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INTERPOLATION OF OPERATORS ON DECREASING FUNCTIONS

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Abstract.

Although there is no general way to identify the interpolated class of the pair of cones of decreasing functions of Banach function lattices, it is shown that in many examples, which include all rearrangement invariant spaces and a number of weighted function lattices, real interpolation for decreasing functions is well behaved.

1. Introduction.

Throughout this paper L_0 will be the vector space of all (equivalence classes of) Lebesgue measurable real functions on $\mathrm{R}^+ = [0, \infty)$, and we shall say that X is a *(quasi-) Banach function lattice* if it is a (quasi-) Banach space and a linear subspace of L_0 such that, if $|f| \leqq |g|$, $g \in X$ and $f \in L_0$, then $f \in X$ and $\|f\|_X \leqq \|g\|_X$.

A *decreasing function* will be a nonincreasing and non-negative function on R^+, and $X^d = X \cap L_0^d$ will denote the cone of all decreasing functions of X.

Recently the boundedness of operators between Banach function lattices, like L_p-spaces with weights or Lorentz spaces on R^+, restricted to decreasing functions has been widely considered in several contexts (cf. [AM], [An], [CS], [Ga], [Sw], etc.) and a natural question is whether there are interpolation theorems for operators which are bounded for decreasing functions.

Y. Sagher, in [Sg], and more recently S. Lai, in [La], have observed that the proof of Marcinkiewicz's interpolation theorem is easily adapted to prove that if a quasilinear operator $T\colon L_{p_0} + L_{p_1} \to L_0$ is of weak type (p_0, p_0) and (p_1, p_1) on decreasing functions, then it is of type (p, p) for any $p \in (p_0, p_1)$. This fact has been used by K. Andersen in [An] for the study of some integral operators of the type $Tf(x) = \int_0^\infty a(r) f(xt)\,dt$ acting on spaces $L_p(\omega)^d$, with $0 < p < \infty$.

A. Garciá del Amo [Ga] extends some results of C. J. Neugebauer about the

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boundedness of the operator $Tf(x) = \frac{1}{x^p}\int_0^x f(x-t)d(t^p)$ when acting on increasing functions using a version of Riesz's Convexity Theorem.

When extending these examples to the setting of the real K-method, there is no general way to identify the interpolated class for a pair of cones (X_0^d, X_1^d) of decreasing functions of a couple $\bar{X} = (X_0, X_1)$ of Banach function lattices, even if we know the interpolation space $\bar{X}_{\vartheta,p}$ of this pair of spaces.

Nevertheless, we show that for all pairs of symmetric spaces and a number of couples of weighted function lattices the K-method of interpolation for decreasing functions is well-behaved.

In these cases the K-functional $K(f,t;\bar{X})$, restricted to decreasing functions, is equivalent to the K-functional $K(f,t;\bar{X}^d)$ associated to the corresponding pair of cones of decreasing functions. I.e., these cones are Marcinkiewicz cones in the sense of Y. Sagher [Sg], who applied interpolation properties of operators between cones of Banach lattices to the study of the Fourier coefficients for some classes of functions.

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2. The K-method for decreasing functions.

Let $\bar{X} = (X_0, X_1)$ be a pair of quasi-Banach function lattices and $\bar{X}^d = (X_0^d, X_i^d)$ the corresponding pair of cones of decreasing functions.

If we denote $\Sigma(\bar{X}^d) = X_0^d + X_1^d$, for every $f \in \Sigma(\bar{X}^d)$ we can consider the usual K-functional

$$K(f,t) = K(f,t;\bar{X}) = \inf\{\|f_0\|_0 + t\,\|f_1\|_1; f = f_0 + f_1, f_j \in X_j\}$$

and the K^d-functional,

$$K^d(f,t) = K(f,t;\bar{X}^d) = \inf\{\|f_0\|_0 + t\,\|f_1\|_1; f = f_0 + f_1, f_j \in X_j^d\}.$$

Let now $0 < \vartheta < 1$ and $0 < q \leqq +\infty$. As for the usual K-method we define

$$(\bar{X}^d)_{\vartheta,q} = \left\{f \in \Sigma(\bar{X}^d); \|f\|_{\vartheta,q,d} = \left(\int_0^\infty [t^{-\vartheta}K^d(f,t)]^q \frac{dt}{t}\right)^{1/q} < \infty\right\}.$$

The following *Decomposition Lemma* will be very useful to prove the basic properties of these classes $(\bar{X}^d)_{\vartheta,q}$. It allows to show that $\Sigma(\bar{X}^d)$ and $(\bar{X}^d)_{\vartheta,q}$ are quasi-Banach lattices (stretching the definition to cover cones rather than spaces), since, if $g \leqq f$ are decreasing functions and $f \in \Sigma(\bar{X}^d)$, then $g \in \Sigma(\bar{X}^d)$ and $K^d(g,t) \leqq K^d(f,t)$. Also $g \in (\bar{X}^d)_{\vartheta,q}$ and $\|g\|_{\vartheta,q,d} \leqq \|f\|_{\vartheta,q,d}$ if $f \in (\bar{X}^d)_{\vartheta,q}$.

LEMMA 1. *Let $f, g, h \in L_0$ be three right continuous decreasing functions such that*

$f \leqq g + h$. Then there exists a decomposition $f = f_0 + f_1$ with $f_j \in L_0^d$ $(j = 0, 1)$, $f_0 \leqq g$ and $f_1 \leqq h$.

