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ISOMORPHISM BETWEEN THE ASSOCIATIVE AND NON-ASSOCIATIVE L_p-SPACES OF TYPE II_1 HYPERFINITE FACTORS

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Abstract.

We prove that for each $p \in [1, \infty]$ the non-associative L_p-space associated with any type II_1 hyperfinite JW-factor is isomorphic to the non-commutative L_p-space associated with the type II_1 hyperfinite factor.

0. Introduction.

It is one of the interesting problems of the theory of non-commutative integration, and as well of non-associative integration, to study linear-topological properties of different classes of Banach spaces, associated with von Neumann and Jordan algebras. The L_p-spaces associated with injective factors which are, perhaps, the most important objects in the theory of these algebras, present a special interest. The most well-known objects here are the L_p-spaces associated with the type I_∞ factor, which are nothing but the von Neumann-Schatten p-classes C_p, whose geometric structure has been studied very intensively during the last thirty years. The structure of L_p-spaces associated with non-atomic (non-commutative) von Neumann algebras are less well-known. It appears that the study of their isomorphic structure requires new techniques (see, for example [SC], [FS 1-3], [Su].

Weakly closed Jordan algebras of self-adjoint operators, the (so-called) JW-algebras, present a non-associative real counterpart to von Neumann algebras (see for details [HS], [Io1], [Ayu 1]). The corresponding non-associative integration theory was developed in [Io1], [Io2], [Ayu 1]. In particular, the "non-associative" analogue of Yeadon's description [Ye] of isometries between two non-commutative L_p-spaces, was established in [Ayu 2] and [AA] and it follows that the only possibility for two non-associative L_p-spaces $(p \neq 2)$

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to be isometrically isomorphic is to have Jordan isomorphic associated JW-algebras. It follows, for example, that there exist two *non-isometric* non-associative L_p-spaces for type II_1 hyperfinite JW-factors (see details in Section 2 below).

In the present paper, we study the linear-topological structure of L_p-spaces associated with type II_1 hyperfinite factors and show that, in this case, these two (a priory different) classes of non-commutative and non-associative L_p-spaces coincide as linear topological spaces. The main technical tool of the paper is a certain total and minimal system (D-system or non-commutative Walsh system) in the non-commutative L_p-spaces which is a natural generalization of the classical Walsh system in the usual function spaces $L_p(0, 1)$. These systems were introduced in [FS 1] as the eigenvectors of a representation of the dyadic group D and appear to be useful in the study of geometry of L_p-spaces associated with the type II_1 hyperfinite factor R. For instance, taken in the so-called Walsh-Paley ordering, the form (Schauder) bases in all $L_p(R)$ spaces for $1 < p < \infty$ [FS 1,2] and this appears to be the first example of a basis in such spaces for $p \neq 2$. Another application of these systems can be found in [FS 3], where it was proved that the homotopic type of general linear groups of $L_p(R)$ is trivial for all $1 < p < \infty$. The non-commutative Walsh system used in the present text is closely connected with the D-system from [FS 1], Corollary 8, however there is some difference in the methods of their constructions. Namely, we prefer here to define the non-commutative Walsh system via the corresponding non-commutative Rademacher system (a set of its multiplicative generators) which forms a spin system of R. We construct these systems and study their properties in Section 1 and consider connections with non-associative L_p-spaces of hyperfinite Jordan factors of type II_1 in Section 2 below.

We use the terminology of the theory of von Neumann algebras and non-commutative integration (respectively, JW-algebras and non-associative integration) from [Sa], [Ta], [FK] (respectively from [Ayul]). The existence of a linear-topological isomorphism between Banach spaces X and Y will be denoted by $X \approx Y$.

1. Non-commutative Walsh system and auxiliaries.

Let M be the algebra of complex 2×2 matrices with normalized trace tr and the identity $\mathbf{1}_M$, and let $R = \otimes_{i=1}^{\infty} (M, \mathrm{tr})$ be the infinite tensor product (see [Sa]) with the faithful normal finite trace $\tau = \otimes_{i=1}^{\infty} \mathrm{tr}$. It is well-known that R is the unique hyperfinite type II_1 von Neumann factor. The finite dimensional von Neumann factors $\otimes_{i=1}^{n} (M, \mathrm{tr}) \otimes \otimes_{i=n+1}^{\infty} (\mathrm{C} \cdot \mathbf{1}_M, \mathrm{tr})$, $n \in \mathrm{N}$, will be denoted by R_n. The symbol $\mathbf{1}$ henceforth means the identity of the algebra R.

Throughout the paper, let D be the dyadic group, i.e. the group $\prod_{n=1}^{\infty} \mathbf{Z}_2$, where $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$, and $\hat{D}$ denotes the dual group. Recall that each $\gamma \in \hat{D}$ can be identified with an eventually zero sequence $(\gamma_n)_{n=1}^{\infty}, \gamma_n \in \mathbf{Z}_2$ such that for every $t = (t_n)_{n=1}^{\infty} \in D$ one has $\gamma(t) = \prod_{n=1}^{\infty} \gamma_n(t_n)$. Put $D_n^* = \{\gamma \in \hat{D} \mid \gamma_i = 0 \text{ for every } i > n\}$ and $A_\gamma = \{n \in \mathbf{N} \mid \gamma_n = 1\}$.

Consider the system of generators of the complex Lie group SU(2), which are known as Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{pmatrix},$$

and define the "anticommutative Rademacher system" in R as follows:

$$r_0 = \mathbf{1}, \; r_1 = \sigma_1 \otimes \bigotimes_{i=2}^{\infty} \mathbf{1}_M, \; r_2 = \sigma_2 \otimes \bigotimes_{i=2}^{\infty} \mathbf{1}_M,$$

(1) and for $n > 2$

$$r_n = \begin{cases} \bigotimes_{i=1}^{\frac{n-1}{2}} \sigma_3 \otimes \sigma_1 \otimes \bigotimes_{i=\frac{n+3}{2}}^{\infty} \mathbf{1}_M \\ \bigotimes_{i=1}^{\frac{n}{2}-1} \sigma_3 \otimes \sigma_2 \otimes \bigotimes_{i=\frac{n}{2}+1}^{\infty} \mathbf{1}_M \end{cases}$$

The next lemma shows that the above system is really anticommutative and forms a concrete spin system.

LEMMA 1. *The system $(r_n)_{n=1}^{\infty} \subset R$ has the following properties:*

(i) $r_n r_m = -r_m r_n$ *if* $n \neq m$; *and* $r_n^2 = \mathbf{1}$ *for all* $n, m \in \mathbf{N}$;

(ii) $r_n^* = r_n$ *for every* $n \in \mathbf{N}$.

