,\alpha)\zeta(v+n) + r_N(u,v;\alpha),\end{aligned}$$

where

$$\begin{aligned}(4.4)\qquad r_N(u,v;\alpha) &= \frac{1}{\Gamma(u)\Gamma(v)(e^{2\pi i u}-1)(e^{2\pi i v}-1)}\int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} \\ &\quad \times \int_{\mathscr{C}} \frac{y^{v+N-1}}{e^y-1}\int_{\mathscr{C}} h^{(N)}(x+\tau y;\alpha)x^{u-1}\,dx\,dy\,d\tau.\end{aligned}$$

The above calculations are justified by Fubini's theorem and Lemma 1. Since the integrals in (4.4) are convergent absolutely for $\operatorname{Re} u < N+1$ and any v, (4.3) gives the analytic continuation of $g(u,v;\alpha)$ to $\operatorname{Re} u < N+1$ and any v, for any $\alpha > 0$. Therefore, (3.5) is now valid for $\operatorname{Re} u < N+1$, $\operatorname{Re} v < N+1$ and $\alpha > 0$.

Remark. Here we mention how to treat the discrete mean square (1.4). Substitute (4.3) into (3.4), put $\alpha = a/q$, and sum up with respect to $a = 1, 2, \ldots, q$. Noting the relations

$$\sum_{1 \leqq a \leqq q} \zeta\left(s, \frac{a}{q}\right) = q^s\zeta(s), \qquad \sum_{1 \leqq a \leqq q} h\left(z; \frac{a}{q}\right) = h\left(\frac{z}{q}\right) - 1,$$

where $h(z) = e^z(e^z-1)^{-1} - z^{-1}$, we obtain [7, (2.4)], from which Theorems 1 and 2 in [7] can be easily deduced.

5. Several auxiliary lemmas.

For any $\alpha \geqq 0$, the expression

$$(5.1)\qquad \zeta_1(s,\alpha) = \frac{1}{\Gamma(s)(e^{2\pi i s}-1)}\int_{\mathscr{C}} \frac{z^{s-1}e^{-\alpha z}}{e^z-1}\,dz$$

can be proved in a standard way (similar to the proof of (4.1)), and from which the following two lemmas can be deduced easily.

LEMMA 3. *Let* $\alpha_1 \geqq 0, \alpha_2 \geqq 0$, *and* $s\,(\neq 1)$ *be a complex number. Then we have*

$$\int_{\alpha_1}^{\alpha_2} \zeta_1(s,\alpha)\,d\alpha = \frac{1}{s-1}(\zeta_1(s-1,\alpha_1) - \zeta_1(s-1,\alpha_2)), \tag{5.2}$$

in particular,

$$\int_0^1 \zeta_1(s,\alpha)\,d\alpha = \frac{1}{s-1}. \tag{5.3}$$

LEMMA 4. *For any* $\alpha \geqq 0$ *and any complex* $s\,(\neq 1)$, $\zeta_1(s,\alpha)$ *is differentiable with respect to* α, *and*

$$\left(\frac{\partial}{\partial\alpha}\right)^n \zeta_1(s,\alpha) = (-1)^n (s)_n \zeta_1(s+n,\alpha)$$

for any non-negative integer n.

From (5.3) and the relation $\zeta(s,\alpha) = \zeta_1(s,\alpha) + \alpha^{-s}$, we have

LEMMA 5. *For* $\mathrm{Re}\, s < 1$,

$$\int_0^1 \zeta(s,\alpha)\,d\alpha = 0.$$

We note that this result was noticed by Mikolás [11, (5.1)]. (He misprinted the condition $\mathrm{Re}\, u < 1$ as $\mathrm{Re}\, u > 1$.)

LEMMA 6. *For any complex* $s\,(\neq 1)$ *and any* $\alpha \geqq 0$, *we have*

$$\zeta_1(s,\alpha) = \sum_{n=0}^{N-1} \frac{(-1)^n (s)_n \alpha^n}{n!} \zeta(s+n) + (-1)^N (s)_N \alpha^N \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} \zeta_1(s+N,\alpha\tau)\,d\tau$$

for any positive integer N.

