MATH. SCAND. 78 (1996), 213–232

L^1 NORM OF LÉVY PROCESSES WITH EXPONENTIAL TAILS*

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Abstract.

For a Lévy process $\{X(t), t \geqq 0\}$ for which both left and right tails of the Lévy measure are in the exponential class $S(\alpha)$, and so is their sum, we compute the tail distribution of the sample path L^1 norm on an interval of finite length.

1. Introduction.

Throughout this paper $\mathbf{X} = \{X(t), 0 \leqq t \leqq 1\}$, $X(0) = 0$ a.s., is a process with stationary independent increments (*Lévy process*). Its characteristic function can then be written in the form

$$E\exp(i\theta X(t)) = \exp(t\psi(\theta)), \tag{1.1}$$

where

$$\psi(\theta) = ib\theta - \sigma^2\theta^2/2 + \int_{-\infty}^{\infty} (e^{i\theta x} - 1 - i\theta x \mathbf{1}(|x| \leqq 1))\rho(dx) \tag{1.2}$$

with $b \in R$, $\sigma \geqq 0$ and ρ a Borel measure such that $\int_{-\infty}^{\infty}(1 \wedge x^2)\rho(dx) < \infty$ (the Lévy measure of $\mathbf{X}$).

It is well known that a Lévy process has a measurable version, and in the sequel we will without any further notice take a measurable version of X and any other process with stationary independent increments. Studying the distributional properties of the integrals of the absolute values of Lévy processes is not an easy task. In the case of Browian motion this can be done using Kac's formula (which can be even made to work for the integral of the Brownian bridge), as demon-

* Samorodnitsky's research was supported by the NSF grant DMS-94-00535 and United States – Israel Binational Science Foundation (BSF) Grant 92-00074/1. Part of Samorodnitsky's research was done at Chalmers University, Göteborg, under support of Swedish Natural Science Research Council grant F-GF 09207-305, while the other part was done at ETH Zürich, under support of Forschungsinstitut für Mathematik and Institut für Operations Research.

Received October 14, 1994.

strated by Shepp [She82]. However, this approach does not seeem to be convenient to use in the case of more general Lévy processes, for the resulting equations become too complicated. Therefore, other approaches are called for. For Lévy processes with *subexponential* tails of the Lévy measure, such integrals can be treated using the general results of Rosinski and Samorodnitsky [RS93]. The corresponding case of *exponential* tails of Lévy measure could not be precisely described by the existing results, with only partial information available (see Braverman and Samorodnitsky [BS95]), and it is our goal in this paper to find the eact asymptotic distribution of the sample path integral $I(T) = \int_0^T |X(t)|\,dt$ under the assumption that the the tails of both positive and negative parts of the Lévy measure ρ and their sum belong in the appropriate sense to the exponential class $\mathscr{S}(\alpha)$.

We remind the reader that a distribution F on $[0, \infty)$ belongs to the exponential class $\mathscr{S}(\alpha)$, $\alpha > 0$ if

$$l := \lim_{\lambda \to \infty} \frac{\overline{F * F}(\lambda)}{\bar{F}(\lambda)} \text{ exists and is finite} \tag{1.3}$$

and $F \in \mathscr{L}(\alpha)$, where

$$\mathscr{L}(\alpha) = \left\{ F: \lim_{u \to \infty} \frac{\bar{F}(u+v)}{\bar{F}(u)} = e^{-\alpha v}, \text{ any } v > 0 \right\}. \tag{1.4}$$

Occasionally we will abuse the terminology a bit and apply the expression $\mu \in \mathscr{L}(\alpha)$ to finite (not necessarily probability) measures on $[0, \infty)$.

We will use several well-known facts about distributions with exponential tails, which are collected for convenience below.

First of all, in the remainder of this paper $\mathscr{S}(\alpha)$ refers to the collection of distributions on *the whole of* R which are in $\mathscr{L}(\alpha)$ and for which (1.3) holds. The extensions of the quoted results to this more general case are entirely straightforward. See Willekens [Wil86], and also Bertoin and Doney [BD93].

LEMMA 1.1. *Let* $F \in \mathscr{S}(\alpha)$, $\alpha > 0$. *Then*

(i) (Chover, Ney and Wainger [CNW73], Cline [Cli87]) $m_F(\alpha) = \int_{-\infty}^{\infty} e^{\alpha x} F(dx) < \infty$ *and* $l = 2m_F(\alpha)$ *in* (1.3).

(ii) (Embrechts and Goldie [EG82], Cline [Cli87]) *If the limit* $c_i = \lim_{\lambda \to \infty} \frac{\bar{G}_i(\lambda)}{\bar{F}(\lambda)}$ *exists and is finite for two distribution functions* G_1, G_2 *then*

$$\lim_{\lambda \to \infty} \frac{\overline{G_1 * G_2}(\lambda)}{\bar{F}(\lambda)} = c_1 m_{G_2}(\alpha) + c_2 m_{G_1}(\alpha).$$

Moreover, $G_i \in \mathscr{S}(\alpha)$ if $c_i > 0$.

(iii) (Chover, Ney and Wainger [CNW73], Embrechts and Goldie [EG82]) *For every $n \geqq 1$, $\lim_{\lambda\to\infty} \dfrac{\overline{F^{*n}}(\lambda)}{\bar{F}(\lambda)} = nm_F(\alpha)^{n-1}$. Furthermore, there is a $K < \infty$ such that for every $n \geqq 1$ and $\lambda > 0$*

$$\overline{F^{*n}}(\lambda)/\bar{F}(\lambda) \leqq K(1 + m_F(\alpha))^{n-1}.$$

(iv) (Embrechts and Goldie [EG82]) *For a $\mu > 0$ let $G(x) = e^{-\mu}\sum_{n=0}^{\infty} \dfrac{\mu^n}{n!} F^{*n}(x)$. Then $\lim_{\lambda\to\infty} \dfrac{\bar{G}(\lambda)}{\bar{F}(\lambda)} = \mu m_G(\alpha)$. More generally, if G is an infinitely divisible distribution such that the right tail $\bar{\rho}$ of the corresponding Lévy measure ρ is equivalent to the tail of a distribution in $S(\alpha)$, then $\lim_{\lambda\to\infty} \dfrac{\bar{G}(\lambda)}{\bar{\rho}(\lambda)} = m_G(\alpha)$.*

(v) (Cline [Cli86]) *Let $G \in \mathscr{L}(\alpha)$, and $\sup_{\lambda>0} \bar{G}(\lambda)/\bar{F}(\lambda) < \infty$. Then $H = F * G$ is in $S(\alpha)$ and $\bar{H}(\lambda) \sim m_G(\alpha)\bar{F}(\lambda) + m_F(\alpha)\,\bar{G}(\lambda)$ as $\lambda \to \infty$.*

A function $h: R \to R_+$ is said to be equivalent to the tail of a distribution in $S(\alpha)$ if there is an $F \in S(\alpha)$ such that

$$\lim_{\lambda\to\infty} \frac{h(\lambda)}{\bar{F}(\lambda)} = 1.$$

In the case when h is right continuous, non-increasing and converges to zero at infinity, it follows from Lemma 1.1 (ii) that h is equivalent to the tail of a distribution in $S(\alpha)$ if and only if the distribution G on $[0, \infty)$ defined by $G(x) = 1 - \min(1, h(x))$ is in $\mathscr{S}(\alpha)$.