PROOF. By considering $g' = \min(f, g)$ and $h' = \min(f, h)$ we can suppose $g, h \leqq f$.

We begin with the case of simple funtions. Obviously we can assume that f, g and h are constant and non zero on the same intervals $[x_i, x_{i+1})$ $(0 \leqq i \leqq n)$ and we will use the notation $f = \sum_{i=0}^{n} a_i \chi_{[x_i, x_{i+1})} = (a_0, a_1, \ldots, a_n)$ $(a_{i+1} \leqq a_i)$.

Then we have $g = (c_0, c_1, \ldots, c_n)$ and $h = (d_0, d_1, \ldots, d_n)$ with $a_i \leqq c_i + d_i$. We can find $0 \leqq \varepsilon_i \leqq 1$ $(0 \leqq i \leqq n)$ such that

(i) $\varepsilon_i a_i \leqq c_i$ and $(1 - \varepsilon_i) a_i \leqq d_i$, and

(ii) $\varepsilon_0 a_0 \geqq \varepsilon_1 a_1 \geqq \ldots \geqq \varepsilon_n a_n$, $(1 - \varepsilon_0) a_0 \geqq (q - \varepsilon_1) a_1 \geqq \ldots \geqq (1 - \varepsilon_n) a_n$.

We can take $\varepsilon_n \in [1 - d_n/a_n, c_n/a_n]$ and

$$\varepsilon_i \in \left[\max\left(1 - \frac{d_i}{a_i}, \varepsilon_{i+1} \frac{a_{i+1}}{a_i}\right), \min\left(\frac{a_i - a_{i+1} + \varepsilon_{i+1} a_{i+1}}{a_i}, \frac{c_i}{a_i}\right)\right]$$

for $0 \leqq i \leqq n - 1$. Then

$$f_0 = (\varepsilon_0 a_0, \ldots, \varepsilon_n a_n) \quad \text{and} \quad f_1 = ((1 - \varepsilon_0) a_0, \ldots, (1 - \varepsilon_n) a_n)$$

satisfy the conditions of the lemma.

For the general case let $\operatorname{supp} f = [0, +\infty)$. We wil consider $D = \bigcup_{n=0}^{\infty} D_n$, with $D_n = \{k + j2^{-n}; 0 \leqq k \leqq n, 1 \leqq j \leqq 2^n\}$, and on this set of diadic points we write $p \precsim q$ when $n_p < n_q$, or $n_p = n_q$ and $p \leqq q$, where $n_p = \min\{n; p \in D_n\}$. Let $D = \{p_i\}_{i=1}^{\infty}$ with $p_1 \precsim p_2 \precsim \ldots$ and

$$f_n = \sum_{i=1}^{(n+1)2^n} f(p_i) \chi_{[q_i, p_i)}, \; g_n = \sum_{i=1}^{(n+1)2^n} g(p_i) \chi_{[q_i, p_i)}, \; h_n = \sum_{i=1}^{(n+1)2^n} h(p_i) \chi_{[q_i, p_i)}.$$

We have seen that there exist $0 \leqq \varepsilon_n(p_i) \leqq 1$ $(1 \leqq i \leqq (n + 1)2^n)$ such that

$$f_n^0 = \sum_{i=1}^{(n+1)2^n} \varepsilon_n(p_i) f(p_i) \chi_{(q_i, p_i)}$$

and

$$f_n^1 = \sum_{i=1}^{(n+1)2^n} (1 - \varepsilon_n(p_i)) f(p_i) \chi_{(q_i, p_i)}$$

are simple decreasing functions, and

$$f_n = f_n^0 + f_n^1, \; f_n^0 \leqq g_n, \; f_n^1 \leqq h_n.$$

If $i > (n + 1)2^n$ we define $\varepsilon_n(p_i) = 0$, and by a compactness argument we find a subsequence $\{\varepsilon_{n_k}\}$ of $\{\varepsilon_n\}$ such that $\lim_{k \to \infty} \varepsilon_{n_k}(p)$ for all $p \in D$. Then we define

$$f_0(p) = \varepsilon(p)f(p), f_1(p) = (1 - \varepsilon(p))f(p)$$

over D and we extend them over $[0, \infty)$ by right continuity,

$$f_i(x) = \lim_{p \downarrow x} f_i(p) \qquad (i = 0, 1)$$

to obtain f_0 and f_1 with the announced properties.

Obviously

(1) $$\Sigma(\bar{X}^d) \subset \Sigma(\bar{X})^d$$

and

(2) $$K(f, t) \leqq K^d(f, t) \quad \text{for all} \quad f \in \Sigma(\bar{X}^d).$$

Moreover

$$X_0^d \cap X_1^d \hookrightarrow (\bar{X}^d)_{\vartheta,q} \hookrightarrow (\bar{X}^d)_{\vartheta,r} \hookrightarrow \Sigma(\bar{X}^d) \quad \text{if} \quad q \leqq r.$$

From (2) we get $(\bar{X}^d)_{\vartheta,q} \hookrightarrow (\bar{X}_{\vartheta,q})^d$.

We say that $T: \bar{X}^d \to \bar{Y}$ is *quasilinear* if $T: \Sigma(\bar{X}^d) \to \Sigma(\bar{Y})$, $T(X_j^d) \subset Y_j$ $(j = 0, 1)$ and $|T(f + d)| \leqq k(|Tf| + |Tg|)$. There is a similar definition for a quasilinear mapping from $\bar{X}$ to $\bar{Y}^d$, or from $\bar{X}^d$ to $\bar{Y}^d$.