PROOF. Integrating by parts repeatedly, we have

$$\begin{aligned}
\zeta_1(s,\alpha) &= \zeta_1(s,0) + \int_0^\alpha \frac{\partial}{\partial\xi}\zeta_1(s,\xi)\,d\xi \\
&= \zeta_1(s,0) + \sum_{n=1}^{N-1}\left[-\frac{(\alpha-\xi)^n}{n!}\frac{\partial^n}{\partial\xi^n}\zeta_1(s,\xi)\right]_{\xi=0}^{\alpha} \\
&\quad + \int_0^\alpha \frac{(\alpha-\xi)^{N-1}}{(N-1)!}\frac{\partial^N}{\partial\xi^N}\zeta_1(s,\xi)\,d\xi.
\end{aligned}$$

Therefore, using Lemma 4 and changing the variable by $\tau = \alpha^{-1}\xi$ in the last integral, we obtain the assertion of the lemma.

LEMMA 7. *Let* $s(\neq 1)$ *be complex with* $\sigma = \operatorname{Re} s > 0$. *Then for any* $\alpha \geqq 0$, *we have*

$$\zeta_1(s,\alpha) = O((1+\alpha)^{1-\sigma}),$$

where the implied constant depends on σ *and* t.

PROOF. If $\sigma > 1$, then using the Euler-Maclaurin summation formula we have

$$\zeta_1(s,\alpha) = \frac{(1+\alpha)^{1-s}}{s-1} + \tfrac{1}{2}(1+\alpha)^{-s} - s\int_1^\infty (x - [x] - \tfrac{1}{2})(x+\alpha)^{-s-1}\,dx. \tag{5.4}$$

The last integral is convergent absolutely for $\sigma > 0$, hence (5.4) gives the meromorphic continuation of $\zeta_1(s,\alpha)$ to the region $\sigma > 0$. Lemma 7 follows by estimating the right-hand side of (5.4) trivially.

6. A vanishing result and a formula of Mikolás.

By using (4.3) and Lemma 6, it follows that

$$\begin{aligned}
&g(u,v;\alpha) - \alpha^{-u}\zeta_1(v,\alpha) \\
&= \sum_{n=0}^{N-1} \frac{(-1)^n(v)_n}{n!}\zeta(u-n,\alpha)\zeta(v+n) + r_N(u,v;\alpha) \\
&\quad -\alpha^{-u}\left\{\sum_{n=0}^{N-1} \frac{(-1)^n(v)_n\alpha^n}{n!}\zeta(v+n)\right. \\
&\quad \left. + (-1)^N(v)_N\alpha^N\int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!}\zeta_1(v+N,\alpha\tau)\,d\tau\right\} \\
&= \sum_{n=0}^{N-1} \frac{(-1)^n(v)_n}{n!}\zeta_1(u-n,\alpha)\zeta(v+n) + r_N(u,v;\alpha) \\
&\quad -(-1)^N(v)_N\alpha^{N-u}\int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!}\zeta_1(v+N,\alpha\tau)\,d\tau
\end{aligned} \tag{6.1}$$

for $\operatorname{Re} u < N+1$ and any v. It should be noted that in the resulting expression, the singularities in $\zeta(u-n,\alpha)$ $(0 \leqq n \leqq N-1)$ with respect to α at $\alpha = 0$ are cancelled away, so we can integrate both sides with respect to α from 0 to 1. To carry out the integration, we need the following

LEMMA 8. *For any positive integer* N, *we have*

$$\int_0^1 r_N(u, v; \alpha)\, d\alpha = 0$$

for Re $u < N + 1$ *and any* v.

REMARK. Using Lemma 1, we can see that the integrals in (4.4) are convergent absolutely when $\alpha = 0$, so $r_N(u, v; 0)$ can be well-defined.