In the following section we collect and prove auxiliary results needed for the proof of the main result of the paper, which stated and proved in Section 3.

2. Preliminary results and estimates.

Let ρ be a finite measure on $[0, \infty)$. We use the usual notation for the tail of ρ, $\bar{\rho}(x) = \rho(x, \infty)$, $x > 0$, and we introduce further the integrated tail of ρ by

$$\tilde{\rho}(x) = \int_0^1 \bar{\rho}\left(\frac{x}{t}\right) dt. \tag{2.1}$$

More generally, given a probability measure H on $[0, 1]$ we set

$$\tilde{\rho}_H(x) = \int_0^1 \bar{\rho}\left(\frac{x}{t}\right) H(dt). \tag{2.2}$$

LEMMA 2.1. *Let $\rho \in \mathscr{L}(\alpha)$, $\alpha > 0$. For $0 < \delta < 1$*

$$\lim_{x\to\infty}\frac{\int_{1-\delta}^{1}\bar{\rho}\left(\frac{x}{t}\right)dt}{\tilde{\rho}(x)}=1. \tag{2.3}$$

PROOF. For any $0<\varepsilon<\delta$

$$\tilde{\rho}(x)\geqq\int_{1-\varepsilon}^{1}\bar{\rho}\left(\frac{x}{t}\right)dt\geqq\varepsilon\bar{\rho}\left(\frac{x}{1-\varepsilon}\right).$$

One the other hand,

$$\int_{0}^{1-\delta}\bar{\rho}\left(\frac{x}{t}\right)dt\leqq(1-\delta)\bar{\rho}\left(\frac{x}{1-\delta}\right).$$

Since $\rho\in\mathscr{S}(\alpha)$ implies

$$\lim_{x\to\infty}\frac{\bar{\rho}(ax)}{\bar{\rho}(x)}=0$$

for every $a>1$, (2.3) follows.

Let X be a random variable with a distribution in $\mathscr{S}(\alpha)$, and U be a *bounded* non-negative random variable, independent of X and not identically equal to zero. It has been proven by Cline and Samorodnitsky [CS91] that the distribution of $Y=XU$ belongs to the class $\mathscr{S}(\alpha)$ if $\alpha=0$. The following proposition is an extension of this statement to the case $\alpha>0$.

PROPOSITION 2.1. *Let $\rho\in\mathscr{S}(\alpha)$, $\alpha>0$. If $H((0,1])>0$ then $\tilde{\rho}_H$ is equivalent to the tail of a distribution in the class $\mathscr{S}(\alpha)$.*

PROOF. We may assume without loss of generality that the $H((\theta,1])>0$ for every $\theta\in(0,1)$. The case $H(\{1\})>0$ being quite simple, we will consider the more interesting case $H(\{1\})=0$. It is obviously enough to prove our statement in the case $H(\{0\})=0$. We may assume further, without loss of generality, that ρ is a probability measure. Let F be the cumulative distribution function (c.d.f.) of ρ. Then $\tilde{\rho}_H$ is the tail of a probability measure as well, and we denote its c.d.f. by G. Since it is easy to see that $G\in\mathscr{L}(\alpha)$, we only have to show that

$$\lim_{x\to\infty}\frac{\overline{G*G}(x)}{\bar{G}(x)}=2m_G(\alpha)<\infty.$$

An obvious application of Fatou's lemma shows that

$$\varliminf_{x\to\infty}\frac{\overline{G*G}(x)}{\bar{G}(x)}\geqq 2m_G(\alpha)$$

for every $G\in\mathscr{L}(\alpha)$, and therefore our remaining task is to prove that

$$\overline{\lim_{x\to\infty}} \frac{\overline{G*G}(x)}{\bar{G}(x)} \leqq 2m_G(\alpha) < \infty. \tag{2.4}$$

It is well known (see e.g. Cline [Cli86]) that for every $F \in \mathscr{L}(\alpha)$

$$\bar{F}(x) = k(x)\exp\left(-\int_0^x \alpha(v)dv\right), \tag{2.5}$$

where $k(x) \to k > 0$ and $\alpha(x) \geqq 0$, $\alpha(x) \to \alpha$ as $x \to \infty$.

Keeping the same $\alpha(\cdot)$ in (2.5) but replacing $k(\cdot)$ with $k_1(\cdot) \equiv k$, we obtain the tail of yet another distribution in $\mathscr{L}(\alpha)$, say F_1. Let G_1 be the c.d.f. of $1 - \tilde{\rho}_H$ corresponding to that case. Observe that

$$\lim_{x\to\infty} \frac{\bar{F}(x)}{\bar{F}_1(x)} = \lim_{x\to\infty} \frac{\bar{G}(x)}{\bar{G}_1(x)} = 1.$$

If $F_1 \in \mathscr{S}(\alpha)$ implies $G_1 \in \mathscr{S}(\alpha)$, then Lemma 1.1 (ii) shows that $F \in \mathscr{S}(\alpha)$ implies $G \in \mathscr{S}(\alpha)$. Therefore, it is enough to prove the proposition under the assumption $k(x) \equiv k$ in (2.5).

Observe that in the latter case F is absolutely continuous and

$$\lim_{x\to\infty} \frac{F'(x)}{\bar{F}(x)} = \alpha. \tag{2.6}$$

For a $b > 0$ denote

$$F_b(x) = F\left(\frac{x}{b}\right).$$

Then

$$\overline{G*G}(x) = \int_0^1 \int_0^1 \overline{F_t * F_s}(x)H(dt)H(ds) \tag{2.7}$$

and

$$\bar{G}(x) = \int_0^1 \bar{F}_t(x)H(dt). \tag{2.8}$$

It easily follows from the relation $\bar{F}(ax) = o(\bar{F}(x)), a > 1$, that for every $0 < b < 1$

$$\int_0^b \int_0^b \overline{F_t * F_s}(x)H(dt)H(ds) = o(\overline{G*G}(x)). \tag{2.9}$$

Taking into account the symmetric role of t and s, we get

$$\overline{G * G}(x) \sim 2\int_b^1 \int_0^t \left(\int_0^\infty \bar{F}\left(\frac{x - sy}{t}\right) F(dy) \right) H(ds)H(dt). \tag{2.10}$$

Let $a > 0$. For $x > a$ divide the region of integration over y into two parts: $[0, s^{-1}a]$ and $(s^{-1}a, \infty)$ and denote the corresponding integrals by I_1 and I_2. In the sequel C will be a generic finite positive constant that may change between appearances.

