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ELLIPTIC BOUNDARY PROBLEMS AND THE BOUTET DE MONVEL CALCULUS IN BESOV AND TRIEBEL-LIZORKIN SPACES

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Abstract.

The Boutet de Monvel calculus of pseudo-differential boundary operators is generalised to the scales of Besov and Triebel-Lizorkin spaces, $B^s_{p,q}$ and $F^s_{p,q}$, with $s \in \mathbb{R}$ and p and $q \in]0, \infty]$ (though with $p < \infty$ for the $F^s_{p,q}$ spaces).

The continuity and Fredholm properties proved here extend those in [Fra86a] and [Gru90], and the results on range complements of surjectively elliptic Green operators improve the earlier known, even for the classical spaces with $1 < p < \infty$.

The symbol classes treated are the x-uniformly estimated ones. On $\overline{\mathbb{R}}^n_+$ a trace operator T and a singular Green operator G, both of class 0, are defined in general to be

$$T = K^* e^+, \qquad G = r^+ G_1^* e^+ \tag{0.1}$$

where the Poisson operator K is $\mathrm{OPK}(e^{iD_{x'} \cdot D_{\xi'}} \bar{\tilde{t}}(x', x_n, \xi'))$ and the singular Green operator G_1 equals $\mathrm{OPG}(e^{iD_{x'} \cdot D_{\xi'}} \bar{\tilde{g}}(x', y_n, x_n, \xi'))$, respectively.

1. Summary.

As a main example of the considerations in this article one may take an elliptic differential operator $A = \sum_{|\alpha| \leq d} a_\alpha(x) D^\alpha$ on an open bounded set $\Omega \subset \mathbb{R}^n$ with C^∞ boundary $\Gamma := \partial\Omega$ and a trace operator T for which

$$\begin{aligned} Au(x) &= f(x) \quad \text{in} \quad \Omega, \\ Tu(x) &= \varphi(x) \quad \text{on} \quad \Gamma, \end{aligned} \tag{1.1}$$

is a boundary value problem that is elliptic in the sense of Agmon, Douglis and Nirenberg [ADN59].

The topics to be discussed in this connection are:

(I) solvability and regularity results in the *Besov* and *Triebel-Lizorkin* spaces $B^s_{p,q}$ and $F^s_{p,q}$ with $s \in \mathbb{R}$ and p and $q \in]0, \infty]$ (and $p < \infty$ in the F case),

(II) a generalisation of the pseudo-differential calculus of Boutet de Monvel [BdM71] (for problems like that in (1.1), e.g.) to the setting in (I).

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In fact (I) and (II) are treated simultaneously in order to give a *unified* treatment of these two systematic points of view.

It is also the purpose to present modifications of some basic facts in the calculus with full proofs. The arguments should be of interest, partly because they are fairly elementary, and partly since they bring the $\mathbb{R}^n_+$-part of the calculus closer to the pseudo-differential theory of Hörmander [Hör85, Sect. 18.1].

The results are presented in the rest of this section, and Section 2 settles the notation and the prerequisites on the $B^s_{p,q}$ and $F^s_{p,q}$ spaces. Section 3 describes the operator classes of the calculus relatively to the half-space $\mathbb{R}^n_+$, whereas the continuity properties in the $\mathbb{R}^n_+$-case are proved in Section 4. Fredholm properties are treated in Section 5, even for multi-order systems acting in vector bundles, and applications are indicated in Section 6.

On the spaces. The project in (I) covers many different spaces at the same time, since:

- $C^s = B^s_{\infty,\infty}$ for $s > 0$ (Hölder-Zygmund spaces),
- $W^s_p = B^s_{p,p}$ for $s \in \mathbb{R}_+ \setminus \mathbb{N}_0$ and $1 < p < \infty$ (Slobodetskiĭ spaces),
- $W^k_p = F^k_{p,2}$ for $k \in \mathbb{N}_0$ and $1 < p < \infty$ (Sobolev spaces),
- $H^s_p = F^s_{p,2}$ for $s \in \mathbb{R}$ and $1 < p < \infty$ (Bessel potential spaces).

In particular $F^0_{p,2} = L_p$ for $1 < p < \infty$ and $B^s_{2,2} = F^s_{2,2} = H^s$ for $s \in \mathbb{R}$ (Lebesgue and Sobolev spaces).

For these relations the reader is referred to the books of H. Triebel [Tri83, Tri92]. As a further motivation, note that the local Hardy space h_p equals $F^0_{p,2}$ for $0 < p < \infty$, and that the relation to Morrey-Campanato spaces is explained in [Tri92], together with the fact that also $F^s_{p,\infty}$ has been considered earlier on in [Chr84, DS84]. (The local BMO space bmo equals $F^0_{\infty,2}$, which is not treated here. However, a recent work of J. Marschall [Mar] may provide a point of departure for an extension of the present results to the $F^s_{\infty,q}$ spaces.)

For a presentation of the results the reader is referred to Theorems 1.1 and 1.3 and to Corollaries 1.2 and 1.4 below.

The calculus of Boutet de Monvel was originally worked out for the H^s Sobolev spaces [BdM71], and an extension to $B^s_{p,q}$ and $F^s_{p,2}$ with $1 < p < \infty$ and $1 \leqq q \leqq \infty$ was given by G. Grubb [Gru90]. Among the earlier attempts at an L_p theory for the calculus the shortcomings of [RS82] are accounted for in [Gru90, Rem. 3.2], so the reader may refer to the details there.

An extension of the calculus to the $B^s_{p,q}$ and $F^s_{p,q}$ scales, with the restriction $p < \infty$ in the F case, has been worked out already by J. Franke. However, at the central point the arguments are not contained in his thesis [Fra86a], and the only published material on this work is the review article [Fra85], which does not contain proofs. In addition to this, the concept of negative class for operators of the form $P_\Omega + G$ — and hence for general Green operators $\mathcal{A}$ — was first introduced in [Gru90], and already for classical problems like (1.1) this notion is indispensable for an optimal description of parametrices, cf. [Gru90, Thm. 5.4 ff.].

On these grounds the author has included in his thesis [Joh93] an extension to the $B^s_{p,q}$ and $F^s_{p,q}$ spaces. It is presented here with some improvements.

It should also be mentioned that the L_p theory of *differential* boundary problems, studied in [ADN59, ADN64], [Sol66] . . . , was considered first in the $B^s_{p,q}$ and $F^s_{p,q}$ scales by H. Triebel in [Tri78], cf. also [Tri83], although with some restrictions for p and $q < 1$. An extension to the case with p and q also in the whole of $]0, 1]$ has been worked out by Franke and T. Runst [FR95].

On the calculus. Boundary problems like (1.1) are represented here by Green operators in the pseudo-differential boundary operator calculus of Boutet de Monvel, cf. (II). A typical example is obtained from a matrix operator

$$\mathscr{A} = \begin{pmatrix} P_\Omega + G & K \\ T & S \end{pmatrix} : \begin{matrix} C^\infty(\bar\Omega)^N \\ \oplus \\ C^\infty(\Gamma)^M \end{matrix} \to \begin{matrix} C^\infty(\bar\Omega)^{N'} \\ \oplus \\ C^\infty(\Gamma)^{M'} \end{matrix} \tag{1.2}$$

where

- $P_\Omega = r_\Omega P e_\Omega$ is the truncation to Ω of a pseudo-differential operator on R^n with the symbol lying in $S^d_{1,0}$ and satisfying a transmission condition at $\Gamma \subset \mathsf{R}^n$;
- G is a singular Green operator,
- K is a Poisson (or potential) operator,
- T is a trace operator,
- S is a pseudo-differential operator on Γ.

(See Section 3 below for the expressions of these operators in local coordinates.) Then $\mathscr{A}$ is said to have *order d* and *class r*, for numbers $d \in \mathsf{R}$ and $r \in \mathsf{Z}$, if all entries have this order and both T and $P_\Omega + G$ have class r.

As an example, if in (1.1) $P_\Omega = A = \Delta^2$ (the biharmonic operator) and $Tu = (\gamma_0 u, \gamma_1 u)$ with $\gamma_0 u = u|_\Gamma$ and $\gamma_1 u = -i\dfrac{\partial u}{\partial \vec n}\Big|_\Gamma$, $\vec n$ being the outward unit normal vector field at Γ, then $N = 1 = N'$, $M = 0$ and $M' = 2$ together with $G = 0$, $K = 0$ and $S = 0$ allows one to read (1.1) as an equation for the operator $\mathscr{A}$. (This is actually a slightly more general situation with multi-order and multi-class.)