THEOREM 1. *Let $\bar{X}$, $\bar{Y}$ be two pairs of quasi-Banach function lattices.*

(a) *If $T: \bar{X}^d \to \bar{Y}^d$ is quasilinear and $\|Tf\|_j \leqq M_j \|f\|_j$ $(j = 0, 1)$, then $T: (\bar{X}^d)_{\vartheta,q} \to (\bar{Y})_{\vartheta,d}$ and $\|Tf\|_{\vartheta,q,d} \leqq kM_0^{1-\vartheta}M_1^{\vartheta} \|f\|_{\vartheta,q,d}$.*

(b) *If $T: \bar{X}^d - \bar{Y}$ is quasilinear and $\|Tf\|_j \leqq M_j \|f\|_j$, then $T: (\bar{X}^d)_{\vartheta,q} \to \bar{Y}_{\vartheta,q}$ and $\|Tf\|_{\vartheta,q} \leqq kM_0^{1-\vartheta}M_1^{\vartheta} \|f\|_{\vartheta,q,d}$.*

(c) *If $T: \bar{X} \to \bar{Y}^d$ is quasilinear and $\|Tf\|_j \leqq M_j \|f\|_j$, then $T: \bar{X}_{\vartheta,q} \to (\bar{Y}^d)_{\vartheta,q}$ and $\|Tf\|_{\vartheta,q,d} \leqq kM_0^{1-\vartheta}M_1^{\vartheta} \|f\|_{\vartheta,q}$.*

PROOF. If $f = f_0 + f_1$ with $f_j \in X_j^d$, then $Tf \leqq k(Tf_0 + Tf_1)$ and from the Decomposition Lemma we obtain a decomposition $Tf = g_0 + g_1$ such that $g_j \in Y_j^d$ and $g_j \leqq kTf_j$. It follows that

$$K^d(Tf, t) \leqq \|g_0\|_0 + t\,\|g_1\|_1 \leqq k(\|Tf_0\|_0 + t\,\|Tf_1\|_1),$$

and $K^d(Tf, t) \leqq kM_0(\|f_0\|_0 + tM_1/M_0\,\|f_1\|_1)$. Then $K^d(Tf, t) \leqq kM_0K^d(f, M_1t/M_0)$ and $\|Tf\|_{\vartheta,q,d} \leqq kM_0^{1-\vartheta}M_1^{\vartheta} \|f\|_{\vartheta,q,d}$. The other cases are similar.

Since the main problem is now to identify $(\bar{X}^d)_{\vartheta,q}$, at least when the usual interpolated space $\bar{X}_{\vartheta,q}$ is known, it is useful to know if

(3) $$\Sigma(\bar{X})^d = \Sigma(\bar{X}^d)$$

and if there exists a constant $C \geqq 0$ such that

$$K^d(f,t) \leqq CK(f,t) \tag{4}$$

for any $f \in \Sigma(\bar{X}^d)$ and $t > 0$. In this case we say that $\bar{X}$ is a *Marcinkiewicz pair* and we have the equality $(\bar{X}^d)_{\vartheta,q} = (\bar{X}_{\vartheta,q})^d$ with equivalent "norms". This definition is related to the definition of "Marcinkiewicz quasi-cones" in [Sg], where the equivalence of the K functionals for $L_{p,q}$ spaces is done.

In the proof of the following theorem the key fact is the existence of a bounded linear operator T on $\bar{X}$ such that Tf is decreasing if $f \geqq 0$, and $Tf \geqq f$ for any $f \in \Sigma(\bar{X})^d$.

THEOREM 2. *If the Hardy operators*

$$S_1 f(t) = \frac{1}{t}\int_0^t f(s)\,ds \quad \text{and} \quad S_2 f(t) = \int_t^\infty f(s)\frac{ds}{s}$$

are bounded on X_0 and on X_1, then $\bar{X}$ is a Marcinkiewicz pair.

PROOF. Define $T = S_2 S_1$. We observe that

$$Tf(t) = \int_t^\infty \frac{ds}{s^2}\int_0^s f(\tau)\,d\tau = \frac{1}{t}\int_0^t f(s)\,ds + \int_t^\infty f(\tau)\frac{d\tau}{\tau} = S_1 f(t) + S_2 f(t)$$

and, if f is decreasing, $Tf \geqq S_1 f \geqq f$. Moreover, since $S_1 f \geqq 0$ if $f \geqq 0$, Tf is decreasing. Let now $f = f_0 + f_1 \in \Sigma(\bar{X})^d$, with $0 \leqq f_j \in X_j$. Then $f \leqq Tf = Tf_0 + Tf_1$ with $Tf_j \in X_j^d$, and from the Decomposition Lemma we know that there exist $g_j \in X_j^d$ such that $g_j \leqq Tf_j$ and $f = g_0 + g_1$. It follows that $\Sigma(\bar{X})^d = \Sigma(\bar{X}^d)$.