Calculation for I_1. Fix $0 < \varepsilon < 1$. We have $sy/t < a/b$ since $y < s^{-1}a$ and $b < t < 1$. The assumption $F \in \mathscr{S}(\alpha)$ implies that there is $x_1 = x_1(a, b, \varepsilon)$ such that for $z \in [0, a/b]$ and $x > x_1$

$$\frac{\bar{F}(x - z)}{\bar{F}(x)} \leqq (1 + \varepsilon)e^{\alpha z}, \tag{2.11}$$

see e.g. Cline [Cli86]. On the other hand,

$$\alpha s y t^{-1} - \alpha s y = \alpha \frac{sy}{t}(1 - t) \leqq \alpha a(b^{-1} - 1). \tag{2.12}$$

Hence, for all $x > x_1$

$$\begin{aligned} I_1 &\leqq 2(1 + \varepsilon)\int_b^1 H(dt) \int_0^t \left(\int_0^{s^{-1}a} \bar{F}\left(\frac{x}{t}\right) e^{\alpha s y/t} F(dy) \right) H(ds) \\ &\leqq 2(1 + \varepsilon)e^{\alpha a(b^{-1} - 1)} \int_0^1 \bar{F}\left(\frac{x}{t}\right) H(dt) \int_0^1 \int_0^{s^{-1}a} e^{\alpha s y} F(dy)H(ds) \\ &\leqq 2(1 + \varepsilon)e^{\alpha a(b^{-1} - 1)} m_G(\alpha)\bar{G}(x). \end{aligned} \tag{2.13}$$

Calculation for I_2. Now divide the segment $[0, t]$ into three parts: $[0, \varepsilon t]$, $(\varepsilon t, (1 - \varepsilon)t]$ and $((1 - \varepsilon)t, t]$ and denote the corresponding integrals by I_{21}, I_{22} and I_{23}. Since $F \in \mathscr{S}(\alpha)$, then $\overline{F * F}(x) \leqq C\bar{F}(x)$. Therefore,

$$\begin{aligned} I_{21} &\leqq \int_0^1 \int_0^{\varepsilon t} \overline{F * F}\left(\frac{x}{t}\right) H(ds)H(dt) \\ &\leqq C \int_0^1 \int_0^\varepsilon \bar{F}\left(\frac{x}{t}\right) H(ds)H(dt) = CH((0, \varepsilon))\bar{G}(x). \end{aligned} \tag{2.14}$$

Reasoning similarly we obtain

$$I_{23} \leqq CH(b(1 - \varepsilon), 1))\bar{G}(x). \tag{2.15}$$

Now turn to the integral I_{22}. We have

$$\int_{s^{-1}a}^\infty \bar{F}\left(\frac{x - sy}{t}\right) F(dy) = \int_a^\infty \bar{F}\left(\frac{x - y}{t}\right) F'(y/s)dy/s. \tag{2.16}$$

It follows from (2.6) that for $y > a$ and $0 < s \leqq (1-\varepsilon)t$,

$$\frac{F'\left(\frac{y}{s}\right)}{F'\left(\frac{y}{t}\right)} \leqq C\frac{\bar{F}\left(\frac{y}{s}\right)}{\bar{F}\left(\frac{y}{t}\right)} \leqq C\sup_{z>a}\frac{\bar{F}\left(\frac{z}{1-\varepsilon}\right)}{\bar{F}(z)}.$$

Therefore, by (2.16), for every $\varepsilon t < s \leqq (1-\varepsilon)t$,

$$\int_{s^{-1}a}^{\infty} \bar{F}\left(\frac{x-sy}{t}\right)F(dy) \leqq C\varepsilon^{-1}\sup_{z>a}\frac{\bar{F}\left(\frac{z}{1-\varepsilon}\right)}{\bar{F}(z)}\int_a^{\infty}\bar{F}\left(\frac{x-y}{t}\right)F'(y/t)dy/t$$

$$\leqq C\varepsilon^{-1}\sup_{z>a}\frac{\bar{F}\left(\frac{z}{1-\varepsilon}\right)}{\bar{F}(z)}\overline{F_t * F_t}(x) \leqq C\varepsilon^{-1}\sup_{z>a}\frac{\bar{F}\left(\frac{z}{1-\varepsilon}\right)}{\bar{F}(z)}\bar{F}\left(\frac{x}{t}\right),$$

using once again the fact that $F \in \mathscr{S}(\alpha)$. We conclude finally that

$$I_{22}(x) \leqq Ce^{-1}\sup_{z>a}\frac{\bar{F}\left(\frac{z}{1-\varepsilon}\right)}{\bar{F}(z)}\bar{G}(x). \tag{2.17}$$

It follows now from (2.13), (2.14), (2.15) and (2.17) that

$$\varlimsup_{x\to\infty}\frac{\overline{G*G}(x)}{\bar{G}(x)} \leqq C\left(H(0,\varepsilon)) + H((b(1-\varepsilon),1))\right.$$

$$\left. + \varepsilon^{-1}\sup_{z>a}\frac{\bar{F}\left(\frac{z}{1-\varepsilon}\right)}{\bar{F}(z)}\right) + 2(1+\varepsilon)e^{\alpha a(b^{-1}-1)}m_G(\alpha). \tag{2.18}$$

Letting first $b \to 1$, then $a \to \infty$ and then $\varepsilon \to 0$ establishes the only important part of (2.4). The finiteness of $m_G(\alpha)$ follows from that of $m_F(\alpha)$. This completes the proof.

The next assertion is an easy consequence of Lemma 2.1.

LEMMA 2.2. *Suppose H has a density h on an interval $(\varepsilon, 1)$ such that the limit*

$$\lim_{x\to 1-0} h(x) = h(1)$$

exists. Then

$$\lim_{x\to\infty}\frac{\tilde{\rho}_H(x)}{\tilde{\rho}(x)} = h(1).$$

We conclude this section with some elementary estimates involving Poisson arrivals. Let $\{Z_k, k \geqq 1\}$ be i.i.d. random variables with exponential distribution with mean $1/\mu$.