To prove Lemma 8, integrating by parts repeatedly and using Lemma 1, we have

$$\int_{\mathscr{C}} h^{(N)}(x + \tau y; \alpha)x^{u-1}\, dx = (-1)^N(u-1)\cdots(u-N)\int_{\mathscr{C}} h(x + \tau y; \alpha)x^{u-N-1}\, dx.$$

Substituting this into (4.4), changing the order of integrations and using the fact

$$\int_0^1 h(z; \alpha)\, d\alpha = \left[-\frac{e^{(1-\alpha)z}}{z(e^z - 1)} - \frac{\alpha}{z}\right]_{\alpha=0}^1 = 0,$$

we obtain Lemma 8. To verify the change of the order of integrations we note that

$$\int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!}\int_{\mathscr{C}} \frac{y^{v+N-1}}{e^y - 1}\int_{\mathscr{C}} x^{u-N-1}\int_0^1 h(x + \tau y; \alpha)\, d\alpha\, dx\, dy\, d\tau$$

is convergent absolutely for Re $u < N + 1$, because in view of Lemma 1, we have

$$\int_0^1 |h(x + \tau y; \alpha)|\, d\alpha \ll |x|^{-1}.$$

Now, integrating both sides of (6.1) with respect to α and applying Lemmas 3 and 8, we obtain

$$\begin{aligned}(6.2)\quad \int_0^1 (g(u, v; \alpha) - \alpha^{-u}\zeta_1(v, \alpha))\, d\alpha &= \sum_{n=0}^{N-1} \frac{(-1)^n(v)_n}{n!}\cdot\frac{\zeta(v+n)}{u-n-1} \\ &\quad -(-1)^N(v)_N\int_0^1 \alpha^{N-u}\int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!}\zeta_1(v + N, \alpha\tau)\, d\tau\, d\alpha.\end{aligned}$$

If Re $u < N + 1$ and Re $v > -N$, then by using Lemma 7, we can see that the last double integral is convergent absolutely.

Here we make a little digression. Taking $N = 1$ in (4.3), and applying Lemmas 5 and 8, we find

$$(6.3)\qquad \int_0^1 g(u, v; \alpha)\, d\alpha = 0$$

for *Re* $u < 1$ and any v. This result clarifies why the situation becomes so simple by taking the mean value with respect to α, and such sharp results as Corollaries 2 and 5 can be proved. In fact, in the right hand side of (3.4) or (3.5), the most difficult part to analyze for individual α is $g(u,v;\alpha)+g(v,u;\alpha)$, but (6.3) implies that, after integrating with respect to α, the contribution of this part vanishes in the region $\operatorname{Re} u<1$, $\operatorname{Re} v<1$. It may be interesting to recall that in our original proof [7] of the conjecture (1.2), the relation (6.3) has worked as a key lemma.

Next we mention the connection between our method and a formula of Mikolás. For $\operatorname{Re} u<1$, $\operatorname{Re} v<1$, $\operatorname{Re}(u+v)<1$, the formula

$$\begin{aligned}(6.4)\qquad &\int_0^1 \zeta(u,\alpha)\zeta(v,\alpha)\,d\alpha\\ &=\Gamma(u+v-1)\zeta(u+v-1)\left\{\frac{\Gamma(1-u)}{\Gamma(v)}+\frac{\Gamma(1-v)}{\Gamma(u)}\right\}\\ &=2(2\pi)^{u+v-2}\zeta(2-u-v)\Gamma(1-u)\Gamma(1-v)\cos\left(\frac{\pi}{2}(u-v)\right)\end{aligned}$$

is proved by Mikolás [11, (5.2)]. (The second equality follows by a simple application of the functional equation of $\zeta(s)$.) Here we give two proofs of (6.4) in the frame of our method. In fact, integrating both sides of (3.4) with respect to α from 0 to 1, and using Lemma 5 and (6.3), the relation (6.4) immediately follows. It is also possible to deduce (6.4) as the limit $q\to\infty$ of our discrete mean value result [7, Theorem 2].

