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A NOTE ON TOLSTOV'S THEOREM ON HARMONIC FUNCTIONS IN THE PLANE

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Abstract.

Tolstov has proved in [T] that if u is any bounded function which has second order derivatives separately in x and y and satisfies the Laplace equation in a plane region Ω, then it is harmonic in the classical sense, i.e., u is twice continuously differentiable jointly in x and y and satisfies the Laplace equation. In this note I shall show that if u belongs to $L^{1,\mathrm{loc}}$ and satisfies the rest of the hypothesis of Tolstov, then u is harmonic in the classical sense. Also I shall show that this is the best possible result.

1. Introduction.

The aim of this note is to clearly establish the best of the theorems of a certain kind which belong in the realm of regularity of solutions of the Laplace equation in the plane. To this end we need to abuse some well-established notation.

Let C denote the space of functions u defined on a region Ω in the Cartesian x, y plane with the following properties:

a) u is continuous,

b) u is twice differentable in x, y separately,

c) $u_{zz} + u_{yy} = 0$ everywhere in Ω.

Let $L^{p,\mathrm{loc}}$ denote the space of functions u which satisfy b) and c) above and instead of a), satisfy

a) $|u|^p$ is locally summable.

We assume that $0 < p \leqq \infty$. Let H denote the space of real-analytic functions that satisfy the Laplace equation in Ω. Now we can state the classical theorem as proved in [P, pp. 239–241] as follows:

THEOREM 1.1. $H = C$.

In the same book on page 238, Petrovskii presents the example,

$$u = \operatorname{Re} e^{-\frac{1}{z^4}}$$

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to show that the theorem cannot be improved. This particular function is presented by various authors [L, p. 107], [T, p. 559] to name a few, for a similar purpose in each case. For example Tolstov [T] proves,

THEOREM 1.2. $L^{\infty,\mathrm{loc}} = H$.

He cites the same example in order to show that his result is the best possible. Another interesting fact is that Tolstov makes use of the theorem 1.1 to derive theorem 1.2. Betterment is obtained froma strengthening of the maximum modulus principle. In this note we shall establish an even better maximum principle than Tolstov's and assuming the theorem of Tolstov, derive

THEOREM 1.3. $L^{1,\mathrm{loc}} = H$.

This is not unusual as can be seen from the various versions of the Phragmen-Lindelöf theorems, one successively better than and based on the previous one.

The example of

$$u = \frac{xy}{|z|^4}$$

shows that u satisfies b) and c) everywhere in the plane and belongs to every $L^{p,\mathrm{loc}}$ for $0 < p < 1$ and does not belong to H. This proves that our theorem cannot be improved.

2. The proof of Theorem 1.3.

We need a lemma before the actual proof.

LEMMA PL. *Suppose u is sub-harmonic in the square*

$$D = \{(x, y): 0 < x < R, 0 < y < R\}$$

and

$$\limsup_{q \to q_0} u \leqq M \quad \text{for every} \quad q_0 \in \partial D \backslash \{O\}$$

where $q \in D$, O is the origin, and further

$$\limsup_{r \to 0} r^2 M(r) \leqq 0$$

where $M(r) = \sup u$ on the lines $x = r$ and $y = r$ in D. Then $u \leqq M$ on all of D.

This is a non-traditional version of the Phragmen-Lindeloff theorem. Normally ∞ is the exceptional point and in our case it is the origin. For the case of holomorphic functions, one can find the proof in Boas [B, pp. 242–245] for example. Our lemma is easily established by imitating the proof.