LEMMA 2.3. *Let m and k be positive integers and*

$$U = \sum_{j=1}^{k} Z_j \mathbf{1}_{\{\Sigma_{j=1}^{k+m} Z_j \leqq 1\}}.$$

Then the continuous part of U has the density given by the formula

$$h(x) = \mu^k e^{-\mu}\left(e^{\mu(1-x)} - \sum_{j=0}^{m-1} \frac{\mu^j(1-x)^j}{j!}\right)\frac{x^{k-1}}{(k-1)!}, \tag{2.19}$$

$0 < x < 1$. *If $m = 0$, then*

$$h(x) = \mu^k e^{-\mu x}\frac{x^{k-1}}{(k-1)!} \tag{2.20}$$

for $0 < x < 1$.

PROOF. Let $\Gamma_n = Z_1 + \ldots + Z_n$, $n \geqq 1$ and $N(t) = \max\{n: \Gamma_n \leqq t\}$, $t > 0$. Using the notation P^* for the continuous part of the distribution of U we have by the standard properties of Poisson processes,

$$P^*(U \leqq x) = P(\Gamma_k \leqq x, \Gamma_{k+m} \leqq 1)$$

$$= \int_0^x \mu^k e^{-\mu t}\frac{t^{k-1}}{(k-1)!} P(N(1-t)) \geqq m)dt$$

for $0 < x < 1$, and so

$$h(x) = \mu^k e^{-\mu x}\frac{x^{k-1}}{(k-1)!} P(N(1-x)) \geqq m)$$

$$= \mu^k e^{-\mu x}\frac{x^{k-1}}{(k-1)!} e^{-\mu(1-x)} \sum_{j=m}^{\infty} (\mu(1-x))^j/j!,$$

as required.

The following lemma is an immediate corollary of Lemma 2.3.

LEMMA 2.4. *Let*

$$W = \left(1 - \sum_{j=1}^{k} Z_j\right)\mathbf{1}_{\{\Sigma_{j=1}^{k+m} Z_j \leqq 1\}}.$$

Then the continuous part of W has the density given by the formula

$$g(x) = \mu^k e^{-\mu}\left(e^{-\mu x} - \sum_{j=0}^{m-1} \frac{x^j}{j!}\right)\frac{(1-x)^{k-1}}{(k-1)!} \tag{2.21}$$

for $0 < x < 1$ *if* $m > 0$ *and*

$$g(x) = \mu^k e^{-\mu(1-x)}\frac{(1-x)^{k-1}}{(k-1)!} \tag{2.22}$$

for $0 < x < 1$ *if* $m = 0$.

REMARK. It follows from Lemma 2.3 that for $0 < x < 1$

$$h(x) \leqq \frac{\mu^{k+m}}{(m-1)!(k-1)!} \tag{2.23}$$

where we put $(-1)! = 1$. Lemma 2.4 gives us the same bound for $g(x)$.

3. Tail distribution of the L^1 norm.

Let $\mathbf{X}$ be a Lèvy process as defined by (1.1) and (1.2). For an $x > 0$ we define $\bar{\rho}_+(x) = \rho(x, \infty)$ and $\bar{\rho}_-(x) = \rho(-\infty, -x)$. Let

$$\tilde{\rho}_-(x) = \int_0^1 \bar{\rho}_-\left(\frac{x}{t}\right)dt, \quad \tilde{\rho}_+(x) = \int_0^1 \bar{\rho}_+\left(\frac{x}{t}\right)dt. \tag{3.1}$$

The following is the main theorem of this paper. It gives the exact tail behavior for the probability

$$P(I(1) > \lambda) = P\left(\int_0^1 |X(t)|\,dt > \lambda\right)$$

when the left the right tails $\bar{\rho}_-$ and $\bar{\rho}_+$ and their sum are equivalent to the tails of distributions in $\mathscr{S}(\alpha)$. We remark that a weaker result,

$$0 < \varliminf_{x\to\infty} \frac{P\left(\int_0^1 |X(t)|\,dt > x\right)}{\tilde{\rho}_+(x) + \tilde{\rho}_-(x)} \leqq \varlimsup_{x\to\infty} \frac{P\left(\int_0^1 |X(t)|\,dt > x\right)}{\tilde{\rho}_+(x) + \tilde{\rho}_-(x)} < \infty$$

follows from the general theory of subadditive functionals of infinitely divisible processes with exponential tais developed in Braverman and Samorodnitsky [BS95]. The result of the present paper gives the exact weights one needs to put on the positive and negative parts of the integrated Lévy measure, $\tilde{\rho}_+(x)$ and $\tilde{\rho}_-(x)$ to make the limit exist, and equal to 1.

THEOREM 3.1. *Suppose that* $\bar{\rho}_+$, $\bar{\rho}_-$ *and* $\bar{\rho}_+ + \bar{\rho}_-$ *are equivalent to the tails of distributions in* $\mathscr{S}(\alpha)$. *Then*

$$\lim_{x\to\infty} \frac{P\left(\int_0^1 |X(t)|\,dt > x\right)}{A_+(\rho)\tilde{\rho}_+(x) + A_-(\rho)\tilde{\rho}_-(x)} = 1, \tag{3.2}$$

where

$$A_+(\rho) = Ee^{\alpha I_0} := Ee^{\alpha \int_0^1 X(t)dt}$$

$$= \exp\left(\alpha b/2 + \alpha^2\sigma^2/6 + \int_{-\infty}^{\infty} \frac{e^{\alpha x} - 1 - \alpha x - \frac{1}{2}\alpha^2 x^2 \mathbf{1}(|x| \leqq 1)}{\alpha x}\rho(dx)\right),$$

$$A_-(\rho) = Ee^{-\alpha I_0} = Ee^{-\alpha \int_0^1 X(t)dt}$$

$$= \exp\left(-\alpha b/2 + \alpha^2\sigma^2/6 - \int_{-\infty}^{\infty} \frac{e^{-\alpha x} - 1 + \alpha x - \frac{1}{2}\alpha^2 x^2 \mathbf{1}(|x| \leqq 1)}{\alpha x}\rho(dx)\right),$$

and where $\tilde{\rho}_+$ *and* $\tilde{\rho}_-$ *are defined by* (3.1).

Theorem 3.1 is closely connected with the next theorem. In fact, the two are reformulations of each other. We state the two theorems separately both because the latter one exhibits a rather unexpected property of Lévy processes with exponential tails and because our strategy in the proof will be to switch from one formulation to another at the appropriate moments.