The five types of operators listed above are also defined on more general distributions than C^∞ functions. In this respect the following theorem is proved here:

THEOREM 1.1. *Let $s \in \mathsf{R}$ and p and $q \in]0, \infty]$, and suppose that $\mathscr{A}$ has order $d \in \mathsf{R}$ and class $r \in \mathsf{Z}$.*

If $s > r + \max\left(\frac{1}{p} - 1, \frac{n}{p} - n\right)$, *then* $\mathscr{A}$ *has the continuity properties*

$$\mathscr{A}: \begin{array}{c} B^s_{p,q}(\bar\Omega)^N \\ \oplus \\ B^{s-\frac{1}{p}}_{p,q}(\Gamma)^M \end{array} \to \begin{array}{c} B^{s-d}_{p,q}(\bar\Omega)^{N'} \\ \oplus \\ B^{s-d-\frac{1}{p}}_{p,q}(\Gamma)^{M'} \end{array} \tag{1.3}$$

$$\mathscr{A}: \begin{array}{c} F^s_{p,q}(\bar\Omega)^N \\ \oplus \\ B^{s-\frac{1}{p}}_{p,p}(\Gamma)^M \end{array} \to \begin{array}{c} F^{s-d}_{p,q}(\bar\Omega)^{N'} \\ \oplus \\ B^{s-d-\frac{1}{p}}_{p,p}(\Gamma)^{M'} \end{array} \tag{1.4}$$

provided $p < \infty$ *in* (1.4).

Furthermore, if $\mathscr{A}$ *for some* $s_1 < r + \max\left(\frac{1}{p_1} - 1, \frac{n}{p_1} - n\right)$ *is continuous from either* $B^{s_1}_{p_1,q_1}(\bar\Omega)^N \oplus B^{s_1-\frac{1}{p_1}}_{p_1,q_1}(\Gamma)^M$ *or* $F^{s_1}_{p_1,q_1}(\bar\Omega)^N \oplus B^{s_1-\frac{1}{p_1}}_{p_1,p_1}(\Gamma)^M$ *to the space* $\mathscr{D}'(\Omega)^{N'} \times \mathscr{D}'(\Gamma)^{M'}$, *then both* T *and* $P_\Omega + G$ *have* class $\leq r - 1$.

The theorem also gives statements for each kind of the operators P_Ω, G, K, T and S by consideration of examples of $\mathscr{A}$ with suitable zero-entries. Thus the "working definition" of the class concept – namely that an operator is of class r if and only if it is continuous from $H^r(\bar\Omega)$ to $\mathscr{D}'$ – is generalised to the $B^s_{p,q}$ and $F^s_{p,q}$ setting.

When $\mathscr{A}$ is elliptic (and in particular has polyhomogeneous symbols of order $d \in \mathbb{Z}$), the theorem applies equally well to its parametrix $\tilde{\mathscr{A}}$, which may be taken of order $-d$ and class $r - d$ (cf. [Gru90]). Then $\tilde{\mathscr{A}}$ is is continuous from the right to the left in (1.3) and (1.4) for every $s > r + \max\left(\frac{1}{p} - 1, \frac{n}{p} - n\right)$.

Consequently $\mathscr{A}$ has the expected *inverse regularity* properties:

COROLLARY 1.2. *Let* $\mathscr{A}$ *be elliptic of order* d *and class* r. *Let* (u, ψ) *belong to* $B^s_{p,q}(\Omega)^N \oplus B^{s-\frac{1}{p}}_{p,q}(\Gamma)^M$ *for some* $s > r + \max\left(\frac{1}{p} - 1, \frac{n}{p} - n\right)$ *and assume that* (u, ψ) *–for a parameter with* $s_1 > r + \max\left(\frac{1}{p_1} - 1, \frac{n}{p_1} - n\right)$ *– satisfies*

$$\begin{pmatrix} P_\Omega + G & K \\ T & S \end{pmatrix} \begin{pmatrix} u \\ \psi \end{pmatrix} = \begin{pmatrix} f \\ \phi \end{pmatrix} \in \begin{array}{c} B^{s_1-d}_{p_1,q_1}(\bar\Omega)^{N'} \\ \oplus \\ B^{s_1-d-1/p_1}_{p_1,q_1}(\Gamma)^{M'} \end{array} \tag{1.5}$$

Then (u, φ) *is also an element of* $B^{s_1}_{p_1,q_1}(\bar\Omega)^N \oplus B^{s_1-\frac{1}{p_1}}_{p_1,q_1}(\Gamma)^M$.

In the $F^s_{p,q}$ *spaces* $\mathscr{A}$ *has analogous inverse regularity properties (if* $q = p$ *in the spaces over* Γ*), and the statements likewise carry over to the mixed cases with* (u, ψ) *given in Besov spaces and* (f, φ) *prescribed in Triebel-Lizorkin spaces, or vice versa.*

In the elliptic case, the Fredholm properties of $\mathscr{A}$ are improved and extended to the following Theorem 1.3. Thus an elliptic Green operator is a Fredholm operator with (s, p, q)-independent kernel and index (1° and 2°), for which each choice of a smooth range-complement (by 3°) can be used as such for every admissible (s, p, q).

THEOREM 1.3. *Let $\mathscr{A}$ be elliptic of order $d \in \mathbf{Z}$ and class $r \in \mathbf{Z}$. Consider for each p and $q \in]0, \infty]$ and $s > r + \max\left(\frac{1}{p} - 1, \frac{n}{p} - n\right)$ the two operators in formulae* (1.3) *and* (1.4) *above.*

1° *For each such (s, p, q) the operators in* (1.3) *and* (1.4) *have the same kernel,* $\ker \mathscr{A}$. *Here* $\ker \mathscr{A}$ *is a finite dimensional subspace of $C^\infty(\Omega)^N \oplus C^\infty(\Gamma)^M$, which is independent of (s, p, q).*

2° *For each (s, p, q) the operators have closed ranges. Moreover, there exists a finite dimensional subspace $\mathscr{N} \subset C^\infty(\bar{\Omega})^{N'} \oplus C^\infty(\Gamma)^{M'}$ which for each (s, p, q) is a range-complement of both operators. That is,*

$$\mathscr{N} \oplus \mathscr{A}(B^s_{p,q}(\bar{\Omega})^N \oplus B^{s-\frac{1}{p}}_{p,q}(\Gamma)^M) = B^{s-d}_{p,q}(\bar{\Omega})^{N'} \oplus B^{s-d-\frac{1}{p}}_{p,q}(\Gamma)^{M'} \tag{1.6}$$

$$\mathscr{N} \oplus \mathscr{A}(F^s_{p,q}(\bar{\Omega})^N \oplus B^{s-\frac{1}{p}}_{p,p}(\Gamma)^M) = F^{s-d}_{p,q}(\bar{\Omega})^{N'} \oplus B^{s-d-\frac{1}{p}}_{p,p}(\Gamma)^{M'} \tag{1.7}$$

whenever $s > r + \max\left(\frac{1}{p} - 1, \frac{n}{p} - n\right)$.

3° *If $\mathscr{N} \subset C^\infty(\bar{\Omega})^{N'} \oplus C^\infty(\Gamma)^{M'}$ is any subspace such that either* (1.6) *or* (1.7) *holds for some parameter (s_1, p_1, q_1) with $s_1 > r + \max\left(\frac{n}{p_1} - 1, \frac{n}{p_1} - n\right)$, then $\mathscr{N}$ has finite dimension and both* (1.6) *and* (1.7) *hold for every (s, p, q) that satisfies $s > r + \max\left(\frac{1}{p} - 1, \frac{n}{p} - n\right)$.*

In the determination of specific examples of $\mathscr{N}$ the following result concerning annihilation should be of importance.

For a given subspace $\mathscr{N} \subset C^\infty(\bar{\Omega})^{N'} \oplus C^\infty(\Gamma)^{M'}$ it is convenient to let $\mathscr{N}^\perp$ denote the distributions f and φ for which $\langle f, g \rangle_\Omega + \langle \varphi, \eta \rangle_\Gamma$ makes sense and equals 0 for all (g, η) in $\mathscr{N}$. These questions are meaningful for $\mathscr{A}$'s codomains provided each element (g, η) has sufficiently many vanishing traces $\gamma_j g$:

COROLLARY 1.4. *Let $\mathscr{A}$ be as in Theorem* 1.3, *and let $\mathscr{N} \subset C^\infty(\bar{\Omega})^{N'} \oplus C^\infty(\Gamma)^{M'}$ be a subspace for which each element (g, η) satisfies $\gamma_j g = 0$ for $j < d - r$ (void if $d \leqq r$). Moreover, let one of the identities*

$$\mathscr{N}^\perp \cap (B^{s-d}_{p,q}(\bar{\Omega})^{N'} \oplus B^{s-d-\frac{1}{p}}_{p,q}(\Gamma)^{M'}) = \mathscr{A}(B^s_{p,q}(\bar{\Omega})^N \oplus B^{s-\frac{1}{p}}_{p,q}(\Gamma)^M) \tag{1.8}$$

$$(1.9)\qquad \mathcal{N}^{\perp} \cap (F^{s-d}_{p,q}(\bar{\Omega})^{N'} \oplus B^{s-d-\frac{1}{p}}_{p,p}(\Gamma)^{M'}) = \mathcal{A}(F^{s}_{p,q}(\bar{\Omega})^{N} \oplus B^{s-\frac{1}{p}}_{p,p}(\Gamma)^{M})$$

hold for a parameter (s_1, p_1, q_1) *with* $s_1 > r + \max\left(\frac{1}{p_1} - 1, \frac{n}{p_1} - n\right)$.