PROOF. (i) We can consider inly the case $n > m$ that falls into two subcases:
a) m is odd and $n = m + 1$;
b) $n = 2k + s$, $m = 2l + t$, $k > l$ and $s, t \in \{1, 2\}$.
In the case a) we have

$$r_n r_m = r_{m+1} r_m =$$

$$\Big(\bigotimes_{i=1}^{\frac{m+1}{2}-1} \sigma_3 \otimes \sigma_2 \otimes \bigotimes_{i=\frac{m+1}{2}+1}^{\infty} \mathbf{1}_M\Big) \cdot \Big(\bigotimes_{i=1}^{\frac{m-1}{2}} \sigma_3 \otimes \sigma_1 \otimes \bigotimes_{i=\frac{m+3}{2}}^{\infty} \mathbf{1}_M\Big) =$$

$$\Big(\bigotimes_{i=1}^{\frac{m-1}{2}} \sigma_3^2 \otimes (\sigma_2 \sigma_1) \otimes \bigotimes_{i=\frac{m+3}{2}}^{\infty} \mathbf{1}_M\Big) = -\Big(\bigotimes_{i=1}^{\frac{m-1}{2}} \sigma_3^2 \otimes (\sigma_1 \sigma_2) \otimes \bigotimes_{i=\frac{m+3}{2}}^{\infty} \mathbf{1}_M\Big) =$$

$$-\Big(\bigotimes_{i=1}^{\frac{m-1}{2}} \sigma_3 \otimes \sigma_1 \otimes \bigotimes_{i=\frac{m+3}{2}}^{\infty} \mathbf{1}_M\Big) \cdot \Big(\bigotimes_{i=1}^{\frac{m-1}{2}} \sigma_3 \otimes \sigma_2 \otimes \bigotimes_{i=\frac{m+3}{2}}^{\infty} \mathbf{1}_M\Big) =$$

$$-r_m r_{m+1} = -r_m r_n,$$

and in the case b) we have

$$r_n r_m = (\otimes_{i=1}^{k} \sigma_3 \otimes \sigma_s \otimes \otimes_{i=k+2}^{\infty} \mathbf{1}_M)\cdot(\otimes_{i=1}^{l} \sigma_3 \otimes \sigma_t \otimes \otimes_{i=l+2}^{\infty} \mathbf{1}_M) =$$
$$(\otimes_{i=1}^{l} \sigma_3 \otimes \otimes_{i=l+1}^{k} \sigma_3 \otimes \sigma_s \otimes \otimes_{i=k+2}^{\infty} \mathbf{1}_M)\cdot(\otimes_{i=1}^{l} \sigma_3 \otimes \sigma_t \otimes \otimes_{i=l+2}^{\infty} \mathbf{1}_M) =$$
$$\otimes_{i=1}^{l} \sigma_3^2 \otimes (\sigma_3\sigma_t) \otimes \otimes_{i=l+2}^{k} \sigma_3 \otimes \sigma_s \otimes \otimes_{i=k+2}^{\infty} \mathbf{1}_M =$$
$$-(\otimes_{i=1}^{l} \sigma_3^2 \otimes (\sigma_t\sigma_3) \otimes \otimes_{i=l+2}^{k} \sigma_3 \otimes \sigma_s \otimes \otimes_{i=k+2}^{\infty} \mathbf{1}_M) =$$
$$-(\otimes_{i=1}^{l} \sigma_3 \otimes \sigma_t \otimes \otimes_{i=l+2}^{\infty} \mathbf{1}_M)\cdot(\otimes_{i=1}^{k} \sigma_3 \otimes \sigma_s \otimes \otimes_{i=k+2}^{\infty} \mathbf{1}_M) = -r_m r_n.$$

This proves the first part of our claim. For the other part it is sufficient to note that if $n = 2k + s$, where $s \in \{1, 2\}$, then

$$r_n^2 = (\otimes_{i=1}^{k} \sigma_3 \otimes \sigma_s \otimes \otimes_{i=k+2}^{\infty} \mathbf{1}_M)^2 = \otimes_{i=1}^{k} \sigma_3^2 \otimes \sigma_s^2 \otimes \otimes_{i=k+2}^{\infty} \mathbf{1}_M = \otimes_{i=1}^{\infty} = \mathbf{1}.$$

Notice that we repeatedly used the properties $\sigma_i\sigma_j = -\sigma_j\sigma_i$, $\sigma_i^2 = \mathbf{1}_M$ for all $i \neq j$, $i, j \in \{1, 2, 3\}$.

(ii) Again for $n = 2k + s$, where $s \in \{1, 2\}$, one has

$$r_n^* = (\otimes_{i=1}^{k} \sigma_3 \otimes \sigma_s \otimes \otimes_{i=k+2}^{\infty} \mathbf{1}_M)^* =$$
$$\otimes_{i=1}^{k} \sigma_3^* \otimes \sigma_s^* \otimes \otimes_{i=k+2}^{\infty} \mathbf{1}_M = \otimes_{i=1}^{k} \sigma_3 \otimes \sigma_s \otimes \otimes_{i=k+2}^{\infty} \mathbf{1}_M = r_n.$$

Now we have a Rademacher system in R and it is natural to define the Walsh system in R, for which this Rademacher system is a system of multplicative generators.

DEFINITION. The *non-commutative Walsh system* is an operator system $\{w_\gamma\}_{\gamma \in \hat{D}}$ in R, such that

$$w_0 = \mathbf{1}, \quad w_\gamma = \prod_{n \in A_\gamma} r_n, \text{ for } 0 \neq \gamma \in \hat{D}.$$

We collect simple but important properties of this system in the following statement.

LEMMA 2. *Let $I(\gamma, \eta)$ be the number of inversions in the sequence*

$$n_1, n_2, \ldots, n_k, m_1 + \tfrac{1}{2}, m_2 + \tfrac{1}{2}, \ldots, m_l + \tfrac{1}{2},$$

where $A_\gamma = \{n_1 < n_2 < \ldots < n_k\}$, and $A_\eta = \{m_1 < m_2 < \ldots < m_l\}$.
Then $w_\gamma w_\eta = (-1)^{I(\gamma,\eta)} w_{\gamma+\eta}$;
(ii) *$\tau(w_0) = 1$, and $\tau(w_\gamma) = 0$ for each $0 \neq \gamma \in \hat{D}$;*
(iii) *$w_\gamma = w_\eta$ if and only if $\gamma = \eta$;*
(iv) *the system $\{w_\gamma\}_{\gamma \in D_{2n}^*}$ is the basis of the linear space R_n;*
(v) *for every $\gamma \in \hat{D}$, $k =$ Card A_γ, one has $w_\gamma^* = (-1)^{\frac{k(k-1)}{2}} w_\gamma$;*
(vi) *the system $\{w_\gamma\}_{\gamma \in \hat{D}}$ is a total and minimal system in every $L_p(R)$, $1 \leqq p < \infty$.*

PROOF. (i) First, observe that if Card $A_\gamma =$ Card $A_\eta = 1$, then our claim is nothing but the statement (i) of Lemma 1. The general case can be easily deduced from it – one should use the double induction: by Card A_γ and by Card A_η.