THEOREM 3.2. *Under assumptions of Theorem* 3.1,

$$\lim_{x\to\infty} \frac{P\left(\int_0^1 |X(t)|\,dt > x\right)}{P\left(\left|\int_0^1 X(t)dt\right| > x\right)} = 1. \tag{3.3}$$

Theorems 3.1 and 3.2 are equivalent to each other because the random variable $I_0 = \int_0^1 X(t)dt$ is clearly infinitely divisible with characteristic function

$$E\exp(i\theta I_0) = \exp\left(i\theta b/2 - \theta^2\sigma^2/6\right. \tag{3.4}$$

$$\left. + \int_{-\infty}^{\infty} \frac{e^{i\theta x} - 1 - i\theta x + \frac{1}{2}\theta^2 x^2 \mathbf{1}(|x| \leqq 1)}{i\theta x}\rho(dx)\right).$$

In particular, the positive and the negative parts of the Lévy measure of I_0 are given by $\tilde{\rho}_-$ and $\tilde{\rho}_+$ accordingly. Under the assumptions of the two theorems above we know by Proposition 2.1 that both $\tilde{\rho}_-$ and $\tilde{\rho}_+$ are equivalent to the tails of distributions in $\mathscr{S}(\alpha)$. By Lemma 1.1 we obtain

$$(3.5)\quad P\left(\left|\int_0^1 X(t)dt\right| > x\right) = P\left(\int_0^1 X(t)dt > x\right) + P\left(\int_0^1 X(t)dt < -x\right)$$
$$\sim A_+(\rho)\tilde{\rho}_+(x) + A_-(\rho)\tilde{\rho}_-(x)$$

as $x \to \infty$.

It is interesting to find out how wide is the class of Lévy processes for which (3.3) holds. We do not know the answer yet.

Our argument will go as follows. We will first prove Theorem 3.2 for the particular case when the Lévy process **X** is a compound Poisson process. Together with (3.5) this will establish the statement of Theorem 3.1 for that particular case. We will then extend the latter to the general case, from which the general case of Theorem 3.2 will follow after appealing once again to (3.5).

PROOF OF THEOREM 3.2 FOR COMPOUND POISSON CASE. Here

$$E\exp(i\theta X(t)) = \exp\left(t\int_{-\infty}^{\infty}(e^{i\theta x} - 1)\rho(dx)\right),$$

where ρ is now a finite measure. We write

$$(3.6)\quad X(t) = \sum_{k=1}^{N(t)} Y_k, \quad t \geqq 0,$$

where $\{N(t), t \geqq 0\}$ is a Poisson process with the rate $\mu = \rho(R)$ independent of a sequence of i.i.d. random variables $\{Y_k, k \geqq 1\}$ with the distribution $\mu^{-1}\rho$. Denote $N = N(1)$ and consider the events

$$B_n = \{\omega: N = n\}, \quad n = 0, 1, \ldots.$$

Let $\{\Gamma_k, k \geqq 1\}$ be the arrival times of $N(\cdot)$. Put

$$Z_0 = \Gamma_1, Z_k = \Gamma_{k+1} - \Gamma_k, \quad k \geqq 1.$$

Then $\{Z_k, k \geqq 0\}$ are independent exponential random variables with mean $1/\mu$. If $\omega \in B_n$, then

$$(3.7)\quad \int_0^1 |X(t)|\,dt = |Y_1|Z_1 + |Y_1 + Y_2|Z_2 + \ldots + |Y_1 + \ldots + Y_{n-1}|Z_{n-1}$$
$$+ |Y_1 + \ldots + Y_n|(1 - Z_0 - Z_1 - \ldots - Z_{n-1}) \equiv S_n.$$

Define for $k \geqq 2$

$$A_k = \{\omega \in B_n: \operatorname{sign}(Y_1) = \ldots = \operatorname{sign}(Y_1 + \ldots + Y_{k-1}) = -\operatorname{sign}(Y_1 + \ldots + Y_k)\}.$$

The events $A_2, A_3, \ldots$ are obviously disjoint. Let

$$D_n = B_n \setminus \left(\bigcup_{k=2}^{n} A_k \right).$$

Observe that for every $\omega \in D_n$

$$S_n = \int_0^1 |X(t)|\, dt = \left| \int_0^1 X(t) dt \right|. \tag{3.8}$$

Denote

$$A_n^k(x) = \left\{ \omega \in A_k : \int_0^1 |X(t)|\, dt > x \right\},$$

$k = 2, \ldots, n$. The next assertion is crucial:

$$\lim_{x \to \infty} \frac{\sum_{n=2}^{\infty} \sum_{k=2}^{n} P(A_n^k(x))}{\tilde{\rho}_-(x) + \tilde{\rho}_+(x)} = 0. \tag{3.9}$$

We now prove (3.9). Denote $\varepsilon_k = \operatorname{sign}(Y_1 + \ldots + Y_k)$, $k \geqq 1$ and

$$U_n = (1 - Z_0 - Z_1 - \ldots - Z_{n-1}).$$

Fix an $\omega \in \bigcup_{k=2}^{n} A_k$. We get from (3.7)

$$\begin{aligned} S_n &= Y_1 \left(\varepsilon_n U_n + \sum_{j=1}^{n-1} \varepsilon_j Z_j \right) + Y_2 \left(\varepsilon_n U_n + \sum_{j=2}^{n-1} \varepsilon_j Z_j \right) + \ldots \\ &\quad + Y_{n-1} \left(\varepsilon_n U_n + \varepsilon_{n-1} Z_{n-1} \right) + Y_n \varepsilon_n U_n \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\leqq |Y_1| \left[\max_{i \geqq 2} (1 - Z_0 - Z_i) \vee (Z_1 + \ldots Z_{n-1}) \right] + Z_2 |Y_2| + Z_3 |Y_2 + Y_3| + \ldots \\ &\quad + Z_{n-1} |Y_2 + \ldots + Y_{n-1}| + U_n |Y_2 + \ldots + Y_n| \\ &\leqq |Y_1| \left[\max_{i \geqq 2} (1 - Z_0 - Z_i) \vee (Z_1 + \ldots Z_{n-1}) \right] \\ &\quad + (Z_2 + Z_3 + \ldots + Z_{n-1} + U_n) \max_{j \leqq n} |Y_2 + \ldots + Y_j| \\ &\leqq \left(|Y_1| + \max_{j \leqq n} |Y_2 + \ldots + Y_j| \right) \left[\max_{1 \leqq i \leqq n-1} (1 - Z_0 - Z_i) \vee (Z_1 + \ldots Z_{n-1}) \right]. \end{aligned}$$