Then $\mathcal{N}$ *is a range complement and both* (1.8) *and* (1.9) *hold for all* (s, p, q) *with* $s > r + \max\left(\frac{1}{p} - 1, \frac{n}{p} - n\right)$.

EXAMPLE 1.5. $\mathcal{A} = \begin{pmatrix} -\Delta \\ \gamma_1 \end{pmatrix}$ represents the Neumann problem for the Laplacian, and for $(s, p, q) = (2, 2, 2)$ the data (f, φ) belong to the range of $\mathcal{A}$ precisely when $\int_\Omega f + i \int_\Gamma \varphi = 0$. By Corollary 1.4, this annihilation of $(1_\Omega, i1_\Gamma)$ characterises the range of $\mathcal{A}$ for every (s, p, q) with $s > 2 + \max\left(\frac{1}{p} - 1, \frac{n}{p} - n\right)$.

In the preceding exposition the scope has been restricted somewhat for simplicity's sake. In fact $\mathcal{A}$ could equally well have been a multi-order and multi-class system in the Douglis-Nirenberg sense, even with each entry matrixformed over Ω and Γ – i.e. $P_\Omega = (P_{i,j,\Omega})$ where $P_{i,j,\Omega}\colon C^\infty(E_i) \to C^\infty(E'_j)$ etc. – as in [Gru90, Cor. 5.5]. Theorem 1.1 and 1.3 are both proved in this generality below, the latter even in a version for one-sided elliptic operators.

The methods. To carry out the analysis in the $B^s_{p,q}$ and $F^s_{p,q}$ spaces the definitions and results based on Fourier analysis, as presented in [Tri83], are adopted. Together with M. Yamazaki's theorems on convergence of series of distributions satisfying spectral conditions, [Yam86, Thm.s 3.6 and 3.7], these are the main tools used here concerning the function spaces.

For the treatment of the five types of operators in the $B^s_{p,q}$ and $F^s_{p,q}$ spaces it is used that a pseudo-differential operator P on $\mathbb{R}^n$ is known to be bounded

$$(1.10)\qquad P\colon B^s_{p,q}(\mathbb{R}^n) \to B^{s-d}_{p,q}(\mathbb{R}^n), \qquad P\colon F^s_{p,q}(\mathbb{R}^n) \to F^{s-d}_{p,q}(\mathbb{R}^n)$$

for $s \in \mathbb{R}$ and p and $q \in]0, \infty]$ (with p finite in the F case) whenever the symbol belongs to the Hörmander class $S^d_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$; that is, whenever the x-uniform estimate – with $\langle\xi\rangle := (1 + |\xi|^2)^{\frac{1}{2}}$ –

$$(1.11)\qquad \sup\{\langle\xi\rangle^{-(d-|\alpha|)} |D^\beta_x D^\alpha_\xi p(x, \xi)| \mid x, \xi \in \mathbb{R}^n\} =: C_{\alpha\beta} < \infty$$

is valid for $p(x, \xi) \in \mathcal{E}(\mathbb{R}^{2n})$ for all multi-indices α and β.

The central result in (1.10) was obtained for $p < 1$ (even for more general symbols) by Bui Huy Qui and L. Päivärinta [Bui83, Päi83], and it has been reproved (with further generalisations) in M. Yamazaki's paper [Yam86], e.g., where also the history of this L_p theory is outlined.

In order to carry over the continuity in (1.10) to a version of Theorem 1.1 for the halfspace $\mathbb{R}^n_+$, the x-uniformly estimated symbols and symbol-kernels of [Gru90] and [GK93] are treated here. Cf. (3.6) ff. below for these classes.

For the P_Ω's in particular, the so-called uniform two-sided transmission condition at Γ is required to hold for P. In local coordinates this amounts to the fulfilment of (3.5) below.

By and large, the $\mathbb{R}^n_+$-version of Theorem 1.1 is deduced from (1.10) by a method of attack that is rather close to the one adopted in [Gru90], and hence in a way that is quite standard within the calculus. However, to include the interval $]0,1[$ for the integral-exponents p, a fresh approach is needed. For this point J. Franke [Fra86a] has given an argument based on estimates of para-multiplication operators (like those in [Yam86], for example) and on complex interpolation of the Besov and Triebel-Lizorkin spaces.

In addition to this there is a main technical difficulty in the fact that denseness of the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ in either of $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ holds precisely when both $p < \infty$ and $q < \infty$ do so. The approach taken here is to define trace operators $T = \mathrm{OPT}(\tilde{t})$ and singular Green operators $G = \mathrm{OPG}(\tilde{g})$ of class 0 on $\mathbb{R}^n_+$ by the formulae

$$Tu = K^* e^+ u, \qquad Gu = r^+ G_1^* e^+ u \tag{1.12}$$

when e^+ makes sense on $u \in \mathscr{S}'(\bar{\mathbb{R}}^n_+)$. Hereby $K = \mathrm{OPK}(e^{iD_{x'}\cdot D_{\xi'}}\bar{\tilde{t}}(x',x_n,\xi'))$ and $G_1 = \mathrm{OPG}(e^{iD_{x'}\cdot D_{\xi'}}\bar{\tilde{g}}(x',y_n,x_n,\xi'))$, that have their adjoints K^* and G_1^* defined on $\mathscr{S}'_0(\mathbb{R}^n_+)$.

Comparison with other works. The continuity properties shown here extend those in [Gru90, Thm. 3.11 ff.], mainly to the case with $p \in]0,\infty]$. (The special results on $B^s_{p,p} \cap H^s_p$ and $B^s_{p,p} \cup H^s_p$ there are recovered by use of the full statements below, see Theorem 4.15 ff.) The results of [Fra86a] are extended to multi-order systems, that can have class $r < 0$, in which case the ranges of s are larger than his. Using techniques from [Gru90], the present restrictions on s are proved to be essentially sharp. For the subscales $B^s_{p,p}$ and $F^s_{p,2}$ the borderline cases $s = r + \dfrac{1}{p} - 1$ were first analysed in [Gru90], but here in the more complicated situation with $q \in]0,\infty]$ this question is only given a rudimentary treatment; cf. Remark 4.12 below. In Section 4, continuity from $\mathscr{S}'(\bar{\mathbb{R}}^n_+)$ is shown to hold precisely for operators of class $-\infty$.

The result on $\ker \mathscr{A}$ extends the one in [Gru90] to the full scales $B^s_{p,q}$ and $F^s_{p,q}$, whereas 1° and 2° in Theorem 1.3 generalise [Fra86a] to the $\mathscr{A}$'s considered here. The exact range characterisations of surjectively elliptic operators in [Gru90, Thm. 5.4] amount to annihilation of a specific (s,p,q)-independent finite dimensional C^∞ space. Extending this, Corollary 1.4 shows that a smooth space

$\mathcal{N}$ need only be annihilated by the range for a *single* (s, p, q), for then it is so for all admissible parameters; cf. also Example 1.5 above. For the existence of such an $\mathcal{N}$ having vanishing traces in case $r \leqq d$, see [Gru90, Rem. 5.3]. Moreover, a *smooth* space $\mathcal{N}$ complements the range either for all (s, p, q) or for none by Theorem 1.3. Even for differential problems with $1 < p < \infty$ this conclusion has seemingly not been formulated before.

After submission of the first version of this paper, I became aware of a work of D.-C. Chang, S. G. Krantz and E. M. Stein [CKS93], which also deals with boundary problems in spaces with $p < 1$. They consider the solution operators R_D and R_N for the boundary homogeneous Dirichlét and Neumann problems, respectively, for $-\Delta$, and they show that $\partial^2_{jk} R_D$ and $\partial^2_{jk} R_N$ are bounded from the 'minimal' local Hardy space $r_\Omega F^0_{p,2;0}(\bar{\Omega})$ for every $p > 0$, whilst for $F^0_{p,2}(\bar{\Omega})$ this holds for $\partial^2_{jk} R_D$ if and only if $p > \dfrac{n}{n+1}$ and for $\partial^2_{jk} R_N$ if and only if $p > 1$.

On one hand, by application of Theorem 1.1 or 5.2 to $\partial^2_{jk} R_D$ and $\partial^2_{jk} R_N$ as special cases, the present general theory also yields the boundedness on $F^0_{p,2}(\bar{\Omega})$ for $p > \dfrac{n}{n+1}$ and $p > 1$ as well as the unboundedness for $p < \dfrac{n}{n+1}$ and $p < 1$ (since the operators are known to have class -1 and 0, respectively). On the other hand, however, scales like $r_\Omega F^s_{p,q;0}(\bar{\Omega})$ are not considered here.