(ii) The first part of the statement is trivial. Consider some $0 \neq \gamma \in \hat{D}$ and let $A_\gamma = \{n_1 < n_2 < \ldots < n_k\}$. Recall that by Lemma 1 for every $m \in \mathbb{N}$ the operator r_m is a unitary operator, therefore $\tau(r_m^* w_\gamma r_m) = \tau(r_m w_\gamma r_m) = \tau(w_\gamma)$. Then, if k is an even number we have

$$\tau(w_\tau) = \tau(r_{n_1} w_\gamma r_{n_1}) = \tau(-w_\gamma) = -\tau(w_\gamma),$$

because the number of inversions in the sequence $n_1, n_1 + \frac{1}{2}, n_2 + \frac{1}{2}, \ldots, n_k + \frac{1}{2}$ is 0, and the number of inversions in the sequence $n_2, \ldots, n_k, n_1 + \frac{1}{2}$ is $k-1$. Therefore the statement (i) of this lemma implies that $r_{n_1} w_\gamma r_{n_1} = -w_\gamma$. It follows that $\tau(w_\gamma) = 0$ if k is even. Similarly, if k is odd then for some $m > n_k$ we obtain

$$\tau(w_\gamma) = \tau(r_m w_\gamma r_m) = \tau(-w_\gamma) = -\tau(w_\gamma),$$

i.e. $\tau(w_\gamma) = 0$ in this case as well.

(iii) Let $w_\gamma = w_\eta$. Then $w_\gamma w_\eta$ and by point (i) of this lemma it means that

$$(-1)^{I(\gamma,\gamma)} w_0 = (-1)^{I(\gamma,\eta)} w_{\gamma+\eta}.$$

Since the trace of the left side of the equality is non-zero, the trace of the right side should be also non-zero. By point (ii) of this lemma it is possible if and only if $\gamma + \eta = 0$, therefore $\gamma = \eta$.

(iv) Since the dimension of the linear space R_n is 4^n and Card $\{w_\gamma\}_{\gamma \in D_{2n}^*}$ is also 4^n, it is sufficient to show that if $\sum_{\gamma \in D_{2n}^*} a_\gamma w_\gamma = 0$ then $a_\gamma = 0$ for each element $\gamma \in D_{2n}^*$.
But

$$0 = \tau((\textstyle\sum_{\gamma \in D_{2n}^*} a_\gamma w_\gamma) w_\eta) = \tau(\textstyle\sum_{\gamma \in D_{2n}^*} a_\gamma w_\gamma w_\eta) =$$
$$\textstyle\sum_{\gamma \in D_{2n}^*} \tau(a_\gamma (-1)^{I(\gamma,\eta)} w_{\gamma+\eta}) = a_\eta (-1)^{I(\eta,\eta)},$$

therefore $a_\eta = 0$ for every $\eta \in D_{2n}^*$

(v) Let $A_\gamma = \{n_1 < n_2 < \ldots < n_k\}$. By the statement (i) of the present lemma we can write

$$w_\gamma^* = r_{n_k}^* r_{n_{k-1}}^* \ldots r_1^* = (-1)^{k-1} r_{n_{k-1}} \ldots r_1 r_{n_k} =$$
$$(-1)^{k-1}(-1)^{k-2} r_{n_{k-2}} \ldots r_1 r_{n_{k-1}} r_{n_k} = \ldots =$$
$$(-1)^{(k-1)+(k-2)+\ldots+1} r_1 r_2 \ldots r_{n_k} = (-1)^{\frac{k(k-1)}{2}} w_\gamma.$$

(vi) It is easy to see that the set $\bigcup_{n=1}^\infty R_n$ is dense in the norm topology in every space $L_p(R)$, $1 \leqq p < \infty$. Now (iv) implies that the system $\{w_\gamma\}_{\gamma \in \hat{D}}$ is total in the mentioned spaces. Suppose that $\{w_\gamma\}_{\gamma \in \hat{D}}$ is not minimal in some $L_p(R)$. Then there exist $n \in \mathbb{N}$, $\eta \in \hat{D}_n$ and $\{a_\gamma\}_{\gamma \neq \eta} \subset \mathbb{C}$ such that $\|w_\eta - \sum_{\gamma \neq \eta} a_\gamma w_\gamma\|_{L_p(R)} \leqq \frac{1}{2}$, hence $\|w_\eta(w_\eta - \sum_{\gamma \neq \eta} a_\gamma w_\gamma)\|_{L_p(R)} \leqq \frac{1}{2}$ or $\|1 - \sum_{\gamma \neq 0} b_\gamma w_\gamma\|_{L_p(R)} \leqq \frac{1}{2}$ where $b_\gamma = a_{\gamma+\eta}(-1)^{I(\eta,\gamma)+I(\eta,\eta)}$ for every $\gamma \neq 0$. Consider the *-subalgebra $Z = \mathbf{C}1$ of the factor R, and let $E^Z: L_p(R) \mapsto L_p(R)$ be conditional expectation on the subalgebra Z. Since the conditional expectation is a trace preserving operator and taking into account the statement (ii) of this lemma we can conclude that $E^Z w_\gamma = 0$ for every $\gamma \neq 0$. Since E^Z is a contractive projection we obtain

$$1 = \|\mathbf{1}\|_{L_p(R)} = \|E^Z(\mathbf{1} - \textstyle\sum_{\gamma \neq 0} b_\gamma w_\gamma)\|_{L_p(R)} \leqq \|\mathbf{1} - \textstyle\sum_{\gamma \neq 0} b_\gamma w_\gamma\|_{L_p(R)} \leqq \tfrac{1}{2}.$$

This contradiction implies that the system $\{w_\gamma\}_{\gamma \in \hat{D}}$ is minimal.

Let A be a subset of $\hat{D}$, and let χ_A be the indicator of A. The bounded operator $T(A): L_p(R) \mapsto L_p(R)$, $1 < p < \infty$, is called *a projection along A* if $T(A)w_\gamma = \chi_A(\gamma)w_\gamma$ for each $\gamma \in \hat{D}$ (note that in this case it is indeed a projection). By point (vi) of Lemma 2 if such operator exists then it is determined uniquely. The following properties of projections along sets are straightforward:

$$\begin{gathered} T(A_2)T(A_1) = T(A_1 \cap A_2) = T(A_1)T(A_2), \\ T(\hat{D}/A_1) = \mathrm{Id} - T(A_1), \\ T(A_1 \cup A_2) = T(A_1) + T(A_2) - T(A_1)T(A_2) \end{gathered} \tag{2}$$

for every $A_1, A_2 \subset \hat{D}$.