Moreover $K^d(f,t)$ and $K(f,t)$ are equivalent, since

$$K^d(f,t) \leqq K^d(Tf,t) \leqq \inf_{f=f_0+f_1} (\|Tf_0\|_0 + t\,\|Tf_1\|_1 \leqq \max(M_0, M_1)K(f,t).$$

It is easy to obtain examples of weighted L_p-spaces over R^+ without properties (3) and (4):

EXAMPLE 1. If $\bar{X} = (L_1(\omega_0), L_1(\omega_1))$ with

$$\omega_0(x) = \chi_{(0,2)}(x) + \frac{1}{x-2}\chi_{(2,+\infty)}(x)$$

and

$$\omega_1(x) = \chi_{(0,1)}(x) + \frac{1}{x-1}\chi_{(1,+\infty)}(x),$$

then $\Sigma(\bar{X}^d) \neq \Sigma(\bar{X})^d$, since $\Sigma(\bar{X}) = L_1(\min(\omega_0,\omega_1))$, $\chi_{(0,3)} \in L_1(\min(\omega_0,\omega_1))^d$ and $\chi_{(0,3)} \notin L_1(\omega_0)^d + L_1(\omega_1)^d$.

Moreover $(\bar{X}^d)_{1/2,1} \neq \bar{X}_{1/2,1} \cap \Sigma(\bar{X}^d) \neq (\bar{X}_{1/2,1})^d$, since

$$K^d(f,t) = \inf_{f=f_0+f_1}\left(\int_0^2 f_0 + t\int_0^1 f_1\right) = \inf_{f=f_0+f_1}\int_0^1 (f_0+tf_1) + \int_1^2 f$$

$$\geqq K(f\chi_{(0,1)}, t; L_1(0,1)^d, L_1(0,1)^d) + \int_1^2 f = \min(1,t)\int_0^1 f + \int_1^2 f$$

and it follows that $f = 0$ on $(1,2)$ for any $f \in (\bar{X}^d)_{1/2,1}$, and $(\bar{X}^d)_{1/2,1} = L_1(0,1)^d$; but it is well known that $(L_1(\omega_0), L_1(\omega_1))_{\vartheta,1} = L_1(\omega_0^{1-\vartheta}\omega_1^{\vartheta})$, and we have

$$\omega_0^{1/2}(x)\omega_1^{1/2}(x) = \chi_{(0,1]}(x) + (x-1)^{-1/2}\chi_{(1,2]}(x) + (x-1)^{-1/2}(x-2)^{-1/2}\chi_{(2,\infty)}(x)$$

so that $\chi_{(0,2)} \in \bar{X}_{1/2,1} \cap \Sigma(\bar{X}^d)$ but $\chi_{(0,2)} \notin (\bar{X}^d)_{1/3,1}$. Also, $\chi_{(0,3)} \in (\bar{X}_{(1/2,1)})^d$.

If $\bar{X} = (L_1, L_1(e^{-x}))$, then $\Sigma(\bar{X}^d) = \Sigma(\bar{X})^d = L_1(e^{-x})^d$, but for $f = 1$, $K(f,t)$ and $K^d(f,t)$ are not equivalent. Notice that $1 \in (\bar{X}_{\vartheta,1})^d$ but $1 \notin (\bar{X}^d)_{\vartheta,1}$, since $f = 1$ only can be decomposed as a sum of two decreasing functions as $\varepsilon + (1-\varepsilon)$ with $0 \leqq \varepsilon \leqq 1$, and $\varepsilon \notin L_1$.

3. Reiteration.

In this section we adapt the beautiful Holmstedt Theorem (cf. [Ho]) to our setting. Let $\bar{X}$ be a fixed pair of quasi-Banach function lattices, $0 < \vartheta_0 < \vartheta_1 < 1$, $\delta = \vartheta_1 - \vartheta_0$ and $0 < q_0, q_1 \leqq \infty$. For all $f \in \Sigma(\bar{X}^d)$ and $j = 0,1$ we define

$$P_j^d f(t) = \left(\int_0^t [s^{-\vartheta_j}K^d(f,s)]^{q_j}\frac{ds}{s}\right)^{1/q_j} \text{ and } Q_j^d f(t) = \left(\int_t^\infty [s^{-\vartheta_j}K^d(f,s)]^{q_j}\frac{ds}{s}\right)^{1/q_j}.$$

The following Lemma will allow us to avoid taking differences $f - g$ of decreasing functions, which need not be decreasing.

LEMMA 2. *Let $f \in \Sigma(\bar{X}^d)$ be such that $P_0^d f(t_0) < \infty$ and $Q_1^d f(t_0) < \infty$, for some $t_0 > 0$. Then there exist a decomposition $f = g + h$, with $g \in (\bar{X}_d)_{\vartheta_0,q_0}$ and $h \in (\bar{X}^d)_{\vartheta_1,q_1}$, and a constant $c > 0$ such that $\|g\|_{\vartheta_1,q_1,d} \leqq cP_0^d f(t_0)$, $\|h\|_{\vartheta_1,q_1,d} \leqq cQ_1^d f(t_0)$.*

PROOF. Let $f = g + h \in X_0^d + X_1^d$ and $\|g\|_0 + t_0\|h\|_1 \leqq 2K^d(f,t_0)$.

Then $K^d(g,s) \leqq 2K^d(f,t_0)$ and $K^d(h,s) \leqq (2s/t_0)K^d(f,t_0)$ for any $s > 0$. Now, $K^d(f,t)/t$ being decreasing and $K^d(f,t)$ increasing, we find that

$$Q_0^d g(t_0) \leqq cP_0^d f(t_0) < \infty,\ P_1^d h(t_0) \leqq cQ_1^d f(t_0) < \infty.$$

On the other hand, if $f = g + h$ and g, h are decreasing, then $g \leqq f$ and $K^d(g,s) \leqq K^d(f,s)$, thus $P_0^d g \leqq P_0^d f$, $Q_1^d h \leqq Q_1^d f$. It follows that

$$\|g\|_{\vartheta_0,q_0,d}^{q_0} \leqq (P_0^d g(t_0))^{q_0} + (Q_0^d g(t_0))^{q_0} \leqq C(P_0^d f(t_0))^{q_0} < \infty.$$

Similarly, $\|h\|_{\vartheta_0,q_0,d} \leqq CQ_1^d f(t_0) < \infty$.