Perhaps the spaces with p and $q \in]0, 1[$ deserve some extra attention in view of the fact that $B^s_{p,q}$ and $F^s_{p,q}$ with such exponents are merely quasi-Banach spaces. For continuity questions it is well known that (2.9) below can be applied with succes instead of the quasi-triangle inequality; cf. also Remark 2.2 below. So for Theorem 1.1 the essential difficulties lie in the case with $p = \infty$, which is handled by means of (1.12).

To prove Theorem 1.3 it might seem necessary to generalise the notion of Fredholm operators to quasi-Banach spaces (as in [Fra86a] and [FR95]). However, this approach is neither necessary nor particularly useful here. In fact the restriction to $s > r + \max\left(\dfrac{1}{p} - 1, \dfrac{n}{p} - n\right)$ when T and $P_\Omega + G$ are of class r allows embeddings into spaces with p and $q \in]1, \infty[$ on which the operators are defined. This gives a way to deduce the various properties from the Banach space cases. Cf. also Remark 5.1.

In reality the "extra" spaces over Ω with p and $q \in]0, 1[\cup \{\infty\}$ treated here do not in comparison provide any "new" functions to which a given operator can be applied, cf. Remark 5.1.

From this point of view the achievement in the present article consists rather of continuity with respect to new topologies and of more detailed Fredholm properties.

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2. Preliminaries.

In this section an overview of the Besov spaces $B^s_{p,q}$ and Triebel-Liorkin spaces $F^s_{p,q}$ is given. Subsections 2.4, 2.5, 2.7 and 2.8 are vital for the treatment of spaces with integral-exponent $p < 1$, in particular because they provide a substitute for the duality arguments in [Gru90], that only work for $p > 1$.

2.1. *Notation.* For a normed or quasi-normed space X, $\| x \,|X\|$ denotes the quasi-norm of the vector x. Recall that X is quasi-normed when the triangle inequality is weakened to $\| x + y \,|X\| \leqq c(\| x \,|X\| + \| y \,|X\|)$ for some $c \geqq 1$ independent of x and y. (The prefix "quasi-" is omitted when confusion is unlikely to occur.)

As simple examples there are $L_p(\mathsf{R}^n)$ and $\ell_p := \ell_p(\mathsf{N}_0)$ for $p \in]0, \infty]$, where $c = 2^{\frac{1}{p}-1}$ is possible for $p < 1$. However, it is a stronger fact that

$$(2.1) \qquad \| f + g \,|L_p\| \leqq (\| f \,|L_p\|^p + \| g \,|L_p\|^p)^{\frac{1}{p}}, \quad \text{for} \quad 0 < p \leqq 1,$$

and this inequality has an exact analogue for the ℓ_p spaces.

The vector space of bounded linear operators from X to Y is denoted $\mathsf{L}(X, Y)$; the operator quasi-norm $\| \cdot \,|\mathsf{L}(X, Y)\|$ satisfies the quasi-triangle inequality with the same constant as $\| \cdot \,|Y\|$.

The space of compactly supported smooth functions is written $C_0^\infty(\Omega)$ or $\mathscr{D}(\Omega)$ when $\Omega \subset \mathsf{R}^n$ is open, and $\mathscr{D}'(\Omega)$ is the dual space of distributions on Ω. The duality between $u \in \mathscr{D}'(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$ is denoted $\langle u, \varphi \rangle$.

The Schwartz space of rapidly decreasing functions is denoted by $\mathscr{S} = \mathscr{S}(\mathsf{R}^n)$, and the dual space of tempered distributions by $\mathscr{S}' = \mathscr{S}'(\mathsf{R}^n)$. The seminorms on $\mathscr{S}(\mathsf{R}^n)$ are taken to be $\| \psi \,|\mathscr{S}, \alpha, \beta\| = \sup\{|x^\alpha D^\beta \psi| \mid x \in \mathsf{R}^n\}$ for $\alpha, \beta \in \mathsf{N}_0^n$ or equivalently $\| \psi \,|\mathscr{S}, N\| = \max\{\| \psi \,|\mathscr{S}, \alpha, \beta\| \mid |\alpha|, |\beta| \leqq N\}$ for $N \in \mathsf{N}_0$.

Throughout $D^\alpha = (-i)^{|\alpha|} \dfrac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \dfrac{\partial^{\alpha_n}}{\partial x_1^{\alpha_n}}$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for $\alpha \in \mathsf{N}_0^n$.

With the norm $\| f \,|C(\mathsf{R}^n)\| = \sup|f|$, it is convenient to let

$$(2.2) \qquad C(\mathsf{R}^n) = \{f \in L_\infty(\mathsf{R}^n) \mid f \text{ is uniformly continuous}\}.$$

Moreover, $C_b^k(\mathsf{R}^n) = \{f \mid D^\alpha f \in C(\mathsf{R}^n), |\alpha| \leqq k\}$ and $C^\infty(\mathsf{R}^n) = \cap\, C_b^k(\mathsf{R}^n)$ is the space of smooth functions with bounded derivatives of any order; it is equipped with the semi-norms $\sup\{|D^\alpha f(x)| \mid x \in \mathsf{R}^n,\ |\alpha| \leqq k\}$. (This distinguishes the space C_b^k from the Hölder-Zygmund space $B^s_{\infty,\infty} = C^s$, $s > 0$. Then $C^k \subset C_b^{k-1}$ and $C^\infty = \cap\, B^s_{\infty,\infty}$.)

For the Besov spaces $B^s_{p,q}(\mathsf{R}^n)$, where $s \in \mathsf{R}$ and $0 < p, q \leqq \infty$, and the

Triebel-Lizorkin spaces $F^s_{p,q}$ (considered for $0 < p < \infty$ only) the notation of H. Triebel in [Tri83] is adopted.

When $\Omega \subset \mathbb{R}^n$ is open, $C^\infty(\bar{\Omega})$, $B^s_{p,q}(\bar{\Omega})$ and $F^s_{p,q}(\bar{\Omega})$ etc. are defined by restriction to Ω. E.g., $C(\bar{\Omega}) = r_\Omega C(\mathbb{R}^n)$ where $r_\Omega\colon \mathscr{D}'(\mathbb{R}^n) \to \mathscr{D}'(\Omega)$ is the transpose of the extension by 0 outside of Ω, denoted $e_\Omega\colon C_0^\infty(\Omega) \to C_0^\infty(\mathbb{R}^n)$. When $\Omega = \mathbb{R}^n_\pm$ the abbreviations $r^\pm = r_{\mathbb{R}^n_\pm}$ and $e^\pm = e_{\mathbb{R}^n_\pm}$ are used. Here $\mathbb{R}^n_\pm$ denotes the halfspace where $x_n \gtrless 0$ and $\bar{\mathbb{R}}^n_\pm := \{x \in \mathbb{R}^n \mid x_n \gtreqless 0\}$ its closure.

Moreover, $B^s_{p,q;\,0}(\bar{\Omega})$, $\mathscr{S}'_0(\bar{\Omega})$ etc. denote subspaces supported by $\bar{\Omega}$, e.g.,

$$\mathscr{S}'_0(\bar{\mathbb{R}}^n_+) = \{u \in \mathscr{S}'(\mathbb{R}^n) \mid \operatorname{supp} u \subset \bar{\mathbb{R}}^n_+\}. \tag{2.3}$$

The Fourier transform is denoted by $\mathscr{F}u(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x)dx$, and the notation $\mathscr{F}^{-1}v(x) = \check{v}(x)$ is used for its inverse; the co-Fourier transform is written $\bar{\mathscr{F}}u(\xi) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}u(x)dx$ and its inverse is denoted $\bar{\mathscr{F}}^{-1}v(x)$. For functions $u(x', x_n) \in \mathscr{S}(\mathbb{R}^n)$, where $x' = (x_1, \dots, x_{n-1})$, a partial transformation in x' is indicated by $\mathscr{F}_{x'\to\xi'}u(x', x_n) = \acute{u}(\xi', x_n) = \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'}u(x', x_n)dx'$. Indexations like this are also used for the other transformations and for functions of, say, $n-1$ variables. However, in any case "$\wedge$" indicates a Fourier transformation with respect to all variables; when the meaning is clear this replaces $\mathscr{F}_{x'\to\xi'}v(x')$ etc.

For $u \in C^\infty(\bar{\mathbb{R}}^n_+)$ we let $\gamma_0 u(x') = u(x', 0)$ and $\gamma_j u = \gamma_0 D^j_{x_n} u$. As usual $\langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}$ and $\langle x'\rangle = \langle (x', 0)\rangle$, where $|\cdot|$ is the euclidean norm on $\mathbb{R}^n$. The measure $(2\pi)^{-n}\,dx$ is abbreviated $đx$, and $đx' := (2\pi)^{1-n}\,dx'$ on $\mathbb{R}^{n-1}$. Usually it is clear from the context whether p denotes an integral-exponent in $]0, \infty]$ or a symbol $p = p(x, \xi)$ (in $S^d_{1,0}$).

