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A NEW PROOF OF THE FUNDAMENTAL LEMMA OF INTERPOLATION THEORY

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Introduction.

One of the basic results in the real method of interpolation is the equivalence of the J-method and the K-method. The crucial step in the proof of this is the so called fundamental lemma. This is usually proved by using the discrete versions of the real methods. In this note we give a direct proof of the fundamental lemma using only the continuous methods. Our method also gives an improved estimate for the constant.

Results.

1. We shall in the following use the notations of [BL]. We shall prove the following

THEOREM. *Let $\bar{X}$ be a Banach couple and let $\varepsilon > 0$ and $x \in E(\bar{X})$ be given, such that $\min(1, 1/t)K(t, x) \to 0$ when $t \to 0, \infty$. There exists then a measurable $\Delta(\bar{X})$-valued function $u(t), 0 < t < \infty$, such that for all t, $J(t, u(t)) \leqq \frac{e}{2}(1 + \varepsilon)K(t, x)$ and furthermore*

$$(*) \qquad \int_0^\infty u(t)\frac{dt}{t} = x.$$

In order to prove the theorem we shall need some notations.

Given x and $\bar{X}$ as in the theorem a pair $\kappa = \kappa(t) = (a_0(t), a_1(t))$ of (measurable) functions will be called a *K-decomposition* of x if for all t, $0 < t < \infty$, $a_0(t) \in X_0$, $a_1(t) \in X_1$ and $a_0(t) + a_1(t) = x$. Given κ we define

$$k(t, x, \kappa) = \|a_0(t)\|_0 + t\,\|a_1(t)\|_1.$$

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We shall then say that κ is *C-bounded* if $\sup_{0<t<\infty} k(t,x,\kappa)/K(t,x) \leqq C$ and we shall say that κ is bounded if it is C-bounded for some C.

To prove out theorem we shall need two lemmas starting with

LEMMA 1. *Let x and $\bar{X}$ be as in the theorem and let κ be a bounded K-decomposition of x. Let further $u(t) = \frac{1}{2}(a_0(et) - a_0(t/e))$. Then*

$$\int_r^R u(t)\frac{dt}{t} = x - \tfrac{1}{2}\left(\int_{r/e}^{re} a_0(t)\frac{dt}{t} + \int_{R/e}^{Re} a_1(t)\frac{dt}{t}\right).$$

PROOF. Since $a_0(et) - a_0(t/e) = a_1(t/e) - a_1(et)$ we have

$$\int_r^R u(t)\frac{dt}{t} = \tfrac{1}{2}\left(\int_r^1 (a_0(et) - a_0(t/e))\frac{dt}{t} + \int_1^R (a_1(t/e) - a_1(et))\frac{dt}{t}\right)$$

$$= \tfrac{1}{2}\left(-\int_{r/e}^{er} a_0(t)\frac{dt}{t} + \int_{1/e}^{e} (a_0(t) + a_1(t))\frac{dt}{t} - \int_{R/e}^{eR} a_1(t)\frac{dt}{t}\right)$$

$$= x - \tfrac{1}{2}\left(\int_{r/e}^{er} a_0(t)\frac{dt}{t} + \int_{R/e}^{eR} a_1(t)\frac{dt}{t}\right) = x - I_r - I_R.$$

While our first lemma was used to find a function having the correct integral the purpose of our seond will be to show that if we start from a good K-decomposition, then we get a good estimate for $J(t,u(t))$.

LEMMA 2. *Let x, $\bar{X}$ and ε be as in the theorem and let κ be a $(1+\varepsilon)$-bounded K-decomposition of x. Let further $u(t) = \frac{1}{2}(a_0(et) - a_0(t/e))$ be as in lemma 1. Then $J(t,u(t)) \leqq (1+2\varepsilon)\dfrac{e}{2}K(t,x)$.*

PROOF. For the proof we shall use the fact that if we have $x = x_0 + x_1$, where $x_k \in X_k$ then for all t, $\|x_0\|_0 + t\|x_1\|_1 \geqq K(t,x)$. We shall use the following linear inequalities.

(1) $$\|a_0(t/e)\|_0 + t/e\,\|a_1(t/e)\|_1 \leqq (1+\varepsilon)K(t/e,x)$$

(2) $$\|a_0(et)\|_0 + et\,\|a_1(et)\|_1 \leqq (1+\varepsilon)K(et,x)$$

(3) $$\|a_0(t/e)\|_0 + t\,\|a_1(t/e)\|_1 \geqq K(t,x)$$

(4) $$\|a_0(et)\|_0 + t\,\|a_1(et)\|_1 \geqq K(t,x)$$

We shall also need the following standard (and obvious) monotonicity and concavity properties of the K-function.

(5) $$eK(t/e,x) + K(et,x) \leqq (e+1)K(t,x)$$

(6) $$K(t/e,x) \leqq K(t,x)$$

$$(7) \qquad K(et,x) \leqq eK(t,x)$$

Combining these inequalities we get from the expression $e[1] + (e-1)[2] - [3] + (1+\varepsilon)[5] + (1+\varepsilon)[7]$ the inequality

$$(8) \qquad (e-1)(\|a_0(t/e)\|_0 + \|a_0(et)\|_0 + et\,\|a_1(et)\|_1) \leqq (e-1)e\left(1+\varepsilon+\frac{\varepsilon}{e(e-1)}\right)K(t,x)$$

and from $e(e-1)[1] + [2] - [4] + (1+\varepsilon)[5] + (1+\varepsilon)e(e-2)[6]$ the inequality

$$(9) \qquad (e-1)(e\,\|a_0(t/e)\|_0 + t\,\|a_1(t/e)\|_1 + t\,\|a_1(et)\|_1) \leqq (e-1)e\left(1+\varepsilon+\frac{\varepsilon}{e(e-1)}\right)K(t,x).$$

Together [8] and [9] imply that $J(t,u(t)) \leqq \frac{e}{2}\left(1+\varepsilon+\frac{\varepsilon}{e(e-1)}\right)K(t,x)$ and this proves the lemma.