In general, there are no reasons for the projection along an arbitrary set A to exist. However, there are some important classes of subsets for which these projections always exist.

LEMMA 3. *Let G be a subgroup of $\hat{D}$. Then the following projections exist and are bounded in every space $L_p(R)$, $1 \leqq p < \infty$:*

(i) *the projection along G;*

(ii) *the projection along $\gamma + G$ for each $\gamma \in \hat{D}$.*

PROOF. (i) Let us consider the subsets

$$M_n = \{\textstyle\sum_{\gamma \in G \cap D_n^*} a_\gamma w_\gamma \mid a_\gamma \in \mathbf{C}\}, \quad n \in \mathbf{N},$$

of the factor R. By (i),(v) of Lemma 2, every M_n is a *-subalgebra of R. Let T_n: $L_p(R) \mapsto L_p(R)$ be conditional expectation on M_n. Fix some $\mu \in \hat{D}$. Since $T_n w_\mu \in M_n$, we can put $T_n w_\mu = \sum_{\rho \in G \cap D_n^*} a_{\rho\mu} w_\rho$. Properties of conditional expectations imply that

$$\tau(w_\gamma w_\mu) = \tau(w_\gamma T_n w_\mu) = \tau(w_\gamma \textstyle\sum_{\rho \in G \cap D_n^*} a_{\rho\mu} w_\rho) =$$

$$\tau(\textstyle\sum_{\rho \in G \cap D_n^*} a_{\rho\mu}(-1)^{I(\gamma,\rho)} w_{\gamma+\rho}) = (-1)^{I(\gamma,\gamma)} a_{\gamma\mu}$$

for every $\gamma \in G \cap D_n^*$. In particular, if $\mu \notin G \cap D_n^*$ then $\tau(w_\gamma w_\mu) = 0$, i.e. $a_{\gamma\mu} = 0$ for every $\gamma \in G \cap D_n^*$, so that $T_n w_\mu = 0$; and if $\mu \in G \cap D_n^*$ then $a_{\gamma\mu} = 0$ for $\gamma \neq \mu$ as before and

$$(-1)^{I(\mu,\mu)} = \tau(w_\mu w_\mu) = (-1)^{I(\mu,\mu)} a_{\mu\mu},$$

i.e. $a_{\mu\mu} = 1$, so that $T_n w_\mu = w_\mu$.

Now consider the sequence $\{T_n\}_{n=1}^\infty$. For every $x = \sum_{\gamma\in\hat{D}} x_\gamma w_\gamma$, where the sum is finite, the sequence $\{T_n x\}_{n=1}^\infty$ is eventually equal to $\sum_{\gamma\in\hat{D}} \chi_G(\gamma) x_\gamma w_\gamma$ therefore it is a convergent sequence. By point (vi) of Lemma 2 the set of all above elements x is dense in the space $L_p(R)$. So, taking into account that T_n are contractive projective projections in $L_p(R)$ we obtain that there exists a strong limit, say T, of the operator sequence $\{T_n\}_{n=1}^\infty$ and, in addition, $\|T\|_{L_p(R)\to L_p(R)} \leqq 1$. Now it is sufficient to notice that $T = T(G)$, because

$$Tw_\gamma = \lim_{n\to\infty} T_n w_\gamma = \lim_{n\to\infty} \chi_G(\gamma) w_\gamma = \chi_G(\gamma) w_\gamma.$$

(ii) It is a simple corollary from the preceding point. If $\gamma \in G$ then $T(\gamma + G) = T(G)$. In the other case consider the set $\Gamma = G \cup (\gamma + G) \in \hat{D}$ which is evidently a subgroup. Then the projection $T(\Gamma) - T(G)$ is just $T(\gamma + G)$ by (2).

Given an increasing sequence $(n_k)_{k=1}^\infty \subset \mathrm{N}$, consider the following group homomorphisms $\varphi(n_k): \hat{D} \mapsto \hat{D}$:

$$\varphi(n_k)(\gamma_1, \gamma_2, \ldots, \gamma_k, \ldots) = (0, \ldots, \gamma_1, 0, \ldots, 0, \gamma_2, 0, \ldots 0, \gamma_k, 0, \ldots),$$

where γ_k is on the n_kth place.

There are trivial but significant observation, namely

$$\text{(3)} \qquad \operatorname{Card} A_\gamma = \operatorname{Card} A_{\varphi(n_k)\gamma}; \quad I(\gamma, \mu) = I(\varphi(n_k)\gamma, \varphi(n_k)\mu).$$

Next, let $\Phi(n_k): \bigcup_{n=1}^\infty R_n \mapsto \bigcup_{n=1}^\infty R_n$, where

$$\Phi(n_k)\left(\sum_{\gamma\in D_n^*} a_\gamma w_\gamma\right) = \sum_{\gamma\in D_n^*} a_\gamma w_{\varphi(n_k)\gamma}.$$

By (3) we have

$$\Phi(n_k)(xy) = \Phi(n_k)\left(\sum_{\gamma\in D_n^*} x_\gamma w_\gamma \cdot \sum_{\mu\in D_n^*} y_\mu w_\mu\right) =$$

$$\Phi(n_k)\left(\sum_{\gamma\in D_n^*} \sum_{\mu\in D_n^*} x_\gamma y_\mu \cdot (-1)^{I(\gamma,\mu)} w_{\gamma+\mu}\right) =$$

$$\sum_{\gamma\in D_n^*} \sum_{\mu\in D_n^*} x_\gamma y_\mu (-1)^{I(\gamma,\mu)} w_{\varphi(n_k)(\gamma+\mu)} =$$

$$\sum_{\gamma\in D_n^*} \sum_{\mu\in D_n^*} x_\gamma y_\mu (-1)^{I(\varphi(n_k)\gamma, \varphi(n_k)\mu)} w_{\varphi(n_k)(\gamma+\mu)} =$$

$$\sum_{\gamma\in D_n^*} \sum_{\mu\in D_n^*} x_\gamma y_\mu w_{\varphi(n_k)\gamma} w_{\varphi(n_k)\mu} =$$

$$\sum_{\gamma\in D_n^*} x_\gamma w_{\varphi(n_k)\gamma} \sum_{\mu\in D_n^*} y_\mu w_{\varphi(n_k)\mu} =$$

$$\Phi(n_k)(x)\, \Phi(n_k)(y),$$

and putting $J(\gamma) = \text{Card}\, A_\gamma$, we obtain

$$\Phi(n_k)(x^*) = \Phi(n_k)((\sum_{\gamma \in D_n^*} x_\gamma w_\gamma)^*) = \Phi(n_k)(\sum_{\gamma \in D_n^*} \bar{x}_\gamma w_\gamma^*) =$$