The convention that $t_\pm = \max(0, \pm t)$ is used for $t \in \mathbb{R}$, and $\lfloor t \rfloor$ and $\lceil t \rceil$ denote the largest integer $\leqq t$ and the smallest integer $\geqq t$, respectively. For each given assertion we shall follow D. E. Knuth's suggestion in [Knu92] and let [assertion] denote 1 and 0 when the assertion is true respectively false.

2.2. *The spaces.* For the definition of $B^s_{p,q}$ and $F^s_{p,q}$ the conventions in [Yam86] (that are equivalent to the ones in [Tri83, Tri92]) are employed.

First a partition of unity, $1 = \sum_{j=0}^\infty \Phi_j$, is constructed: From $\Psi \in C^\infty(\mathbb{R})$, such that $\Psi(t) = 1$ for $0 \leqq t \leqq \frac{11}{10}$ and $\Psi(t) = 0$ for $\frac{13}{10} \leqq t$, the functions

$$\Psi_j(\xi) = [j \in \mathbb{N}_0]\Psi(2^{-j}|\xi|) \tag{2.4}$$

are introduced and used to define

$$\Phi_j(\xi) = \Psi_j(\xi) - \Psi_{j-1}(\xi), \quad \text{for} \quad j \in \mathbb{Z}. \tag{2.5}$$

Secondly there is then a decomposition, with (weak) convergence in $\mathscr{S}'$,

$$u = \sum_{j=0}^\infty u_j = \sum_{j=0}^\infty \mathscr{F}^{-1}\Phi_j\mathscr{F}u, \quad \text{for every} \quad u \in \mathscr{S}'. \tag{2.6}$$

Here the convention $u_j := \mathscr{F}^{-1}\Phi_j\mathscr{F}u = \mathscr{F}^{-1}(\Phi_j\hat{u})$ is used, as it is throughout.

Now the Besov space $B^s_{p,q}(\mathsf{R}^n)$ with *smoothness index* $s \in \mathsf{R}$, *integral-exponent* $p \in]0,\infty]$ and *sum-exponent* $q \in]0,\infty]$, is defined as

$$(2.7) \qquad B^s_{p,q}(\mathsf{R}^n) = \left\{ u \in \mathscr{S}'(\mathsf{R}^n) \,\middle|\, \left\| \{2^{sj} \| \mathscr{F}^{-1}\Phi_j\mathscr{F}u \,|L_p\|\}_{j=0}^{\infty} \middle| \ell_q \right\| < \infty \right\},$$

and the Triebel-Lizorkin space $F^s_{p,q}(\mathsf{R}^n)$ with *smoothness index* $s \in \mathsf{R}$, *integral-exponent* $p \in]0,\infty[$ and *sum-exponent* $q \in]0,\infty]$ is defined as

$$(2.8) \qquad F^s_{p,q}(\mathsf{R}^n) = \left\{ u \in \mathscr{S}'(\mathsf{R}^n) \,\middle|\, \left\| \| \{2^{sj}\mathscr{F}^{-1}\Phi_j\mathscr{F}u\}_{j=0}^{\infty} \,|\ell_q\|(\cdot) \middle| L_p \right\| < \infty \right\}.$$

For the history of these spaces we refer to Triebel's books [Tri83, Tri92].

The spaces $B^s_{p,q}$ and $F^s_{p,q}$ are quasi-Banach spaces with the quasi-norms given by the finite expressions in (2.7) and (2.8). Concerning an analogue of (2.1) one has

$$(2.9) \quad \| f + g \,|B^s_{p,q}\| \leqq (\| f \,|B^s_{p,q}\|^{\lambda} + \| g \,|B^s_{p,q}\|^{\lambda})^{\frac{1}{\lambda}}, \quad \text{for} \quad \lambda = \min(1,p,q),$$

with a similar inequality for the Triebel-Lizorkin spaces.

EXAMPLE 2.1. The delta distribution $\delta_0(x)$ belongs to $B^{\frac{n}{p}-n}_{p,\infty}(\mathsf{R}^n)$ for each $p \in]0,\infty]$, since by definition (2.7),

$$(2.10) \qquad \| \delta_0 \,|B^{\frac{n}{p}-n}_{p,\infty}\| = \max_{j=0,1} (2^{j(\frac{n}{p}-n)} \| \check{\Phi}_j \,|L_p\|) < \infty.$$

REMARK 2.2 For the reader's sake a piece of folklore is recalled, namely that (2.9) leads to the fact that, say, $d(u,v) = \| u - v \,|F^s_{p,q}\|^{\lambda}$ for $\lambda = \min(1,p,q)$ is a metric on $F^s_{p,q}(\mathsf{R}^n)$. For this reason both $B^s_{p,q}(\mathsf{R}^n)$ and $F^s_{p,q}(\mathsf{R}^n)$ are topological vector spaces with the topology induced by a translation invariant metric – even when p or q is <1. The same conclusion applies to, say, $\mathsf{L}(B^s_{p,q}, B^t_{r,o})$ (where the operator quasi-norm inherits the constants c and λ from $B^t_{r,o}$).

Concerning functional analysis, this shows that these spaces *in any case* are examples of the F-spaces in W. Rudin's monograph [Rud73], and hence one may refer to the exposition there. In particular the closed graph theorem is applicable.

2.3. *Properties.* In the rest of this subsection the explicit mention of the restriction $p < \infty$ concerning the Triebel-Lizorkin spaces is omitted. E.g., (2.11) below should be read with $p \in]0,\infty]$ in the $B^s_{p,q}$ part and with $p \in]0,\infty[$ in the $F^s_{p,q}$ part. Furthermore, to avoid repetition the underlying set is suppressed when it is R^n.

Identifications with other spaces are found in Section 1.

The spaces $B^s_{p,q}$ and $F^s_{p,q}$ are complete, for p and $q \geqq 1$ they are Banach spaces and in any case $\mathscr{S} \hookrightarrow B^s_{p,q}, F^s_{p,q} \hookrightarrow \mathscr{S}'$ are continuous. Moreover, the image of

$\mathcal{S}$ is dense in $B^s_{p,q}$ and in $F^s_{p,q}$ when both p and $q < \infty$, and C^∞ is so in $B^s_{\infty,q}$ for $q < \infty$ (where the latter assertion is inferred from Triebel's proof of the former [Tri83]).

The definitions imply that $B^s_{p,p} = F^s_{p,p}$, and they imply the existence of *simple* embeddings for $s \in \mathbb{R}$, $p \in]0, \infty]$ and o and $q \in]0, \infty]$,

$$B^s_{p,q} \hookrightarrow B^s_{p,o}, \quad F^s_{p,q} \hookrightarrow F^s_{p,o}, \quad \text{when} \quad q \leqq o, \tag{2.11}$$

$$B^s_{p,q} \hookrightarrow B^{s-\varepsilon}_{p,o}, \quad F^s_{p,q} \hookrightarrow F^{s-\varepsilon}_{p,o}, \quad \text{when} \quad \varepsilon > 0, \tag{2.12}$$

$$B^s_{p,\min(p,q)} \hookrightarrow F^s_{p,q} \hookrightarrow B^s_{p,\max(p,q)}. \tag{2.13}$$

There are Sobolev embeddings if $s - \frac{n}{p} \geqq t - \frac{n}{r}$ and $r > p$, more specifically

$$B^s_{p,q} \hookrightarrow B^t_{r,o}, \quad \text{provided } q \leqq o \text{ when } s - \frac{n}{p} = t - \frac{n}{r}, \tag{2.14}$$

$$F^s_{p,q} \hookrightarrow F^t_{r,o}, \quad \text{for any } o \text{ and } q \in]0, \infty]. \tag{2.15}$$

Furthermore, Sobolev embeddings also exist between the two scales, in fact under the assumptions $\infty \geqq p_1 > p > p_0 > 0$ and $s_0 - \frac{n}{p_0} = s - \frac{n}{p} = s_1 - \frac{n}{p_1}$ one has that

$$B^{s_0}_{p_0,q_0} \hookrightarrow F^s_{p,q} \hookrightarrow B^{s_1}_{p_1,q_1}, \quad \text{for } q_0 \leqq p \text{ and } p \leqq q_1. \tag{2.16}$$

This is obtained from (2.14), (2.15) and (2.13) except for the cases with equality, which are interpolation results due to J. Franke [Fra86b] and B. Jawerth [Jaw77], respectively.