Observe that for any $x > 0$,

$$P\left(|Y_1| + \max_{j \leqq n} |Y_2 + \ldots + Y_j| > x \right) \tag{3.11}$$

$$P(|Y_1| + \ldots |Y_n| > x)$$

It follows from Lemma 1.1 (iii) that there is a finite positive constant K such that for all $x > 0$ and $n \geqq 1$,

$$(3.12)\quad P\left(\sum_{k=1}^{n} |Y_k| > x\right) \leqq K\mu^{-1}(1 + m_{\mu^{-1}\rho}(\alpha) + m_{\mu^{-1}\rho}(-\alpha))^{n-1}(\bar{\rho}_+(x) + \bar{\rho}_-(x)).$$

Let V be a random variable with distribution H concentrated on $[0, 1]$, and independent of $Y_1, \ldots, Y_n$. Then by (3.11), (3.12) and (3.13) we obtain

$$(3.13)\quad P\left(V\left(|Y_1| + \max_{j \leqq n} |Y_2 + \ldots + Y_j|\right) > x\right)$$

$$\leqq K\mu^{-1}(1 + m_{\mu^{-1}\rho}(\alpha) + m_{\mu^{-1}\rho}(-\alpha))^{n-1} \int_0^1 \left(\bar{\rho}_-\left(\frac{x}{t}\right) + \bar{\rho}_+\left(\frac{x}{t}\right)\right) H(dt).$$

In particular, using for $i = 1, \ldots, n-1$

$$(3.14)\quad V = (1 - Z_0 - Z_i)\mathbf{1}_{(\Sigma_{j=0}^{n-1} Z_j \leqq 1)}$$

and (2.23) with $k = 2$, $m = n - 2$ we conclude that for every $i = 1, \ldots, n-1$,

$$(3.15)\quad P\left(\left(|Y_1| + \max_{j \leqq n} |Y_2 + \ldots + Y_j|\right)(1 - Z_0 - Z_i) > x\right)$$

$$\leqq K\mu^{n-1}(1 + m_{\mu^{-1}\rho}(\alpha) + m_{\mu^{-1}\rho}(-\alpha))^{n-1}((n-3)!)^{-1}(\tilde{\rho}_-(x) + \tilde{\rho}_+(x)).$$

Similarly, using

$$V = Z_1 + \ldots + Z_{n-1}$$

and (2.23) with $k = n - 2$ and $m = 2$, we obtain the same bound

$$(3.16)\quad P\left(\left(|Y_1| + \max_{j \leqq n} |Y_2 + \ldots + Y_j|\right)(Z_1 + \ldots + Z_{n-1}) > x\right)$$

$$\leqq K\mu^{n-1}(1 + m_{\mu^{-1}\rho}(\alpha) + m_{\mu^{-1}\rho}(-\alpha))^{n-1}((n-3)!)^{-1}(\tilde{\rho}_-(x) + \tilde{\rho}_+(x)).$$

Therefore, we obtain by (3.10), (3.15) and (3.16)

$$\sum_{n=2}^{\infty} \sum_{k=2}^{n} \frac{P(A_n^k(x))}{\tilde{\rho}_-(x) + \tilde{\rho}_+(x)}$$

$$\leqq \sum_{n=2}^{\infty} K^n(n+1)\mu^{n-1}(1 + m_{\mu^{-1}\rho}(\alpha) + m_{\mu^{-1}\rho}(-\alpha))^{n-1}((n-3)!)^{-1} < \infty$$

(recall that $(-1)! = 1$), and so (3.9) will follow if we prove that for every $n \geqq 2$ and $2 \leqq k \leqq n$,

$$\lim_{x\to\infty}\frac{P(A_n^k(x))}{\tilde\rho_-(x)+\tilde\rho_+(x)}=0. \tag{3.17}$$

To this end observe that it follows from (3.10) and (3.14) that for every $n \geqq 2$ and $2 \leqq k \leqq n$,

$$P(A_n^k(x)) \leqq \sum_{i=1}^{n} K\mu^{-1}(1+m_{\mu^{-1}\rho}(-\alpha)+m_{\mu^{-1}\rho}(-\alpha))^{n-1} \int_0^1 \left(\tilde\rho_-\left(\frac{x}{t}\right)+\tilde\rho_+\left(\frac{x}{t}\right)\right) H_i(dt), \tag{3.18}$$

where for $i = 1,\ldots,n-1$, H_i is the law of $(1 - Z_0 - Z_i)\mathbf{1}_{(\Sigma_{j=0}^{n-1} Z_j \leqq 1)}$, and H_n is the law of $(Z_1 + \ldots + Z_{n-1})\mathbf{1}_{(\Sigma_{j=0}^{n-1} Z_j \leqq 1)}$. Now (3.17) follows from (3.18) and Lemmas 2.2, 2.3 and 2.4. This proves (3.9).

Now the conclusion of Theorem 3.2 in the compound Poisson case follows from (3.9), (3.8) and (3.5).

We have, therefore, established the conclusion of Theorem 3.1 for compound Poisson processes. As announced, our next step is to prove this theorem in the general case.

PROOF OF THEOREM 3.1 IN THE GENERAL CASE. We start with observing that $\int_0^1 |X(t)|\,dt \geqq \left|\int_0^1 X(t)dt\right|$. Therefore, (3.5) shows that we only need to prove that

$$\overline{\lim_{x\to\infty}}\frac{P\left(\int_0^1 |X(t)|\,dt > x\right)}{A_+(\rho)\tilde\rho_+(x)+A_-(\rho)\tilde\rho_-(x)} \leqq 1. \tag{3.19}$$

We add first a possibility of a drift. Specifically, let ρ be still finite, and suppose that

$$\psi(\theta) = ib\theta + \int_{-\infty}^{\infty}(e^{i\theta x}-1)\rho(dx). \tag{3.20}$$

For an $n \geqq 1$ *let* $\mathbf{X}_n = (X_n(t), t \geqq 0)$ be a process with stationary independent increments, with Lévy exponent given by

$$\psi_n(\theta) = n(e^{ib\theta/n}-1) + \int_{-\infty}^{\infty}(e^{i\theta x}-1)\rho(dx). \tag{3.21}$$

Observe that

$$\mathbf{X}_n \Rightarrow \mathbf{X} \text{ as } n \to \infty \text{ weakly in } D[0,1].$$