Now the proof of Holmstedt's Reiteration Theorem is easily adapted to our case of pairs of cones of decreasing functions:

THEOREM 3. *Let* $\bar{K}^d(f,t) = K(f,t;(\bar{X}^d)_{\vartheta_0,q_0},(\bar{X}^d)_{\vartheta_1,q_1})$. *Then*

$$\bar{K}^d(f,t^\delta) \simeq P_0^d f(t) + t^\delta Q_1^d f(t)$$

and

$$((\bar{X}^d)_{\vartheta_0,q_0},(\bar{X}^d)_{\vartheta_1,q_1})_{\vartheta,q} = (\bar{X}^d)_{\vartheta',q} \tag{5}$$

with equivalent "norms". Here $0 < q \leqq \infty$, $0 < \vartheta < 1$ *and* $\vartheta' = (1-\vartheta)\vartheta_0 + \vartheta\vartheta_1$.

PROOF. Let $H(f,t^\delta) = P_0^d f(t) + t^\delta Q_1^d f(t)$. Then

$$H(f,t^\delta) \leqq C\bar{K}^d(f,t^\delta)$$

is proved as in [Ho].

To obtain the converse inequality we observe that, if $f \in \Sigma(\bar{X}^d)$ is such that $P_0^d f(t) < \infty$ and $Q_1^d f(t) < \infty$, we know from Lemma 2 that there exists a decomposition $f = g + h$ with

$$\|g\|_{\vartheta_0,q_0,d} \leqq cP_0^d f(t),\ \|h\|_{\vartheta_1,q_1,d} \leqq cQ_1^d f(t)$$

and

$$\bar{K}^d(f,t^\delta) \leqq c(P_0^d f(t) + Q_1^d f(t)) \leqq cH(f,t^d).$$

Finally, (5) is proved exactly as in Theorem 3.1 of [Ho].

COROLLARY 2. *If* $\bar{X}$ *is a Marcinkiewicz pair, then* $(\bar{X}_{\vartheta_0,q_0},\bar{X}_{\vartheta_1,q_1})$ *is also a Marcinkiewicz pair and* $((\bar{X}^d)_{\vartheta_0,q_0},(\bar{X}^d)_{\vartheta_1,q_1})_{\vartheta,q} = (\bar{X}_{\vartheta',q})^d$, *where* $\vartheta' = (1-\vartheta)\vartheta_0 + \vartheta\vartheta_1$.

PROOF. Let $f \in (\bar{X}^d)_{\vartheta_0,q_0} + (\bar{X}^d)_{\vartheta_1,q_1}$. We know that $\bar{K}^d(f,t^\delta) \simeq P_0^d f(t) + t^\delta Q_1^d f(t)$, and from $\bar{K}^d(f,t) \simeq \bar{K}(f,t)$ it follows that

$$P_0^d f(t) = \left(\int_0^t (s^{-\vartheta_0}K^d(f,s))^{q_0}\frac{ds}{s}\right)^{1/q_0} \simeq \left(\int_0^t (s^{-\vartheta_0}K(f,s))^{q_0}\frac{ds}{s}\right)^{1/q_0} = P_0 f(t)$$

and also $Q_1^q f(t) \simeq Q_1 f(t)$. So, $\bar{K}^d(f,t) \simeq \bar{K}(f,t)$. Let us prove now that

$$(\bar{X}^d)_{\vartheta_0,q_0} + (\bar{X}^d)_{\vartheta_1,q_1} = [\bar{X}_{\vartheta_0,q_0} + \bar{X}_{\vartheta_1,q_1}]^d.$$

Suppose that

$$f \in [\bar{X}_{\vartheta_0,q_0} + \bar{X}_{\vartheta_1,q_1}]^d \subset \Sigma(\bar{X})^d = \Sigma(\bar{X}^d).$$

Then $P_0(t) < \infty$ and $P_1(t) < \infty$ for any $t > 0$. Since $K^d(f,t) \simeq K(f,t)$, we also have $P_0^d f(t) < \infty$ and $Q_1^d f(t) < \infty$, and from Lemma 2 if follows that $f \in \bar{X}_{\vartheta_0,q_0} + \bar{X}_{\vartheta_1,q_1}$.

The reiteration property follows from the above reiteration theorem and from the equality $(\bar{X}^d)_{\vartheta',q} = (\bar{X}_{\vartheta',q})^d$.

The following example is a simple application of the above reiteration results.

EXAMPLE 2. For any $0 < p < \infty$, (L_p, L_∞) is a simple of a Marcinkiewicz part, and $(L^d_{p_0}, L^d_\infty)_{\vartheta,q} = (L_{p_0}, L_\infty)^d_{\vartheta,q}$, with $1/p = (1-\vartheta)/p_0$.