By use of (2.6), (2.7) and (2.14), it is found when $0 < p, q \leqq \infty$ that

$$\begin{gathered} B^s_{p,q} \hookrightarrow B^0_{\infty,1} \hookrightarrow C \hookrightarrow L_\infty \hookrightarrow B^0_{\infty,\infty}, \\ \text{if } s > \frac{n}{p}, \text{ or if } s = \frac{n}{p} \text{ and } q \leqq 1. \end{gathered} \tag{2.17}$$

$$\begin{gathered} F^s_{p,q} \hookrightarrow B^0_{\infty,1} \hookrightarrow C \hookrightarrow L_\infty, \\ \text{if } s > \frac{n}{p}, \text{ or if } s = \frac{n}{p} \text{ and } p \leqq 1. \end{gathered} \tag{2.18}$$

Moreover, when $n\left(\frac{1}{p} - 1\right)_+ \leqq s < \frac{n}{p}$ one has, with $\frac{n}{t} = \frac{n}{p} - s$, that

$$\begin{gathered} F^s_{p,q} \hookrightarrow \cap \{L_r \mid p \leqq r \leqq t\}, \\ \text{provided } q \leqq 1 + [1 < p] \text{ if } s = 0. \end{gathered} \tag{2.19}$$

See [Joh93] or [Joh95a] for a proof of this and of the corresponding fact that

$$B^s_{p,q} \hookrightarrow \cap \{L_r \mid p \leqq r < t\}, \tag{2.20}$$

where $r = t$ can be included in general when $q \leqq t$. For $s = 0$ one has $B^s_{p,q} \hookrightarrow L_p$ for $q \leqq \min(2, p)$ and $p \geqq 1$. (Cf. [Tri92, p. 97] for the pitfalls in the case $p < 1$.)

For an open set $\Omega \subset \mathsf{R}^n$ the space $B^s_{p,q}(\bar{\Omega})$ is defined by restriction,

$$B^s_{p,q}(\bar{\Omega}) = r_\Omega B^s_{p,q} = \{u \in \mathscr{D}'(\Omega) \mid \exists v \in B^s_{p,q} : r_\Omega v = u\} \tag{2.21}$$

$$\| u \mid B^s_{p,q}(\bar{\Omega})\| = \inf\{\| v \mid B^s_{p,q}\| \mid r_\Omega v = u\}, \tag{2.22}$$

and $F^s_{p,q}(\bar{\Omega})$ is defined analogously. By the definitions all the embeddings in (2.11)–(2.20) carry over to the corresponding scales over Ω.

Moreover, when Ω is a suitable set of finite measure and $\infty \geqq p \geqq r > 0$ the inclusion $L_p(\Omega) \hookrightarrow L_r(\Omega)$ carries over to the embeddings

$$B^s_{p,q}(\bar{\Omega}) \hookrightarrow B^s_{r,q}(\bar{\Omega}), \quad F^s_{p,q}(\bar{\Omega}) \hookrightarrow F^s_{r,q}(\bar{\Omega}). \tag{2.23}$$

When Ω is bounded this is shown in [Tri83, 3.3.1], except for the case $q = \infty$ for the $F^s_{p,q}$ spaces. In [Joh95a] there is a (simpler) proof of (2.23) in its full generality.

For $m \in \mathsf{Z}$ the order-reducing operator $\Xi^m := \mathscr{F}^{-1}\langle\xi\rangle\mathscr{F}$ is bounded

$$\Xi^m \colon B^s_{p,q} \xrightarrow{\sim} B^{s-m}_{p,q}, \quad \Xi^m \colon F^s_{p,q} \xrightarrow{\sim} F^{s-m}_{p,q} \tag{2.24}$$

and bijective for any (s, p, q), cf. [Tri83]. On R^{n-1} the corresponding operator is denoted Ξ'^m.

2.4. *Convergence theorems.* Yamazaki's theorems are recalled from the article [Yam86], where the convergence of the series in the following two theorems was proved first.

THEOREM 2.3. *Let $s \in \mathsf{R}$, p and $q \in]0, \infty]$ and suppose $u_j \in \mathscr{S}'(\mathsf{R}^n)$ satisfies*

$$\operatorname{supp} \hat{u}_j \subset \{\xi \mid [j > 0]A^{-1}2^j \leqq |\xi| \leqq A2^j\}, \quad \text{for} \quad j \in \mathsf{N}_0, \tag{2.25}$$

for some $A > 0$. Then the following holds, if $p < \infty$ in (2):

(1) *If $\|\{2^{sj} \| u_j | L_p \|\}^\infty_{j=0} | \ell_q \| = B < \infty$, then the series $\sum^\infty_{j=0} u_j$ converges in $\mathscr{S}'(\mathsf{R}^n)$ to a limit $u \in B^s_{p,q}(\mathsf{R}^n)$ and the estimate $\| u | B^s_{p,q}\| \leqq CB$ holds for some constant $C = C(n, A, s, p, q)$.*

(2) *If $\| \|\{2^{sj}u_j\}^\infty_{j=0} | \ell_q \| (\cdot) | L_p \| = B < \infty$, then the series $\sum^\infty_{j=0} u_j$ converges in $\mathscr{S}'(\mathsf{R}^n)$ to a limit $u \in F^s_{p,q}(\mathsf{R}^n)$ and the estimate $\| u | F^s_{p,q}\| \leqq CB$ holds for some constant $C = C(n, A, s, p, q)$.*

Hence, when $q < \infty$ the series in (2.6) converges in $B^s_{p,q}$ for $u \in B^s_{p,q}$, and similarly for $u \in F^s_{p,q}$.

The second of these theorems states that the spectral conditions on the series $\sum^\infty_{j=0} u_j$ can be relaxed if the smoothness index s is sufficiently large.

THEOREM 2.4. *Let $s \in \mathsf{R}$, p and $q \in]0, \infty]$ and suppose $u_j \in \mathscr{S}'(\mathsf{R}^n)$ satisfies*

$$\operatorname{supp} \hat{u}_j \subset \{\xi \mid |\xi| \leqq A2^j\}, \quad \text{for} \quad j \in \mathbb{N}_0, \tag{2.26}$$

for some $A > 0$. Then the following holds, if $p < \infty$ in (2):

(1) *If $s > n\left(\frac{1}{p} - 1\right)_+$ and if $\| \{2^{sj} \| u_j | L_p \| \}_{j=0}^{\infty} | \ell_q \| = B < \infty$, then the series $\sum_{j=0}^{\infty} u_j$ converges in $\mathscr{S}'(\mathbb{R}^n)$ to a limit $u \in B^s_{p,q}(\mathbb{R}^n)$ and the estimate $\| u \mid B^s_{p,q} \| \leqq CB$ holds for some constant $C = C(n, A, s, p, q)$.*

(2) *If $s > n\left(\frac{1}{\min(p,q)} - 1\right)_+$, and if $\| \, \| \{2^{sj} u_j\}_{j=0}^{\infty} | \ell_q \| (\cdot) | L_p \| = B < \infty$ then the series $\sum_{j=0}^{\infty} u_j$ converges in $\mathscr{S}'(\mathbb{R}^n)$ to a limit $u \in F^s_{p,q}(\mathbb{R}^n)$ and the estimate $\| u | F^s_{p,q} \| \leqq CB$ holds for some constant $C = C(n, A, s, p, q)$.*

For the proofs of Theorems 2.3 and 2.4 the reader is referred to [Yam86]. In part Theorem 2.4 is based on [Yam86, Lemma 3.8], which for later reference is stated for $s < 0$ in a slightly generalised version (that is proved analogously):

LEMMA 2.5. *For each $s < 0$ and q and $r \in]0, \infty]$ there exists a $c < \infty$ such that for any sequence $\{a_j\}_{j=0}^{\infty}$ of complex numbers*

$$\| \{2^{sj} (\textstyle\sum_{k=0}^{j} |a_k|^r)^{\frac{1}{r}} \}_{j=0}^{\infty} | \ell_q \| \leqq c \, \| \{2^{sj} a_j\}_{j=0}^{\infty} | \ell_q \| \tag{2.27}$$

(with modification for $r = \infty$).

2.5. *Tensor products.* As a tool in connection with the Poisson operators in Section 4.1 a boundedness result for the operator that tensorises with the delta-distribution δ_0 is included here.

PROPOSITION 2.6. *Let p and $q \in]0, \infty]$ and suppose that $s + 1 - \frac{1}{p} < 0$. Then*

$$\| f \otimes \delta_0 | B^s_{p,q}(\mathbb{R}^n) \| \leqq c \, \| \delta_0 | B^{\frac{1}{p} - 1}_{p,\infty}(\mathbb{R}) \| \, \| f | B^{s+1-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}) \|, \tag{2.28}$$

$$\| f \otimes \delta_0 | F^s_{p,q}(\mathbb{R}^n) \| \leqq c(p,q) \, \| f | B^{s+1-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}) \|, \tag{2.29}$$

when $p < \infty$ holds in (2.29).