Now for the proof of theorem 1.3. Since u_x, u_y exist everywhere in Ω,

$$\Omega = \cup_1^\infty E_n$$

where E_n is the set of points in Ω for which

$$|u(z+h) - u(z)| \leqq n|h| \quad \text{for} \quad |h| \leqq 1/n.$$

Here we let $z = x + iy$ denote (x, y) and h be either real or purely imaginary, n any positive integer. One can check that each E_n is relatively closed in Ω. See [T] for example. Let F be the set of all points z in Ω such that u is not bounded in any neighbourhood of z. Obviously F is a closed set and also any interior of any E_n does not belong to F. By the Baire category theorem we see that the interior of F is empty. Next we show that F itself is empty. Since F is a locally compact metric space and is a countable union of sets $F \cap E_n$ closed in F, by the Baire category theorem there exists an open square S contained in Ω such that for some positive integer n,

$$S \cap F \subset S \cap E_n \tag{2.1}$$

and $S \cap F$ is non-empty. Choose a point q in $S \cap F$ and a square Q of side length $\delta < 1/n$ entirely contained in S and centered at q. We say that a point ζ is "seen" by a point z in the plane if $z - \zeta$ is real or purely imaginary. Thus if a point in ζ in Q is seen by a point z_0 in $Q \cap F$ then

$$|u(\zeta) - u(z_0)| \leqq n\delta < 1$$

since z_0 belongs to E_n. Since such a z_0 and q both see a point λ, we obtain

$$|u(\zeta) - u(q)| \leqq |u(\zeta) - u(z_0)| + |u(z_0) - u(\lambda)| + |u(\lambda) - u(q)| < 3. \tag{2.2}$$

Hence on all vertical and all horizontal segments of Q that meet F, the estimate (2.2) is valid.

Let $Q_1 \subset Q$ be a closed square centered at q and let G be the union of all vertical and/or horizontal segments running from one end of Q to the other meeting $Q \cap F$. By (2.2), $|u| \leqq |u(q)| + 3$ on G and by the continuity of u on horizontal and vertical lines, u is bounded on ∂Q_1. So there exists an m such that $|u| \leqq m$ on $\gamma = G \cup \partial Q_1$. Since F is closed, G and so γ are closed. If $Q_1 \subset \gamma$, then $|u|$ is bounded by m on Q_1 and hence $q \notin F$ — a contradiction.

Suppose that $Q_1 \backslash \gamma$ is non-empty. Because γ is made up of of vertical and/or horizontal line segments, Q_1/γ is a disjoint union of rectangles. Let R be one such rectangle, and let l_1, l_2, l_3, l_4 denote the segments from γ bounding R on the north, south, west, east respectively. Now the region Λ which is the union of the vertical strip bounded by l_3, l_4 and the horizontal strip bounded by l_1, l_2 is free of points of F and so by theorem 1.2,

$$u \quad \text{is harmonic in} \quad \Lambda. \tag{2.3}$$

Further on each of these lines $|u| \leqq m$. Also since at each of the corners c of R, the disc with center at a point t in R close to c with radius $|t - c|$ lies entirely in Λ and since u belongs to $L^{1,\mathrm{loc}}$, it is easy to see that

$$\pi |t - c|^2 |u(t)| \leqq \int_{|z-t|<|t-c|} |u(z)|\, dx\, dy; \tag{2.4}$$

$$|u(t)| = o(|t - c|^{-2}) \quad \text{as} \quad |t - c| \to 0.$$

We now apply lemma PL to R at each of the corners and obtain that

$$|u| \leqq m \tag{2.5}$$

on R and hence $|u| \leqq m$ on all of Q_1 which means $q \notin F$ – a contradiction. This proves our theorem 1.3.

3. Conclusion.

The theorem of this paper is closely related to the Looman-Menchoff [R] and so does its proof. Also for more information and bibliography, one must refer to [S] and the references there. The open problem that remains interesting in this area is what kind of sets F are sets of singularities for functions u that satisfy b) and c). From our proof here it is obvious that F must be closed and totally disconnected. In fact one can assert that there exists a relatively open and dense subset U of F such that locally any projection of U on to the axes is totally disconnected. One can ask if u belongs to L^p for some $0 < p < 1$, is the set F of measure zero? Would it depend on the p? The same questions remain open in the case of the Looman-Menchoff theorem also.

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