To see that $(L_p + L_\infty)^d = L^d_d + L^d_\infty$, let $f \in (L_p + L_\infty)^d$. We have $f^* = f$ and, if

$$f_0(x) = (f(x) - f(t^p))\chi_{(0,t^p)}(x) \quad \text{and} \quad f_1(x) = f(t^p)\chi_{(0,t^p)}(x) + f(x)\chi_{(t^p,\infty)}(x),$$

then $f = f_0 + f_1 \in L^d_p + L^d_\infty$. Finally, it is well known that $K(f,t) \simeq (\int_0^{t^p} f^*(x)^p\,dx)^{1/p}$ and the equivalence $K(f,t) \simeq K^d(f,t)$ follows from

$$K^d(f,t) \leqq \left(\int_0^{t^p} |f(x) - f^*(t^p)|^p\,dx\right)^{1/p} + ((t^p f^*(t^p))^p)^{1/p}$$

$$\leqq \left(\int_0^{t^p} |f(x) - f^*(t^p)|^p\,dx\right)^{1/p} + \left(\int_0^{t^p} f^*(x)^p\,dx\right)^{1/p}$$

$$\leqq \left(\int_0^{t^p} f(x)^p\,dx\right)^{1/p}$$

Now, by reiteration,

$$(L^d_{p_0}, L^d_{p_1})_{\vartheta,q} = ((L_r, L_\infty)^d_{\vartheta_0,p_0}, (L_r, L_\infty)^d_{\vartheta_1,p_1})_{\vartheta,q} = (L_r, L_\infty)^d_{\eta,q} = L^d_{p.q}$$

for suitable values of r, ϑ_j and η. In the same way, for Lorentz spaces we have

$$(L^d_{p_0,q_0}, L^d_{p_1,q_1})_{\eta,q} = L^d_{p,q}$$

with $1/p = (1-\vartheta)/r$ and $0 < r < \min(p_0, p_1)$ such that $1/p_j = (1-\vartheta_j)/r$ ($j = 0, 1$), since, if $\vartheta = (1-\eta)\vartheta_0 + \eta\vartheta_1$,

$$(L^d_{p_0,q_0}, L^d_{p_1,q_1})_{\eta,q} = ((L_r, L_\infty)^d_{\vartheta_0,q_0}, (L_r, L_\infty)^d_{\vartheta_1,q_1})_{\eta,q} = (L_r, L_\infty)^d_{\vartheta,q}.$$

4. Symmetric spaces.

A *symmetric space* will be a quasi-Banach function lattice X over $(0,\infty)$ with Lebesgue measure such that, if $f \in L_0$ and $g \in X$ are such that $f^* = g^*$, then $f \in X$ and $\|f\|_X = \|g\|_X$, where f^* denotes the decreasing rearrangement of $|f|$. Observe that the operator $Df(t) = f(t/2)$ is bounded on X (cf. [HM]), and it is easily seen that $(Df)^* = D(f^*)$.

THEOREM 4. *If X_0 and X_1 are symmetric spaces, then $\bar{X}$ is a Marcinkiewicz pair.*

PROOF. If $f = f_0 + f_1 \in \Sigma(\bar{X})^d$ with $0 \leqq f_j \in X_j$, then $f^* \leqq Df_0^* + Df_1^*$ and

$Df_j^* \in X_j$. We apply Lemma 1 to obtain a decomposition $f^* = g_0 + g_1$ with g_j decreasing and $g_j \leqq Df_j^*$. It follows that $g_j \in X_j^d$, $\|g_j\|_j \leqq \|D\| \, \|f_j^*\|_j$ and $f = f^* = g_0 + g_1 \in \Sigma(\bar{X}^d)$.

For a given $t > 0$ and $\varepsilon > 0$ we can consider a decomposition $f = f_0 + f_1$ with $0 \leqq f_j \in X_j$ and $\|f_0\|_0 + t\,\|f_1\|_1 \leqq K(f,t) + \varepsilon$. Then

$$K^d(f,t) \leqq \|g_0\|_0 + t\,\|g_1\|_1 \leqq \|D\|\,(\|f_0^*\|_0 + t\,\|f_1^*\|_1) \leqq C(K(f,t) + \varepsilon)$$

and we obtain $K^d(f,t) \simeq K(f,t)$.

As an application we can prove a Wolf's reiteration type theorem for decreasing functions. First we observe that, for any quasi-Banach function lattice X, if we denote

$$X^s = \{f \in L_0;\, f^* \in X\},$$

then $(X^s)^d = X^d$ and that, if X has Fatou property, then X^s with $\|f\|_{X^s} = \|f^*\|_X$ is a quasi-Banach function latice. To see this, observe that for any Cauchy sequence $\{f_n\} \subset X^s$ we can suppose that $\lim_n f_n(t) = f(t)$ a.e. Since X^s has Fatou property, $\|f\|_{X^s} \leqq \liminf_n \|f_n\|_{X^s} < \infty$, thus $f \in X^s$ and similarly $\|f - f_n\|_{X^s} \to 0$.

THEOREM 5. *Let X_j $(j = 1, 2, 3, 4)$ be four quasi-Banach function lattices such that X_j^s $(j = 1, 2, 3, 4)$ are quasi-Banach function lattices and $X_1 \cap X_4 \subset X_2 \cap X_3$.*
If

$$(X_2^d, X_4^d)_{\vartheta,q} = X_3^d \quad \text{and} \quad (X_1^d, X_3^d)_{\varphi,q} = X_2^d,$$

then

$$(X_1^d, X_4^d)_{\varepsilon,r} = X_2^d \quad \text{and} \quad (X_1^d, X_4^d)_{\psi,q} = X_3^d,$$

with $\varepsilon = \varphi\vartheta/(1 - \varphi + \varphi\vartheta)$ and $\psi = \vartheta/(1 - \varphi + \varphi\vartheta)$.