$$\sum_{\gamma \in D_n^*} \bar{x}_\gamma \Phi(n_k) w_\gamma^* = \sum_{\gamma \in D_n^*} \bar{x}_\gamma \Phi(n_k)((-1)^{\frac{J(\gamma)(J(\gamma)-1)}{2}} w_\gamma) =$$

$$\sum_{\gamma \in D_n^*} \bar{x}_\gamma (-1)^{\frac{J(\varphi(n_k)\gamma)(J(\varphi(n_k)\gamma)-1)}{2}} \Phi(n_k) w_\gamma =$$

$$\sum_{\gamma \in D_n^*} \bar{x}_\gamma (-1)^{\frac{J(\varphi(n_k)\gamma)(J(\varphi(n_k)\gamma)-1)}{2}} w_{\varphi(n_k)\gamma} =$$

$$\sum_{\gamma \in D_n^*} \bar{x}_\gamma = \sum_{\gamma \in D_n^*} \bar{x}_\gamma w_{\varphi(n_k)\gamma}^* =$$

$$\sum_{\gamma \in D_n^*} \bar{x}_\gamma (\Phi(n_k) w_\gamma)^* = (\Phi(n_k) \sum_{\gamma \in D_n^*} x_\gamma w_\gamma)^* = (\Phi(n_k)(x))^*$$

for every $x = \sum_{\gamma \in D_n^*} x_\gamma w_\gamma \in \bigcup_{n=1}^\infty R_n$, $y = \sum_{\gamma \in D_n^*} y_\gamma w_\gamma \in \bigcup_{n=1}^\infty R_n$. Therefore $\Phi(n_k)$ is a well-defined $*$-homomorphism of R_n into R_{n_k} preserving the trace, so for every $x \in \bigcup R_n$ one has $\|\Phi(n_k)x\|_{L_p(R)} = \|x\|_{L_p(R)}$, $1 \leqq p < \infty$. Since the set $\bigcup R_n$ is norm-dense in the Banach space $X = L_p(R)$, this $\Phi(n_k)$ uniquely extends to an isometry of whole $L_p(R)$. In the sequel we shall denote this isometry also by $\Phi(n_k)$. Similarly, we will use the symbol $T(n_k)$ having in mind the projection along the subgroup $A = \varphi(n_k)\hat{D}$. It is straightforward that $\Phi(n_k)X = T(n_k)X$, $X = L_p(R)$.

We recall now, that the Pelczynski decomposition principle is the following statement (see, for example [LT]):

Given a Banach space X, if $X \approx l_p(X)$, $p \in [1, \infty)$ and some Y is a complemented subspace in X, and some Z is a complemented subspace in Y and besides $Z \approx X$, then $Y \approx X$.

Here

$$l_p(X) = \{(x_n)_{n=1}^\infty \mid x_n \in X, \quad \|(x_n)_{n=1}^\infty\|_{l_p(X)} = (\sum_{n=1}^\infty \|x_n\|_X^p)^{\frac{1}{p}} < \infty\}.$$

The following lemma shows that the Pelczynski decomposition principle may be applied to the Banach space $X = L_p(R)$ (see also [FS 3], Lemma 2.1).

LEMMA 4. *For every $p \in [1, \infty)$ the spaces $l_p(L_p(R))$ and $L_p(R)$ are isomorphic.*

PROOF. Let $\{p_k\}_{k=1}^\infty$ be an arbitrary family of mutually orthogonal projections from R, and let $P: L_p(R) \mapsto L_p(R)$ be the associated block projection operator which was constructed in [CKS]. It follows from [CKS] that P is a bounded projection and for every $x \in L_p(R)$ we have $Px = \sum_{k=1}^\infty p_k x p_k$. It is clear that $Y = P(L_p(R))$ is a complemented subspace of $L_p(R)$. Since the von Neumann algebras R and $p_k R p_k$ are $*$-isomorphic for every $k \in \mathbb{N}$ (see [Ye]) there exists a bijective isometry $j_k: L_p(R) \mapsto L_p(p_k R p_k)$. Consider the operator $T: l_p(L_p(R)) \mapsto Y$ defined by $T((x_k)_{k=1}^\infty) = \sum_{k=1}^\infty j_k x_k$. Since

$$\|(x_k)\|^p_{l_p(L_p(M,\tau))} = \sum_{k=1}^{\infty} \|x_k\|^p_{L_p(M,\tau)} = \|\sum_{k=1}^{\infty} j_k x_k\|^p_{L_p(M,\tau)},$$

T is an isometric map. It is easily seen that the map

$$Y \ni y \mapsto (j_k^{-1}(p_k y p_k))_{k=1}^{\infty} \in l_p(L_p(R))$$

is the inverse of T. Therefore T isometrically maps the space $l_p(L_p(R))$ onto Y. Now it is sufficient to use the Pelczynski decomposition principle for the space $l_p(L_p(R))$.

2. Non-associative L_p-spaces associated with type II_1 hyperfinite Jordan factors.

The principal objective of this section is to show that for every $p \in [1, \infty]$ all non-associative L_p-spaces, associated with type II_1 hyperfinite JW-factors are topologically isomorphic to $L_p(R)$. Complete classification of injective JW-factors was suggested in [Ayu 1]. In fact, there are only two non-isomorphic type II_1 hyperfinite JW-factors. The first, say R_1, is just the self-adjoint part of R. The more complicated one, say R_0, may be described as the set of all fixed points of R_1 under an involutive *-antiautomorphism (*involution*) of R with the induced algebraic structure. Notice that in spite of R admits a lot of involutions, the corresponding JW-factors R_0 are known to be Jordan isomorphic, because in the injective factor R all involutions are conjugated (see [St], [Gi1], [Gi2].
In order to choose a concrete involution α of R, let us consider the crossed product (see [Ta, ch.V]) $L^\infty(D) \otimes_\omega \hat{D}$ where

$$(\omega(\gamma)f)(t) = f(t + \gamma) \text{ for every } \gamma \in \hat{D} \text{ and } f \in L^\infty(D),$$

and $t + \gamma$ is the element $(t_i + \gamma_i)_{i=1}^{\infty}$ of D. By [Sa, 4.4.6] we have that $L^\infty(D) \otimes_\omega \hat{D}$ is a hyperfinite factor of type II_1 and there is a *-isomorphism $\Psi : L^\infty(D) \otimes_\omega \hat{D} \mapsto R$ such that for each $\gamma = (\gamma_i)_{i=1}^{\infty} \in \hat{D}$

$$\Psi(\pi(\gamma(t))) = \otimes_{i=1}^{\infty} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\gamma_i}, \quad \Psi(U_\gamma) = \otimes_{i=1}^{\infty} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\gamma_i}.$$

Here $\pi(f(t))$, and U_τ are standard generators of $L^\infty(D) \otimes_\omega \hat{D}$.