To complete the proof of the theorem we finally observe that $\|I_r\|_0 \leqq (1+\varepsilon)K(er,x)$ and that similarly $\|I_R\|_1 \leqq (1+\varepsilon)(e/R)K(R/e,x)$, so that (by the assumption on x) the integral (*) is convergent.

Lower bounds.

It seems that the best known lower bound for the fundamental lemma is $\frac{1}{2}$. The simplest instance of this is the case of a 1-dimensional Banach couple.

PROPOSITION 1. *Let X be a 1-dimensional Banach couple and let $x \in \Sigma(\bar{X})$, with $\|x\|_0 = a$, $\|x\|_1 = b$. Then*

$$x = \int_0^\infty u(t)\frac{dt}{t}$$

where $J(t,u(t),\bar{X}) = \frac{1}{2}K(t,x,\bar{X})$.

PROOF. We define $u(t) = (\frac{1}{2}K(t,x,\bar{X})/J(t,x,\bar{X}))\cdot x$. It is then clear that $J(t,u(t))$ is what it should so it remains to compute the integral. Towards this we observe that $J(t,x) = \max(a,bt)$ (for some $a>0, b>0$) while $K(t,x) = \min(a,bt)$. Therefore $u(t) = \frac{1}{2}\min((bt/a),(a/bt))\cdot x$, so that

$$\int_0^\infty u(t)\frac{dt}{t} = \frac{x}{2}\left(\frac{b}{a}\int_0^{a/b} dt + \frac{a}{b}\int_{a/b}^\infty \frac{dt}{t^2}\right) = \tfrac{1}{2}x(1+1) = x$$

and this proves the proposition.

It should be observed that since $\bar{X}$ is 1-dimensional, there is no other freedom in the choice of $u(t)$ than how large it should be. It is therefore obvious from the construction that $\frac{1}{2}$ is the best possible constant for this example.

We shall in the following say that "FL (the Fundamental Lemma) holds with constant C" for an element $x \in \Sigma(\bar{X})$ if there exists (for every $\varepsilon > 0$) a representation

$$(*) \qquad x = \int_0^\infty u(t) \frac{dt}{t}$$

with $J(t, u(t)) < (C + \varepsilon) K(t, x)$. We shall likewise say that FL holds with constant C in the couple $\bar{X} = (X_0, X_1)$ if it holds for all $x \in \Sigma(\bar{X})$.

We shall conclude by giving a condition (satisfied by a large class of Banach couples, including in particular the classical case of (L^1, L^∞)) that guarantees that FL holds with constant $\frac{1}{2}$. Towards this we shall need the following

DEFINITION 1. (i) An element $\alpha \in \Delta(\bar{X})$ is called

a) a *K-atom* if $K(t, a) = \min(a, bt)$ (for some $a > 0$, $b > 0$).

b) an *ε-approximative K-atom* if $(1 - \varepsilon)\min(a, bt) \leqq K(t, a) \leqq \min(a, bt)$ (for all t and some $a > 0$, $b > 0$).

(ii) A Banach couple $\bar{X}$ is called approximately atomic if for every $\varepsilon > 0$ and every $x \in \Sigma(\bar{X})$ there exists a decomposition $x = \sum_{k=1}^\infty a_k$ in such a way that each u_k is an ε-approximative K-atom and such that furthermore $K(t, x, \bar{X}) \leqq (1 + \varepsilon) \sum K(t, u_k)$.

In terms of definition 1, we have now

PROPOSITION 2. *If $\bar{X}$ is approximately atomic, then* FL *holds in $\bar{X}$ with constant* $\frac{1}{2}$.

PROOF. We first observe that if a is an ε-approximative K-atom then

$$\frac{a}{2} \int_0^\infty (K(t, a)/J(t, a))\, dt = (1 - \gamma) a$$

(with $0 \leqq \gamma \leqq \varepsilon$), so that FL holds with constant $(1 - \varepsilon)^{-1}/2$ for a. Let now $\varepsilon > 0$ and $x \in \Sigma(\bar{X})$ be given. We write $x = \sum a_k$ where all a_k are ε-approximative K-atoms. Defining now $u(t) = \sum u_k(t)$ it follows immediately that FL holds with constant $\frac{1}{2} + \varepsilon$ for x and since ε and x are arbitrary FL holds with constant $\frac{1}{2}$ for $\bar{X}$.

REMARK. It is worth observing that in order to prove that FL holds with constant C in a couple $\bar{X}$ it suffices to prove that it holds on a dense subspace.

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REFERENCES

[LP] J. L. Lions, J. Peetre, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Études Sci. Publ. Math. 19 (1964), 5–68.

[BL] J. Berg, J. Löfström, *Interpolation Spaces*, Grundlehren Math. Wiss. 223, 1976.

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