PROOF. The spaces X_j^s are symmetric, so $X_3^s = (X_2^s, X_4^s)_{\vartheta,q}$ and $X_2^s = (X_1^s, X_3^s)_{\varphi,r}$. Since $X_1^s \cap X_4^s \subset X_2^s \cap X_3^s$ it follows from Wolf's theorem that $(X_2^s)^d = (X_1^s, X_4^s)_{\varepsilon,r}^d$ and we obtain

$$X_2^d = (X_2^s)^d = ((X_1^s)^d, (X_4^s)^d)_{\varepsilon,r} = (X_1^d, X_4^d)_{\varepsilon,r}.$$

5. Weighted function lattices.

If X is a quasi-Banach function lattice, for any weight $0 < \omega \in L_0$ the corresponding quasi-Banach weighted function lattice is

$$X(\omega) = \{f \in L_0;\, f\omega \in X\},$$

and $\|f\|_{X(\omega)} = \|f\omega\|_X$.

For a pair (ω_0, ω_1) of weights, we are looking for conditions under which $\bar{X} = (X(\omega_0), X(\omega_1))$ is a Marcinkiewicz pair. We begin with an example without property $\Sigma(\bar{X}^d) = \Sigma(\bar{X})^d$. Recall that X is a Köthe space if $\chi_E \in X$ for any measurable set E with finite measure.

EXAMPLE 3. Let us associate to a fixed $0 < g \in L_1$ the weights

$$\omega_0 = \sum_{n=0}^{\infty} g\chi_{(2n, 2n+1)} + \sum_{n=0}^{\infty} \chi_{(2n+1, 2n+2)}$$

and

$$\omega_1 = \sum_{n=0}^{\infty} \chi_{(2n, 2n+1)} + \sum_{n=0}^{\infty} g\chi_{(2n+1, 2n+2)}.$$

Then $(L_1(\omega_0), L_1(\omega_1))$ is a pair of Köthe spaces such that $1 \in (L_1(\omega_0) + L_1(\omega_1))^d$ but $1 \notin L_1(\omega_0)^d + L_1(\omega_1)^d$.

To prevent such examples we need to impose some restrictions to the behaviour of $\sigma = \omega_1/\omega_0$ at 0 and at ∞. We say that it has *property* 0 if

$$\liminf_{x \to 0} \sigma(x) = 0 \quad \text{and} \quad \limsup_{x \to 0} \sigma(x) = \infty,$$

and *property* ∞ if

$$\liminf_{x \to \infty} \sigma(x) = 0 \quad \text{and} \quad \limsup_{x \to \infty} \sigma(x) = \infty.$$

In Example 3, σ has property ∞.

THEOREM 6. *If $X(\omega_j)$ $(j = 0, 1)$ are Köthe spaces and $\sigma = \omega_1/\omega_0$ has neither property 0 nor property ∞, then $(X(\omega_0) + X(\omega_1))^d = X(\omega_0)^d + X(\omega_1)^d$.*

PROOF. If $f \in (X(\omega_0) + X(\omega_1))^d$, it is known that

$$K(f, t) \simeq \|f\omega_0 \min(1, t\sigma)\|_X \simeq \|f\omega_0 \chi_{\{t\sigma > 1\}}\|_X + \|f\omega_1 \chi_{\{t\sigma \leqq =\}}\|_X. \tag{6}$$

Since σ does not have properties 0 and ∞, there exist $0 < \delta_0 \leqq \delta_1$ and $0 < c_0, c_1 < \infty$ such that at least one of the following facts is true:

(a) $\sigma(x) > c_0$ if $0 \leqq x < \delta_0$, and $\sigma(x) \leqq c_1$ if $\delta_1 \leqq x < \infty$
(b) $\sigma(x) \leqq c_0$ if $0 \leqq x < \delta_0$, and $\sigma(x) > c_1$ if $\delta_1 \leqq x < \infty$
(c) $\sigma(x) \leqq c_0$ if $0 \leqq x < \delta_0$, and $\sigma(x) \leqq c_1$ if $\delta_1 \leqq x < \infty$
(d) $\sigma(x) > c_0$ if $0 \leqq x < \delta_0$, and $\sigma(x) > c_1$ if $\delta_1 \leqq x < \infty$.

In every case it is shown that $f \in X(\omega_0)^d + X(\omega_1)^d$ by taking a suitable decomposition $f = f_0 + f_1$. In case (a), the functions

$$f_0 = [f - f(\delta_1)]\chi_{(0, \delta_1)} \quad \text{and} \quad f_1 = f(\delta_1)\chi_{(0, \delta_1)} + f\chi_{(\delta_1, \infty)}$$

are decreasing and, since $X(\omega_0)$ is a Köthe space, if we apply (6) with $t = 1/c_0$, we obtain

$$\|f_0\|_{X(\omega_0)} \leqq C(K(f, c_0^{-1}) + f(\delta_0)\|\chi_{[\delta_0,\delta_1]}\|_{X(\omega_0)}) < \infty$$

In the same way we obtain $\|f_1\|_{X(\omega_1)} < \infty$.