Let β be the canonical involution of $L^\infty(D) \otimes_\omega \hat{D}$ (see [Ayu], Ch.2, 3, example 1). Recall that if x is a finite sum $\sum \pi(f_\gamma) U_\gamma, f_\gamma \in L^\infty(D)$, then by definition $\beta(x) = \sum U_\gamma \pi(f_\gamma)$. We put for every $x \in R$:

$$\alpha(x) = \Psi \beta \Psi^{-1} x.$$

Clearly, α is an involution of R.

LEMMA 5. *For every $n \in \mathbb{N}$ one has*

$$\alpha(r_n) = \begin{cases} r_n & , \text{ if } n \equiv 1, 2 \bmod 4, \\ -r_n & , \text{ if } n \equiv 0, 3 \bmod 4. \end{cases}$$

PROOF. We shall consider only the case $n \equiv 0 \bmod 4$, because the other cases are quite similar. Let $n = 4k, k \in \mathbf{N}$. Then

$$r_n = \otimes_{i=1}^{2k-1} \sigma_3 \otimes \sigma_2 \otimes \otimes_{i=2k+1}^{\infty} \mathbf{1}_M = \otimes_{i=1}^{2k-1} (\mathbf{i}\sigma_1\sigma_2) \otimes \sigma_2 \otimes \otimes_{i=2k+1}^{\infty} \mathbf{1}_M =$$
$$(\otimes_{i=1}^{2k-1} \mathbf{i}\sigma_1 \otimes \otimes_{i=2k}^{\infty} \mathbf{1}_M) \cdot (\otimes_{i=1}^{2k} \sigma_2 \otimes \otimes_{i=2k+1}^{\infty} \mathbf{1}_M),$$

and therefore

$$\Psi^{-1} r_n = \Psi^{-1}(\otimes_{i=1}^{2k-1} i\sigma_1 \otimes \otimes_{i=2k}^{\infty} \mathbf{1}_M) \cdot \Psi^{-1}(\otimes_{i=1}^{2k} \sigma_2 \otimes \otimes_{i=2k+1}^{\infty} \mathbf{1}_M) =$$
$$\mathbf{i}^{2k-1} \pi(\gamma(t)) U_\mu,$$

where $\gamma = (1, 1, \ldots, 1_{2k-1}, 0, 0, \ldots)$ (first $2k-1$ coordinates are 1, the rest are 0), and $\mu = (1, 1, \ldots, 1_{2k}, 0, 0, \ldots)$ (first $2k$ coordinates equal to 1, others being 0). It follows that

$$\beta(\Psi^{-1} r_n) = \mathbf{i}^{2k-1} \beta(\pi(\gamma(t)) U_\mu) = \mathbf{i}^{2k-1} U_\mu \pi(\gamma(t)).$$

Finally,

$$\alpha(r_n) = \Psi \beta \Psi^{-1} r_n = \mathbf{i}^{2k-1} \Psi(U_\mu \pi(\gamma(t))) =$$
$$\Psi(\Psi^{-1}(\otimes_{i=1}^{2k} \sigma_2 \otimes \otimes_{i=2k+1}^{\infty} \mathbf{1}_M) \cdot \Psi^{-1}(\otimes_{i=1}^{2k-1} \mathbf{i}\sigma_1 \otimes \otimes_{i=2k}^{\infty} \mathbf{1}_M)) =$$
$$(\otimes_{i=1}^{2k} \sigma_2 \otimes \otimes_{i=2k+1}^{\infty} \mathbf{1}_M) \cdot (\otimes_{i=1}^{2k-1} \mathbf{i}\sigma_1 \otimes \otimes_{i=2k}^{\infty} \mathbf{1}_M) =$$
$$\otimes_{i=1}^{2k-1} (\mathbf{i}\sigma_2\sigma_1) \otimes \sigma_2 \otimes \otimes_{i=2k+1}^{\infty} \mathbf{1}_M = \otimes_{i=1}^{2k-1} (-\mathbf{i}\sigma_1\sigma_2) \otimes \sigma_2 \otimes \otimes_{i=2k+1}^{\infty} \mathbf{1}_M =$$
$$(-1)^{2k-1} \otimes_{i=1}^{2k-1} (\mathbf{i}\sigma_1\sigma_2) \otimes \sigma_2 \otimes \otimes_{i=2k+1}^{\infty} \mathbf{1}_M = -\otimes_{i=1}^{2k-1} \sigma_3 \otimes \sigma_2 \otimes \otimes_{i=2k+1}^{\infty} \mathbf{1}_M$$
$$= -r_n.$$

From now on, we set $R_0 = \{x \in R_1 : \alpha(x) = x\}$. We shall prove now the main result of the paper.

THEOREM 1. *For each* $p \in [1, \infty]$ *the spaces* $L_p(R)$, $L_p(R_0)$ *and* $L_p(R_1)$ *are isomorphic as real Banach spaces.*

PROOF. First let $p < \infty$. Below overlines denote the closure of a set in the L_p-norm topology. The main idea of the proof is to consider the following chain of real Banach spaces

$$X = L_p(R) \supset X_0 \supset X_1 \supset X_2 \supset X_3,$$

where

$$X_0 = \overline{\{\textstyle\sum_{\gamma \neq 0} a_\gamma w_\gamma \mid a_\gamma \in \mathbf{C}\}}, \quad X_1 = \overline{\{\textstyle\sum_{\gamma \neq 0} a_\gamma w_\gamma \mid a_\gamma \in \mathbf{R}\}},$$
$$X_2 = \overline{\{\textstyle\sum_{\gamma \in A_2} a_\gamma w_\gamma \mid a_\gamma \in \mathbf{R}\}}, \quad X_3 = \overline{\{\textstyle\sum_{\gamma \in A_3} a_\gamma w_\gamma \mid a_\gamma \in \mathbf{R}\}},$$

$A_2 = \{\gamma \in \hat{D} \mid \operatorname{Card} A_\gamma \equiv 0, 1 \bmod 4\}$, $A_3 = \{\gamma \in A_2 \mid \text{if } n \in A_\gamma \text{ then } n \equiv 1, 2, \bmod 4\}$,

and repeated application of the decomposition principle to show that all these spaces are isomorphic. On the other hand, it turns out that

$$X_3 \subset L_p(R_0) \subset L_p(R_1) \approx L_p(R);$$

using the decomposition principle once more, we will obtain our claim.