The latter space is equipped with Skorohod topology J_1. See e.g. Skorohod [Sko57]. Now, $\mathbf{X}_n$ is a compound Poisson process and its Lévy measure is given by

$$\rho_n = \rho + n\delta_{\{b/n\}}.$$

In particular, ρ_n satisfies the assumptions of the theorem. Therefore, as we know by now,

$$\lim_{x\to\infty} \frac{P\left(\int_0^1 |X_n(t)|\,dt > x\right)}{Ee^{\alpha\int_0^1 X_n(t)dt}\tilde{\rho}_+(x) + Ee^{-\alpha\int_0^1 X_n(t)dt}\tilde{\rho}_-(x)} = 1. \tag{3.22}$$

Observe that

$$\lim_{n\to\infty} Ee^{\alpha\int_0^1 X_n(t)dt} = A_+(\rho), \tag{3.23}$$

and

$$\lim_{n\to\infty} Ee^{-\alpha\int_0^1 X_n(t)dt} = A_-(\rho). \tag{3.24}$$

By an embedding theorem (see e.g. Theorem IV.3.13 of Pollard [Pol84]) we may assume that $(X(t), t \geqq 0)$ and $(X_n(t), t \geqq 0)$ are defined on the same probability space in such a way that

$$\mathbf{X}_n \to \mathbf{X} \text{ as } n \to \infty \text{ a.s. in } D[0,1]$$

and the process $(D_n(t) = X(t) - X_n(t),\ t \geqq 0)$ is independent of the process $(X(t),\ t \geqq 0)$. Then, in particular,

$$\int_0^1 |D_n(t)|\,dt \leqq \max_{0\leqq t\leqq 1} |X(t) - X_n(t)| \to 0 \tag{3.25}$$

a.s. as $n \to \infty$. Moreover, $Ee^{\alpha\int_0^1 |D_n(t)|dt} \leqq Ee^{\alpha\int_0^1 D_n(t)dt} + Ee^{\alpha\int_0^1 D_n(t)dt}$ is easily seen to be bounded in n.

Let $(X_0(t) = X(t) - bt, t \geqq 0)$. Then the process $\mathbf{X}_0 = (X_0(t), t \geqq 0)$ is a compound Poisson process, and so for this process the statement of the theorem has been proved to be true. Fix a $\gamma > 0$. For all $x > 0$ big enough we have

$$P\left(\int_0^1 |X(t)|\,dt > x\right)$$

$$= P\left(\int_0^1 |X(t)|\,dt > x, \int_0^1 |X_n(t)|\,dt + \int_0^1 |X(t) - X_n(t)|\,dt > x\right)$$

$$\leqq P\left(\int_0^1 |X_n(t)|\,dt > x - \gamma\right) + P\left(\int_0^1 |X(t)|\,dt > x, \int_0^1 |D_n(t)|\,dt > \gamma\right)$$

$$\leqq P\left(\int_0^1 |X_n(t)|\,dt > x - \gamma\right) + P\left(\int_0^1 |X_0(t)|\,dt > x - |b|, \int_0^1 |D_n(t)|\,dt > \gamma\right).$$

Therefore, by (3.22) and the properties of exponential distributions,

$$\overline{\lim_{x\to\infty}} \frac{P\left(\int_0^1 |X(t)|\,dt > x\right)}{A_+(\rho)\tilde{\rho}_+(x) + A_-(\rho)\tilde{\rho}_-(x)}$$

$$\leqq e^{\alpha\gamma} \max\left(\frac{Ee^{\alpha\int_0^1 X_n(t)dt}}{Ee^{\alpha\int_0^1 X(t)dt}} \cdot \frac{Ee^{-\alpha\int_0^1 X_n(t)dt}}{Ee^{-\alpha\int_0^1 X(t)dt}}\right)$$

$$+ \max\left(\frac{Ee^{\alpha\int_0^1 X_0(t)dt}}{Ee^{\alpha\int_0^1 X(t)dt}} \cdot \frac{Ee^{-\alpha\int_0^1 X_0(t)dt}}{Ee^{-\alpha\int_0^1 X(t)dt}}\right) e^{\alpha|b|} P\left(\int_0^1 |D_n(t)|\,dt > \gamma\right).$$

We now use (3.25) and let first $n \to \infty$ and then $\gamma \to 0$ to obtain (3.19).

We now consider the general case of characteristic exponent given by (1.2). The argument is similar to the one before. For an $n \geqq 1$ *let* $\mathbf{X}_n = (X_n(t), t \geqq 0)$ be a process with stationary independent increments and characteristic functional given by

$$\psi_n(\theta) = n(e^{ib\theta/n} - 1) + n[n(e^{i\theta\sigma/n} - 1) + n^2(e^{-i\theta\sigma/n^2} - 1)] \tag{3.26}$$

$$+ \int_{|x|>1/n} (e^{i\theta x} - 1 - i\theta x \mathbf{1}(|x| \leqq 1))\rho(dx).$$

Now, $\mathbf{X}_n$ is a process with characteristic functional of the type (3.20), and, moreover, its Lèvy measure is

$$\rho_n = \rho\mathbf{1}_{[-1/n,\,1/n]^c} + n\delta_{\{b/n\}} + n^2\delta_{\{\sigma/n\}} + n^3\delta_{\{-\sigma/n^2\}}.$$

In particular, ρ_n satisfies the assumptions of the theorem. Therefore, (3.22) holds. Of course, both (3.23) and (3.24) hold as well.

Write

$$X(t) = W_n(t) + Z_n(t),\ t \geqq 0, \tag{3.27}$$

$$X_n(t) = W_n(t) + U_n(t),\ t \geqq 0, \tag{3.28}$$

where the processes $\mathbf{W}_n = (W_n(t),\ t \geqq 0)$, $\mathbf{Z}_n = (Z_n(t),\ t \geqq 0)$ and $\mathbf{U}_n = (U_n(t),\ t \geqq 0)$ are processes with stationary independent increments, with corresponding characteristic exponents

$$\psi_{\mathbf{W}_n}(\theta) = \int_{|x| > 1/n} (e^{i\theta x} - 1 - i\theta x \mathbf{1}(|x| \leqq 1))\rho(dx),$$

$$\psi_{\mathbf{Z}_n}(\theta) = ib\theta - \sigma^2\theta^2/2 + \int_{-1/n}^{1/n} (e^{i\theta x} - 1 - i\theta x \mathbf{1}(|x| \leqq 1))\rho(dx)$$

and

$$\psi_{\mathbf{U}_n}(\theta) = n(e^{ib\theta/n} - 1) + n[n(e^{i\theta\sigma/n} - 1) + n^2(e^{-i\theta\sigma/n^2} - 1)].$$

The processes $\mathbf{W}_n$ and $\mathbf{Z}_n$ are independent in (3.27), and the processes $\mathbf{W}_n$ and U_n are independent in (3.28).