7. Completion of the proof of the theorem.

Now we assume $N<\operatorname{Re} u<N+1$ and $N<\operatorname{Re} v<N+1$. Then, by using Lemma 6, we have

$$\begin{aligned}&\sum_{n=0}^{N-1}\frac{(-1)^n(v)_n}{n!}\cdot\frac{\zeta(v+n)}{u-n-1}\\ &=\sum_{n=0}^{N-1}\frac{(-1)^n(v)_n}{n!}\zeta(v+n)\int_1^\infty \alpha^{n-u}\,d\alpha\\ &=\int_1^\infty \alpha^{-u}\left\{\zeta_1(v,\alpha)-(-1)^N(v)_N\alpha^N\int_0^1\frac{(1-\tau)^{N-1}}{(N-1)!}\zeta_1(v+N,\alpha\tau)\,d\tau\right\}d\alpha.\end{aligned}$$

Substituting this result into (6.2), we have

$$\begin{aligned}(7.1)\qquad &\int_0^1 (g(u,v;\alpha)-\alpha^{-u}\zeta_1(v,\alpha))\,d\alpha\\ &=\int_1^\infty \alpha^{-u}\zeta_1(v,\alpha)\,d\alpha-(-1)^N(v)_N\int_0^\infty \alpha^{N-u}\\ &\quad\times\int_0^1\frac{(1-\tau)^{N-1}}{(N-1)!}\zeta_1(v+N,\alpha\tau)\,d\tau\,d\alpha=J_1-J_2,\text{ say,}\end{aligned}$$

and the absolute convergence of the integrals J_1 and J_2 are justified by using Lemma 7.

Putting $\xi = \alpha\tau$, and using a property of the beta-function, we have

$$J_2 = (-1)^N (v)_N \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} \tau^{u-N-1} \int_0^\infty \xi^{N-u} \zeta_1(v+N, \xi) \, d\xi \, d\tau$$

$$= (-1)^N (v)_N \frac{\Gamma(N)\Gamma(u-N)}{(N-1)!\Gamma(u)} \int_0^\infty \xi^{N-u} \zeta_1(v+N, \xi) \, d\xi$$

$$= \frac{(v)_N}{(1-u)_N} \int_0^\infty \xi^{N-u} \zeta_1(v+N, \xi) \, d\xi.$$

Next, in view of Lemmas 4 and 7, integrating by parts N-times we have

$$J_1 = -S_N(v,u) + \frac{(v)_N}{(1-u)_N} \int_1^\infty \alpha^{N-u} \zeta_1(v+N, \alpha) \, d\alpha.$$

Substituting these results into (7.1), we obtain

$$\begin{aligned} &\int_0^1 (g(u,v;\alpha) - \alpha^{-u}\zeta_1(v,\alpha)) \, d\alpha \\ &= -S_N(v,u) - \frac{(v)_N}{(1-u)_N} \int_0^1 \alpha^{N-u} \zeta_1(v+N, \alpha) \, d\alpha \\ &= -S_N(v,u) - \frac{(v)_N}{(1-u)_N} \sum_{l=1}^\infty \int_0^1 \alpha^{N-u} (l+\alpha)^{-v-N} \, d\alpha. \end{aligned} \tag{7.2}$$

To verify this termwise integration, puting $\alpha = l\beta^{-1}$ we observe that

$$\int_0^1 \alpha^{N-u} (l+\alpha)^{-v-N} \, d\alpha = l^{1-u-v} \int_l^\infty \beta^{u+v-2} (1+\beta)^{-v-N} \, d\beta, \tag{7.3}$$

which is estimated by $O(l^{-\mathrm{Re}\, v - N})$. Therefore, the integrals and the infinite series in the right-hand side of (7.2) is convergent absolutely for $\mathrm{Re}\, u < N+1$, $\mathrm{Re}\, v > -N+1$. From (7.2) and (7.3), we obtain

$$\int_0^1 (g(u,v;\alpha) - \alpha^{-u}\zeta_1(v,\alpha)) \, d\alpha = -S_N(v,u) - T_N(v,u). \tag{7.4}$$

Therefore, integrating both sides of (3.5) with respect to α from 0 to 1, and using (5.3) and (7.4), we now arrive at the formula (2.1). Every terms appearing in (2.1) are well-defined in the region $-N+1 < \mathrm{Re}\, u < N+1$, $-N+1 < \mathrm{Re}\, v < N+1$, and the expression (2.2) can be proved by integrating by parts K-times. Thus the proof of the theorem is now completed.

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