The other three cases are similar. In case (c) we consider

$$f_0 = (f - f(\delta_1))\chi_{[\delta_0,\delta_1]} + (f(\delta_0) - f(\delta_1))\chi_{(0,\delta_0)}$$

and

$$f_1 = [f - (f(\delta_0) - f(\delta_1))]\chi_{(0,\delta_0)} + f(\delta_1)\chi_{[\delta_0,\delta_1]} + f\chi_{(\delta_1,\infty)}.$$

If for the couple $\bar{X} = (X(\omega_0), X(\omega_1))$ of Theorem 6 we have $K(f,t) \simeq K^d(f,t)$ over all functions $f \in \Sigma(\bar{X}^d)$, since $K(\chi_{(0,s)}, t) \simeq \|\chi_{(0,s)} \min(\omega_0, t\omega_1)\|_X$ (cf. (6)) and $K^d(\chi_{(0,s)}, t) = \min_{0 \leqq \varepsilon \leqq 1}(\varepsilon \|\chi_{(0,s)}\omega_0\|_X + t(1-\varepsilon)\|\chi_{(0,s)}\omega_1\|_X)$, the condition

$$\|\chi_{(0,s)} \min(\omega_0, t\omega_1)\|_X \simeq \min(\|\chi_{(0,s)}\omega_0\|_X, t\|\chi_{(0,s)}\omega_1\|_X) \tag{7}$$

is necessary for $\bar{X}$ to be a Marcinkiewicz pair.

If $X = L_1$, for any pair (ω_0, ω_1) of decreasing weights of the class $\mathscr{A}_1$ of Muckenhoupt, property (7) is always true, since in this case (cf. [GR])

$$s \min\left(\frac{1}{s}\int_0^s \omega_0, \frac{t}{s}\int_0^s \omega_1\right) \leqq Cs \min(\omega_0(s), t\omega_1(s)) \leqq C\int_0^s \min(\omega_0, t\omega_1).$$

THEOREM 7. *Let (ω_0, ω_1) be a couple of locally integrable weights of class $\Delta_2(\omega_j(2t) \leqq C\omega_j(t))$, with property* (7) *and such that $\sigma = \omega_1/\omega_0$ has neither property* 0 *nor property* ∞. *Then $(L_1(\omega_0), L_1(\omega_1))$ is a Marcinkiewicz pair.*

PROOF. In this case the Lorentz spaces $\Lambda(\omega_j) = \{f \in L_0; \int_0^\infty f^*(s)\omega_j(s)\,ds < \infty\}$ are quasi-normed spaces and, if f and g are decreasing,

$$\|f + g\|_{\Lambda(\omega_0)+\Lambda(\omega_1)} \leqq \|f\|_{\Lambda(\omega_0)+\Lambda(\omega_1)} + \|g\|_{\Lambda(\omega_0)+\Lambda(\omega_1)}.$$

To prove that $K(f,t) \simeq K^d(f,t)$ we can suppose $f = \sum_{j=0}^\infty \alpha_j \chi_{[0,x_j)}$, simple and decreasing. Then

$$\begin{aligned} K(f,t) &\simeq \int_0^\infty f(s)\min(\omega_0(s), t\omega_1(s))\,ds = \sum_{j=0}^n \alpha_j \int_0^{x_j} \min(\omega_0, t\omega_1) \\ &\simeq \sum_{j=0}^n \alpha_j \min\left(\int_0^{x_j}\omega_0, t\int_0^{x_j}\omega_1\right) \simeq \sum_{j=0}^n K(\chi_{[0,x_j)}, t; L_1(\omega_0)^d, L_1(\omega_1)^d) \\ &\simeq \sum_{j=0}^n K(\chi_{(0,x_j)}, t; \Lambda(\omega_0)^d, \Lambda(\omega_1)^d) \simeq \sum_{j=0}^n K(\chi_{(0,x_j)}, t; \Lambda(\omega_0), \Lambda(\omega_1)) \\ &\geqq K(f, t; \Lambda(\omega_0), \Lambda(\omega_1)) \simeq K^d(f,t). \end{aligned}$$

REMARK 1. If (ω_0, ω_1) is a pair of decreasing weights, we can use the fact

$$(\Lambda(\omega_0), \Lambda(\omega_1))_{\vartheta,1} = \Lambda[(\Phi_0^{1-\vartheta}\Phi_1^{\vartheta})']$$

(cf. [Me]) to obtain

$$(L_1(\omega_0)^d, L_1(\omega_1)^d)_{\vartheta,1} = \Lambda[(\Phi_0^{1-\vartheta}\Phi_1^{\vartheta})']^d = L_1[(\Phi_1^{1-\vartheta}\Phi_1^{\vartheta})']^d.$$

REMARK 2. Since

$$K(f^p, t^p; L_1(\omega_0), L_1(\omega_1))^{1/p} \simeq K(f, t; L_p(\omega_0), L_p(\omega_1))$$

and

$$K(f^p, t^p; L_1(\omega_0)^d, L_1(\omega_1)^d)^{1/p} \simeq K(f, t; L_p(\omega_0)^d, L_p(\omega_1)^d),$$

Theorem 7 can be easily extended to the case $X = L_p$ $(1 \leqq p < \infty)$.

REMARK 3. For any quasi-Banach function lattice X, (X, L_∞) is a Marcinkiewicz pair, since in this case it is routine to obtain $(X + L_\infty)^d \subset X^d + L_\infty^d$, and to prove the equivalence $K^d(f, t) \simeq K(f, t)$ for $f \in (X + L_\infty)^d$ we can consider

$$E^d(f, t) = \inf\{\|f_1\|_{X_1}; f = f_0 + f_1, f_i \in X_i^d, \|f_0\|_{X_0} \leqq t\}.$$

Then $K^d(f,t) \simeq K(f,t)$ if and only if $E^d(f,t) \simeq E(f,t)$ and in our case $E(f,t) \simeq \|(f - t)_+\|_X \simeq E^d(f,t)$.

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