For each $x = \sum_{\gamma \in \hat{D}} x_\gamma w_\gamma \in \bigcup_{n=1}^{\infty} R_n$ put $\theta(x) = \sum_{\gamma \in \hat{D}} x_\gamma w_\gamma^*$. Since

$$\theta(xy) = \theta(\textstyle\sum_{\gamma \in \hat{D}} x_\gamma w_\gamma \cdot \sum_{\mu \in \hat{D}} y_\mu w_\mu) = \theta(\sum_{\gamma \in \hat{D}} \sum_{\mu \in \hat{D}} x_\gamma y_\mu (-1)^{I(\gamma,\mu)} w_{\gamma+\mu}) =$$

$$\textstyle\sum_{\gamma \in \hat{D}} \sum_{\mu \in \hat{D}} x_\gamma y_\mu (-1)^{I(\gamma,\mu)} w_{\gamma+\mu}^* = \sum_{\gamma \in \hat{D}} \sum_{\mu \in \hat{D}} x_\gamma y_\mu (w_\gamma w_\mu)^* =$$

$$\textstyle\sum_{\gamma \in \hat{D}} \sum_{\mu \in \hat{D}} x_\gamma y_\mu w_\mu^* w_\gamma^* = \sum_{\mu \in \hat{D}} y_\mu w_\mu^* \cdot \sum_{\gamma \in \hat{D}} x_\gamma w_\gamma^* = \theta(y)\theta(x),$$

$$\theta(x^*) = \theta((\textstyle\sum_{\gamma \in \hat{D}} x_\gamma w_\gamma)^*) = \theta(\sum_{\gamma \in \hat{D}} \bar{x}_\gamma w_\gamma^*) =$$

$$\textstyle\sum_{\gamma \in \hat{D}} \bar{x}_\gamma w_\gamma = (\sum_{\gamma \in \hat{D}} x_\gamma w_\gamma^*)^* = \theta(\sum_{\gamma \in \hat{D}} x_\gamma w_\gamma)^* = \theta(x)^*,$$

and

$$\tau(\theta(x)) = \tau(\theta(\textstyle\sum_{\gamma \in \hat{D}} x_\gamma w_\gamma)) = \tau(\sum_{\gamma \in \hat{D}} x_\gamma w_\gamma^*) = x_0 = \tau(\sum_{\gamma \in \hat{D}} x_\gamma w_\gamma) = \tau(x),$$

θ is a trace preserving *-antiautomorphism of every algebra R_n. Therefore $\|\theta(x)\|_{L_p(R)} = \|x\|_{L_p(R)}$ for every $x \in \bigcup_{n=1}^{\infty} R_n$, so θ can be uniquely extended to an isometry of whole $X = \overline{\bigcup_{n=1}^{\infty} R_n}$. Define the map $Px = \frac{1}{2}(x + \theta(x)^*)$. It is straightforward that P is a contractive projection and for each $x = \sum_{\gamma \in \hat{D}} x_\gamma w_\gamma \in \bigcup_{n=1}^{\infty} R_n$:

$$Px = \tfrac{1}{2}(x + \theta(x)^*) = \tfrac{1}{2}(\textstyle\sum_{\gamma \in \hat{D}} x_\gamma w_\gamma + (\sum_{\gamma \in \hat{D}} x_\gamma w_\gamma^*)^*) = \sum_{\gamma \in \hat{D}} \operatorname{Re}(x_\gamma) w_\gamma.$$

Now let us prove that the spaces X_i, $i = \overline{0, 3}$, are complemented in X. By definitions

$$X_0 = T(\hat{D}/\{0\})X, \quad X_1 = PT(\hat{D}/\{0\})X,$$

$$X_2 = T(A_2)PT(\hat{D}/\{0\})X, \quad X_3 = T(B)T(A_2)PT(\hat{D}/\{0\})X.$$

where $B = \{\gamma \in \hat{D} \mid \text{if } n \in A_\gamma \text{ then } n \equiv 1, 2 \bmod 4\}$ so that $A_3 = B \cap A_2$. Each of the above operators is a projection (this follows from (2) and the simple fact that P commutes with any projection along a set). Next, $T(\hat{D}/\{0\}) = \mathrm{Id} - T(\{0\})$ and since $T(\{0\})$ is a bounded projection along the trivial subgroup $\{0\}$, by Lemma 3 the operator $T(\hat{D}/\{0\})$ is also bounded. For every $x \in PX$ let $Nx = \frac{1}{2}(x + x^*)$. Taking into account the assertion (v) of Lemma 2 one has for every $\gamma \in \hat{D}$, $\operatorname{Card} A_\gamma = k$:

$$Nw_\gamma = \tfrac{1}{2}(w_\gamma + w_\gamma^*) = \tfrac{1}{2}(w_\gamma + (-1)^{\frac{k(k-1)}{2}} w_\gamma) = \chi_{A_2}(\chi) w_\gamma = T(A_2) w_\gamma \in PX$$

and we conclude that N is a contractive projection in PX. Moreover $T(A_2)|_{PX} = N$, i.e. the projection $T(A_2)P$ is bounded. Finally, notice that B is a subgroup of $\hat{D}$ so by Lemma 3 the projection $T(B)$ is bounded. Bringing these facts together we obtain that the spaces X_i, $i = \overline{0,3}$, are complemented in X.

Fix some $0 \neq \gamma \in \varphi(2k-1)\hat{D}$, let $Z = \{aw_\gamma | a \in \mathbf{C}\}$ so that $\dim Z = 1$, and consider the space

$$Y = T(\{\gamma\} \cup \varphi(2k)\hat{D})X_0 = \Phi(2k)X_0 \oplus Z \subset X_0.$$

The projection $T(\{\gamma\} \cup \varphi(2k)\hat{D}) = T(\{\gamma\}) + T(2k)$ is bounded by Lemma 3, hence Y is complemented in X_0. In addition, since $\Phi(2k)$ is an isometry, we have

$$Y = \Phi(2k)X_0 \oplus Z \approx X_0 \oplus T(\{0\})X = X,$$

and by the decomposition principle

$$X_0 \approx X. \tag{4}$$

We have $X_0 = PX_0 \oplus (\mathrm{Id} - P)X_0 = X_1 \oplus \mathbf{i}X_1 \approx X_1 \oplus X_1$.

Put $Y = T(\varphi(2k-1)\hat{D} \cup \varphi(2k)\hat{D})X_1$.