Observe that if $\mathbf{B} = (B(t),\ t \geqq 0)$ is a Brownian motion with drift b and dispersion σ, then, as before, we obtain

$$\mathbf{Z}_n \Rightarrow \mathbf{B}$$

and

$$\mathbf{U}_n \Rightarrow \mathbf{B}$$

as $n \to \infty$ weakly in $D[0, 1]$ equipped with Skorohod's J_1 topology.

We use once again the embedding theorem quoted above to put everything on the same probability space in the following way. Let $\mathbf{W}_n$ be the same in (3.27) and (3.28) and live on $(\Omega_1, \mathcal{F}_1, P_1)$, and let $\mathbf{Z}_n$, $\mathbf{U}_n$ and $\mathbf{B}$ live on another probability space $(\Omega_2, \mathcal{F}_2, P_2)$ in such a way that

$$\mathbf{Z}_n \to \mathbf{B} \text{ a.s.},$$

$$\mathbf{U}_n \to \mathbf{B} \text{ a.s.}$$

in $D[0, 1]$ as $n \to \infty$. Observe that (3.25) holds with $D_n(t) = Z_n(t)$, $t \geqq 0$.

Fix once again as $\gamma > 0$. We have

$$P\left(\int_0^1 |X(t)|\,dt > x\right) = P\left(\int_0^1 |X(t)|\,dt > x,\right.$$

$$\left.\int_0^1 |X_n(t)|\,dt + \int_0^1 |X(t) - X_n(t)|\,dt > x\right)$$

$$\leqq P\left(\int_0^1 |X_n(t)|\,dt > x - \gamma\right) + P\left(\int_0^1 |X(t)|\,dt > x, \int_0^1 |D_n(t)|\,dt > \gamma\right)$$

$$\leqq P\left(\int_0^1 |X_n(t)|\,dt > x - \gamma\right) + P\left(\int_0^1 |W_n(t)|\,dt + \int_0^1 |Z_n(t)|\,dt > x,\right.$$

$$\left.\int_0^1 |D_n(t)|\,dt > \gamma\right).$$

Therefore, by (3.22) and the properties of exponential distributions,

$$\overline{\lim_{x\to\infty}} \frac{P\left(\int_0^1 |X(t)|\,dt > x\right)}{A_+(\rho)\tilde{\rho}_+(x) + A_-(\rho)\tilde{\rho}_-(x)}$$

$$\leqq e^{\alpha\gamma} \max\left(\frac{Ee^{\alpha\int_0^1 X_n(t)dt}}{Ee^{\alpha\int_0^1 X(t)dt}}, \frac{Ee^{-\alpha\int_0^1 X_n(t)dt}}{Ee^{-\alpha\int_0^1 X(t)dt}}\right)$$

$$+ \max\left(\frac{Ee^{\alpha\int_0^1 W_n(t)dt}}{Ee^{\alpha\int_0^1 X(t)dt}}, \frac{Ee^{-\alpha\int_0^1 W_n(t)dt}}{Ee^{-\alpha\int_0^1 X(t)dt}}\right) E\left(e^{\alpha\int_0^1 |Z_n(t)|dt} \mathbf{1}\left(\int_0^1 |D_n(t)|\,dt > \gamma\right)\right).$$

From this (3.19) follows as before once we observe that

$$Ee^{\alpha\int_0^1 W_n(t)\,dt} \to Ee^{\alpha\int_0^1 X_*(t)\,dt},$$

$$Ee^{-\alpha\int_0^1 W_n(t)\,dt} \to Ee^{-\alpha\int_0^1 X_*(t)\,dt},$$

where $(X_*(t),\ t \geqq 0)$ is a process with stationary independent increments and characteristic exponent

$$\psi_*(\theta) = \int_{-\infty}^{\infty} (e^{i\theta x} - 1 - i\theta x \mathbf{1}(|x| \leqq 1))\rho(dx),$$

and that

$$E\left(e^{\alpha\int_0^1|Z_n(t)|dt}\mathbf{1}\left(\int_0^1|D_n(t)|\,dt>\gamma\right)\right)$$

$$\leqq (Ee^{2\alpha\int_0^1|Z_n(t)|dt})^{1/2}\left(P\left(\int_0^1|D_n(t)|\,dt>\gamma\right)\right)^{1/2}$$

$$\leqq \left(E\int_0^1 e^{2\alpha|Z_n(t)|}\,dt\right)^{1/2}\left(P\left(\int_0^1|D_n(t)|\,dt>\gamma\right)\right)^{1/2}$$

$$\leqq \left(E\int_0^1 (e^{2\alpha Z_n(t)}+e^{-2\alpha Z_n(t)})\,dt\right)^{1/2}\left(P\left(\int_0^1|D_n(t)|\,dt>\gamma\right)\right)^{1/2}$$

$$\leqq k\left(P\left(\int_0^1|D_n(t)|\,dt>\gamma\right)\right)^{1/2},$$

where k is an absolute constant. This completes the proof of Theorem 3.1 in its full generality.

REMARK 1. Theorem 3.1 establishes the tail distribution of the L^1 norm of Lévy process on the interval $(0, 1)$. The case of L^1 norm on an interval $(0, T)$ for a $T > 0$ reduces to the above by a simple change of variable. As a corollary, we obtain that under the assumptions of Theorem 3.1 and under its notation we have

$$\lim_{x\to\infty}\frac{P\left(\int_0^T|X(t)|\,dt>x\right)}{(A_+(\rho))^T\tilde{\rho}_+(x/T)+(A_-(\rho))^T\tilde{\rho}_-(x/T)}=T. \tag{3.29}$$

REMARK 2. We cannot characterize so far all Lévy processes for which Theorem 3.2 holds. However, the present argument given for that theorem easily shows that its statement holds when, say, $\tilde{\rho}_+$ is equivalent to the tail of a distribution in $\mathscr{S}(\alpha)$, and $\tilde{\rho}_-$ is of a smaller order. Then, in particular,

$$\lim_{x\to\infty}\frac{P\left(\int_0^T|X(t)|\,dt>x\right)}{(A_+(\rho))^T\tilde{\rho}_+(x/T)}=T. \tag{3.30}$$

The case when $\tilde{\rho}_-$ is equivalent to the tail of a distribution in $\mathscr{S}(\alpha)$, and $\tilde{\rho}_+$ is of a smaller order is, of course, similar.

ACKNOWLEDGMENT. Remarks of the anonymous referee helped the authors to improve the presentation of Proposition 2.1 and to simplify its proof.

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