PROOF. Let $f \in B^{s+1-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$ and introduce the decompositions

$$f(x') = \sum_{k=0}^{\infty} \mathscr{F}^{-1}_{\xi' \to x'} \Phi'_k \mathscr{F}_{x' \to \xi'} f(x') =: \sum_{k=0}^{\infty} f_k(x'), \tag{2.30}$$

$$\delta_0(x_n) = \sum_{k=0}^{\infty} \mathscr{F}^{-1}_{\xi_n \to x_n} \Phi^{(n)}_k(\xi_n) =: \sum_{k=0}^{\infty} \eta_k(x_n), \tag{2.31}$$

where Φ'_k and $\Phi^{(n)}_k$ denote the kth element in the partition of unity associated with the x'- and x_n-space, respectively. In (2.30) and (2.31), and in the following,

$f_k := \mathscr{F}^{-1}_{\xi'\to x'}\Phi'_k\mathscr{F}_{x'\to\xi'}f(x')$ and $\eta_k := \mathscr{F}^{-1}_{\xi_n\to x_n}\Phi^{(n)}_k(\xi_n)$, whereas $f^k = \mathscr{F}^{-1}_{\xi'\to x'}\Psi'_k \mathscr{F}_{x'\to\xi'}f(x')$ and $\eta^k = \mathscr{F}^{-1}_{\xi_n\to x_n}\Psi^{(n)}_k(\xi_n)$. This is used for the central relation

$$f(x') \otimes \delta_0(x_n) = \sum_{k=0}^{\infty} f_k\eta^{k-1} + \sum_{k=0}^{\infty} f^k\eta_k, \tag{2.32}$$

which holds since it is shown below that each of the two sums on the right hand side converges in $\mathscr{S}'$. Indeed, given this convergence it follows that

$$\begin{aligned} \lim_N \sum_{k=0}^{N} f_k\eta^{k-1} + \lim_N \sum_{k=0}^{N} f^k\eta_k &= \lim_N \sum_{k,l=0}^{N} f_k\eta_l \\ &= \lim_N \mathscr{F}^{-1}(\Psi'_N \otimes \Psi^{(n)}_N)\mathscr{F}(f\otimes\delta_0) \\ &= f(x')\otimes\delta_0(x_n), \end{aligned} \tag{2.33}$$

since $\Psi'_N \otimes \Psi^{(n)}_N(\xi)$ equals the C_0^∞ function $\Psi'_0(2^{-N}\xi')\Psi^{(n)}_0(2^{-N}\xi_n)$.

In the following, Theorem 2.3 is applied to each sum in (2.32). The first step is to note the spectral conditions,

$$\begin{aligned} \operatorname{supp}\mathscr{F}(f_k\eta^{k-1}) &\subset \{\xi \mid \tfrac{11}{10}2^{k-1} \leqq |(\xi',0)| \leqq \tfrac{13}{10}2^k,\ |(0,\xi_n)| \leqq \tfrac{13}{10}2^{k-1}\} \\ &\subset \{\xi \mid \tfrac{11}{20}2^k \leqq |\xi| \leqq \tfrac{39}{20}2^k\}, \end{aligned} \tag{2.34}$$

$$\operatorname{supp}\mathscr{F}(f^k\eta_k) \subset \{\xi \mid \tfrac{11}{20}2^k \leqq |\xi| \leqq \tfrac{26}{20}2^k\}. \tag{2.35}$$

Secondly the $\ell^s_q(L_p)$ norms of the sums are estimated. From $\eta^l = 2^l\eta_0(2^l x_n)$ it is seen that $\| f_k\eta^{k-1} \,|L_p(\mathrm{R}^n)\| = 2^{(k-1)(1-\frac{1}{p})} \| \eta_0 \,|L_p(\mathrm{R})\| \, \| f_k \,|L_p(\mathrm{R}^{n-1})\|$, since the L_p norm is multiplicative. Then, with $c = \| \delta_0 \,|B^{\frac{1}{p}-1}_{p,\infty} \|$,

$$\|\{2^{ks}\| f_k\eta^{k-1} \,|L_p\|\}_{k=0}^{\infty} \,|\ell_q\| \leqq 2^{\frac{1}{p}-1}c\,\|\{2^{k(s+1-\frac{1}{p})}\| f_k \,|L_p\|\}_{k=0}^{\infty} \,|\ell_q\|, \tag{2.36}$$

cf. Example 2.1. Concerning the second sum one finds in a similar way that

$$\begin{aligned} 2^{ks}\| f^k\eta_k \,|L_p\| &= 2^{ks}\| f_0 + \cdots + f_k \,|L_p\| \, \| \eta_k \,|L_p\| \\ &\leqq 2^{k(s+1-\frac{1}{p})}(\| f_0 \,|L_p\|^r + \cdots + \| f_k \,|L_p\|^r)^{\frac{1}{r}} \, \| \delta_0 \,|B^{\frac{1}{p}-1}_{p,\infty} \|, \end{aligned} \tag{2.37}$$

when $r = \min(1,p)$. The assumption $s + 1 - \dfrac{1}{p} < 0$ in Proposition 2.6 now allows an application of Lemma 2.5 above, leading to the estimate

$$\|\{2^{ks}\| f^k\eta_k \,|L^p\|\}_{k=0}^{\infty} \,|\ell_q\| \leqq c\,\|\{2^{k(s-1+\frac{1}{p})}\| f_k \,|L_p\|\}_{k=0}^{\infty} \,|\ell_q\|, \tag{2.38}$$

with $c = c' \| \delta_0 \,|B^{\frac{1}{p}-1}_{p,\infty} \|$ for some $c' < \infty$ depending on p and q.

From (2.34), (2.35), (2.36) and (2.38) it follows by Theorem 2.3 that the series in

(2.32) converge in S', and that the sums belong to $B^s_{p,q}(\mathsf{R}^n)$ with norms estimated by constants times the right hand sides of (2.36) and (2.38), respectively. By use of (2.32) it follows that also $f \otimes \delta_0 \in B^s_{p,q}$, and by application of the quasi-triangle inequality this implies (2.28).

For $p \leqq q$ the estimate in (2.29) follows from (2.28) by use of the embedding $B^s_{p,p} \hookrightarrow F^s_{p,q}$. The case $q < p$ is obtained like (2.28) by application of Theorem 2.3 to the sums in (2.32). However, the necessary estimates of the $L_p(\ell^s_q)$ norms are substantially more complicated than (2.36) and (2.38). But with (mainly) notational changes one can proceed as in [Tri83, p. 136], where estimates analogous to (2.28) and (2.29) are shown for a right inverse of $\tilde{\gamma}_0$ (cf. Section 2.6 below).

[To be more specific one can treat the first sum in (2.32) by letting a_k in [Tri83, 2.7.2/31] be equal to f_k, and for simplicity replace the reference to [Tri83, Thm. 1.6.3] by an application of Theorem 2.3. Concerning the second sum in (2.32) one can start by showing an analogue of [Tri83, 2.7.2/34] for η_k and then proceed as before except with $a_k = f^k$ instead; in suitable late stages of the various estimates one can then introduce $\| f^k \,| L_p \| \leqq (\| f_0 \,| L_p\|^r + \cdots + \| f_k \,| L_p \|^r)^{\frac{1}{r}}$, for $r = \min(1, p)$, together with Lemma 2.5.]

2.6. *Traces.* In preparation for Section 4.3 below on general trace operators in the Boutet de Monvel calculus some well-known facts about restriction to hyperplanes is modified to suit the purposes there.

The basic trace operator is the two-sided restriction operator, which takes $v(x)$ in $C^\infty(\mathsf{R}^n)$ to $v(x', 0)$; it is denoted by $\tilde{\gamma}_0 v$. The properties of $\tilde{\gamma}_0$ are investigated in numerous papers, see [FJ90], e.g., and the references therein.

For $u \in C^\infty(\bar{\mathsf{R}}^n_+)$, the *one-sided* restriction operator $\gamma_0 u$ is defined by letting

$$\gamma_0 u = \tilde{\gamma}_0 v, \quad \text{when} \quad r^+ v = u \quad \text{holds for} \quad v \in C^\infty(\mathsf{R}^n). \tag{2.39}$$

Evidently one has the intrinsic description $\gamma_0 u(x') = u(x', +0)$. Moreover, let $\tilde{\gamma}_j = \tilde{\gamma}_0 D^j_{x_n}$ and $\gamma_j = \gamma_0 D^j_{x_n}$.

Henceforth the following simplifying notation is employed: for $k \in \mathsf{Z}$ the parameter (s, p, q) is said to belong to the set D_k if

$$s > k + \max\left(\frac{1}{p} - 1, \frac{n}{p} - n\right), \tag{2.40}$$

cf. Figure 2 below. That $s \geqq k + \max\left(\frac{1}{p} - 1, \frac{n}{p} - n\right)$ means that (s, p, q) belongs to the closure of D_k, so $(s, p, q) \in \bar{\mathsf{D}}_k$ is written then.

For the one-sided trace operator γ_j the following result is needed below (whereas those for $\tilde{\gamma}_0$ in [Tri83, Tri92] do not suffice).