By definition, Y is complemented in X_1. On the other hand

$$Y = T(2k)X_1 \oplus T(2k-1)X_1 = \Phi(2k)X_1 \oplus \Phi(2k-1)X_1 \approx X_1 \oplus X_1 \approx X_0.$$

Using the decomposition principle we obtain that

$$X_0 \approx X_1. \tag{5}$$

The case of X_2 is slightly more complicated. Evidently

$$X_1 = T(A_2)X_1 \oplus T(\hat{D}/A_2)X_1 = X_2 \oplus Z, \tag{6}$$

where $Z = T(\hat{D}/A_2)X_1$. Let $\Delta : X \mapsto X$ be the map $\Delta x = r_1 r_2 x$. It is clear that Δ is an isometry. Consider the spaces

$$Y_1 = \Phi(2k-1)X_2, Y_2 = \Phi(2k)\Delta\Phi(k+2)Z.$$

If we put $\delta\colon \hat{D} \mapsto \hat{D}$, $\delta(\gamma) = (1,1,0,0,\ldots) + \gamma$ and

$$L = \varphi(2k) \circ \delta \circ \varphi(k+2)(\hat{D}/A_2) =$$
$$\{(0,1,0,1,0,\gamma_1,0,\gamma_2,0,\ldots) : \mathrm{Card}\,\{i | \gamma_i = 1\} \equiv 2,3 \bmod 4\},$$

then we can write that

$$Y_2 = \Phi(2k)\Delta\Phi(k+2)T(\hat{D}/A_2)X_1 = \Phi(2k)\Delta T(\varphi(k+2)(\hat{D}/A_2))X_1 =$$
$$\Phi(2k)T(\delta \circ \varphi(k+2)(\hat{D}/A_2))X_1 = T(\varphi(2k) \circ \varphi(k+2)(\hat{D}/A_2))X_1 = T(L)X_1.$$

Of course the last equality is possible only if the projection along L exists. But $L = A_2 \cap (\mu + \varphi(2(k+2))\hat{D})$, where $\mu = (0,1,0,1,0,0,0,\ldots)$ hence by Lemma 3 (ii) and (2) $T(L)$ is bounded. Now consider the space

$$Y = T(\varphi(2k-1)\hat{D} \cup L)X_2$$

which is obviously complemented in X_2. On the other hand, since

$$\varphi(2k-1)\hat{D} \cap L \subset \varphi(2k-1)\hat{D} \cap \varphi(2k)\hat{D} = \{0\},$$

we have

$$Y = T(\varphi(2k-1)\hat{D} \cup L)X_2 = (T(2k-1) + T(L) - T(\{0\}))X_2.$$

Since $T(\{0\})X_2 \subset T(\{0\})X_0 = T(\{0\})T(\hat{D}/\{0\})X = \{0\}$, and using the equality $T(L)X_2 = T(L)X_1$, we obtain from (6) that

$$Y \approx T(2k-1)X_2 \oplus T(L)X_2 = Y_1 \oplus Y_2 =$$

$$\Phi(2k-1)X_2 \oplus \Phi(2k)\Delta\Phi(k+2)Z \approx X_2 \oplus Z = X_1.$$

The decomposition principle implies that

$$X_2 \approx X_1. \tag{7}$$

The case of X_3 now is quite clear. By definition, $X_3 = \Phi(n_k)X_2$ where $n_{2k-1} = 4k-3$, $n_{2k} = 4k-2$ for every $k \in \mathrm{N}$, so that

$$X_3 \approx X_2. \tag{8}$$

The following equalities show that, in fact, the isomorphism $L_p(R_1) \approx L_p(R)$ is obvious in view of (4) and (5):

$$L_p(R_1) \approx T(\{0\})L_p(R_1) \oplus T(\hat{D}/\{0\})L_p(R_1) \approx$$

$$T(\{0\})L_p(R_1) \oplus PT(\hat{D}/\{0\})L_p(R_1) \oplus (\mathrm{Id} - P)T(\hat{D}/\{0\})L_p(R_1) =$$

$$T(\{0\})L_p(R_1) \oplus \{x \mid x \in X_1, x^* = x\} \oplus \{\mathrm{i}x \mid x \in X_1, x^* = -x\} \approx$$

$$T(\{0\})L_p(R_1) \oplus \{x \mid x \in X_1, x^* = x\} \oplus \{x \mid x \in X_1, x^* = -x\} =$$

$$T(\{0\})L_p(R_1) \oplus NX_1 \oplus (\mathrm{Id} - N)X_1 \approx T(\{0\})L_p(R_1) \oplus X_1 \approx$$

$$T(\{0\})L_p(R_1) \oplus X_0 \approx L_p(R).$$

Now consider the map $S: R \mapsto R$, $Sx = \frac{1}{2}(x + \alpha(x))$. Since α is involutive we have that $S^2 = S$, i.e. S is a projection. Moreover, α is a trace-preserving *-anti-automorphism and thus for every $x \in R$ one has $\|\alpha(x)\|_{L_p(R)}$ and S is contractive. It follows that S uniquely extends to a contractive projective projection in $L_p(R)$ and since $SR_1 = R_0$ we obtain $SL_p(R_1) = L_p(R_0)$. Therefore $L_p(R_0)$ is complemented in $L_p(R_1)$.

Finally, let us prove that $X_3 \subset L_p(R_0)$. Recall that by definition of X_3 and $L_p(R_0)$ it is sufficient to prove that if $w_\gamma \in X_3$ then $w_\gamma \in R_0$. Since $k = \text{Card}\, A_\gamma \equiv 0, 1 \bmod 4$ we have

$$w_\gamma^* = (-1)^{\frac{k(k-1)}{2}} w_\gamma = w_\gamma, \tag{9}$$

hence $w_\gamma \in R_1$. By definition, every $w_\gamma \in X_3$ has the form

$$w_\gamma = r_{n_1} r_{n_2} \dots r_{n_k},\ k \equiv 0, 1 \bmod 4,\ n_i \equiv 1, 2 \bmod 4,\ i = \overline{1, k}.$$

Then by Lemma 5 and by (9) we see that

$$\alpha(w_\gamma) = \alpha(r_{n_1} r_{n_2} \dots r_{n_k}) = r_{n_k} r_{n_{k-1}} \dots r_{n_1} = w_\gamma^* = w_\gamma.$$

Therefore if $w_\gamma \in X_3$ then $w_\gamma \in R_0$, i.e. $X_3 \subset L_p(R_0)$.

Now we are in a position to finish the proof. Since $L_p(R_0)$ is complemented in $L_p(R_1) \approx L_p(R)$ and X_3 is complemented in $L_p(R)$ (consequently, in $L_p(R_0)$) and by (4), (5), (7), (8) $X_3 \approx L_p(R)$, so using the Pelczynski decomposition principle we can conclude that $L_p(R_0) \approx L_p(R)$.

Since $L_\infty(R_i) = R_i \approx L_1(R_i)^*$, $i = 0, 1$ (see [Ayu1], Ch.4, Proposition 2.3), the case $p = \infty$ obviously follows by duality.

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