LEMMA 2.7. *For each $j \in \mathbb{N}_0$ the trace γ_j extends uniquely to a bounded operator (when $p < \infty$ in (2.42))*

$$\gamma_j\colon B^s_{p,q}(\bar{\mathbb{R}}^n_+) \to B^{s-j-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}), \quad \text{for} \quad (s,p,q) \in \mathsf{D}_{j+1}, \tag{2.41}$$

$$\gamma_j\colon F^s_{p,q}(\bar{\mathbb{R}}^n_+) \to F^{s-j-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}), \quad \text{for} \quad (s,p,q) \in \mathsf{D}_{j+1}, \tag{2.42}$$

Moreover, when $(s,p,q) \notin \bar{\mathsf{D}}_{j+1}$, there is not any extension of γ_j with the continuity properties in (2.41) *or* (2.42).

PROOF. It is known that $\tilde{\gamma}_0$ has continuity properties corresponding to (2.41) and (2.42), cf. [Tri83, 2.7.2], so it is sufficient to see that γ_0 is well defined by (2.39), i.e., $\tilde{\gamma}_0 v = 0$ should hold whenever v belongs to $B^s_{p,q;0}(\bar{\mathbb{R}}^n_-)$ or $F^s_{p,q;0}(\bar{\mathbb{R}}^n_-)$. It suffices to treat $q < \infty$, and by the continuity of $\tilde{\gamma}_0$, it is enough to prove that $\{\varphi \in \mathcal{S}(\mathbb{R}^n) \mid \bar{\mathbb{R}}^n_- \supset \operatorname{supp} \varphi\}$ is dense in $B^s_{p,q;0}(\bar{\mathbb{R}}^n_-)$ and $F^s_{p,q;0}(\bar{\mathbb{R}}^n_-)$ for $p < \infty$, and that $\{\varphi \in C^\infty(\mathbb{R}^n) \mid \bar{\mathbb{R}}^n_- \supset \operatorname{supp} \varphi\}$ is dense in $B^s_{\infty,q;0}(\bar{\mathbb{R}}^n_-)$.

However, if $\tau_h f = f(\cdot - he_n)$, $\tau_h \to 1$ strongly on $\mathcal{S}(\mathbb{R}^n)$ and on $C^\infty(\mathbb{R}^n)$ for $h \to 0$. By use of the denseness and the relations $\mathcal{F}^{-1}\Phi_j\mathcal{F}\tau_h f = \tau_h\mathcal{F}^{-1}\Phi_j\mathcal{F}f$, it is seen that $\tau_h \to 1$ in the strong operator topology on $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ – for any admissible (s,p,q).

For $u \in B^s_{p,q;0}(\bar{\mathbb{R}}^n_-)$ and $\varepsilon > 0$ we take $h < 0$ so small that $\|u - \tau_h u \mid B^s_{p,q}\| < \frac{\varepsilon}{2}$, and let $g \in C^\infty(\mathbb{R}^n)$ satisfy $\operatorname{supp} g \subset \bar{\mathbb{R}}^n_-$ and $g = 1$ on $\operatorname{supp} \tau_h u$. Because multiplication by g is continuous in $B^s_{p,q}(\mathbb{R}^n)$, we obtain that

$$\| u - gv_k | B^s_{p,q} \| \leqq \frac{\varepsilon}{2} + \| g\tau_h u - gv_k | B^s_{p,q} \| \leqq \varepsilon \tag{2.43}$$

holds eventually, when $v_k \in \mathcal{S}(\mathbb{R}^n)$ (resp. $C^\infty(\mathbb{R}^n)$ for $p = \infty$) converges to $\tau_h u$ in $B^s_{p,q}(\mathbb{R}^n)$. When $u \in F^s_{p,q;0}(\bar{\mathbb{R}}^n_-)$ one can proceed in the same manner.

For $j > 0$ it is now obvious that the composite $\gamma_0 D^j_{x_n}$ is bounded as in (2.41) and (2.42). The uniqueness follows from the denseness of $\mathcal{S}$ or C^∞ when $q < \infty$.

When $(s,p,q) \notin \bar{\mathsf{D}}_{j+1}$ the non-extendability of γ_j follows from Lemma 2.8 below regardless of the choice of $u \neq 0$ and z'. Indeed, for $p < 1$ the existence of r^+v_k shows that γ_j is not continuous at 0 from any space $B^s_{p,q}(\bar{\mathbb{R}}^n_+)$ or $F^s_{p,q}(\bar{\mathbb{R}}^n_+)$ when $s < \frac{n}{p} - (n-1) + j$. For $1 \leqq p < \infty$ the existence of r^+u_k yields the same conclusion $\left(\text{for spaces with } s < \frac{1}{p} + j\right)$, while in the case $p = \infty$ and $s < j$ a Sobolev embedding $B^{\frac{1}{r}+j}_{r,q} \hookrightarrow B^s_{\infty,q}$ reduces the question to the case $p < \infty$.

The next lemma was used in the proof above, and it will later on provide counterexamples that are strong enough to show that each trace operator T, that *has* class $j \in \mathbb{Z}$, is not extendable to spaces with $(s, p, q) \notin \bar{D}_j$, cf. Theorem 4.9 below.

LEMMA 2.8. *For each* $j \in \mathbb{N}_0$, $u \in \mathscr{S}(\mathbb{R}^{n-1})\setminus\{0\}$ *and* $z' \in \mathbb{R}^{n-1}$ *there exists two sequences with elements* u_k *and* $v_k \in \mathscr{S}(\mathbb{R}^n)$ *with the properties*

$$\text{(2.44)} \quad \begin{cases} \tilde{\gamma}_j u_k(x') = u(x') \text{ for each } k \in \mathbb{N}, \\ \lim_{k\to\infty} u_k = 0 \text{ in } B^{\frac{1}{p}+j}_{p,q}(\mathbb{R}^n) \text{ for } 1 < q \leqq \infty, \\ \lim_{k\to\infty} u_k = 0 \text{ in } F^{\frac{1}{p}}_{p,q}(\mathbb{R}^n) \text{ for } 1 < p < \infty, \end{cases}$$

$$\text{(2.45)} \quad \begin{cases} \lim_{k\to\infty} \tilde{\gamma}_j v_k(x') = \delta_{z'}(x') \text{ in } \mathscr{S}'(\mathbb{R}^{n-1}), \\ \lim_{k\to\infty} v_k = 0 \text{ in } B^{\frac{n}{p}-(n-1)+j}_{p,q}(\mathbb{R}^n) \text{ for } 1 < q \leqq \infty, \end{cases}$$

provided $0 < p < \infty$ *in* (2.44) *respectively* $0 < p \leqq 1$ *in* (2.45).

PROOF. In the deduction of (2.44), recall the fact from [Fra86b] that for $s > 0$

$$\text{(2.46)} \qquad \| f(x') \otimes g(x_n) \,|B^s_{p,q}(\mathbb{R}^n)\| \leqq c \, \| f \,|B^s_{p,q}(\mathbb{R}^{n-1})\| \; \| g \,|B^s_{p,q}(\mathbb{R})\|.$$

Here f will play the role of the given $u \in \mathscr{S}(\mathbb{R}^{n-1})$, while for g we shall take $w_k(x_n) = k^{-1}\sum_{l=1}^{k} 2^{-lj} w(2^l x_n)$ for some auxiliary function $w \in \mathscr{S}(\mathbb{R})$ satisfying $\operatorname{supp} \hat{w} \subset \{\frac{3}{4} \leqq |\xi_n| \leqq 1\}$ and $\int \xi^j \hat{w} = 2\pi$. Observe that $\tilde{\gamma}_j w = 1$.

One can let $u_k(x) = u(x') \cdot w_k(x_n)$, for $\tilde{\gamma}_j u_k = u$ and, since $\operatorname{supp} \mathscr{F}(w(2^l \cdot)) \subset \{\Phi^{(n)}_l \equiv 1\}$ where $\Phi^{(n)}_l$ is as in (2.31) ff.,

$$\text{(2.47)} \quad \begin{aligned} \| u_k \,|B^{\frac{1}{p}+j}_{p,q}\| &\leqq c \, \| u \,|B^{\frac{1}{p}+j}_{p,q}(\mathbb{R}^{n-1})\| \; \| \{2^{\frac{1}{p}} \| w(2^l \cdot) \,|L_p(\mathbb{R})\| \}_{l=1}^{k} \,|\ell_q\| \, k^{-1} \\ &= c \, \| u \,|B^{\frac{1}{p}+j}_{p,q}\| \; \| w \,|L_p\| \, k^{\frac{1}{q}-1} \end{aligned}$$

by (2.46). Here $k^{\frac{1}{q}-1} \to 0$ for $k \to \infty$, when $q > 1$.

When $p > 1$ there is an embedding $B^t_{r,r}(\mathbb{R}) \hookrightarrow F^{\frac{1}{p}+j}_{p,q}(\mathbb{R})$ for an $r \in]1, p[$ and $t - \dfrac{1}{r} = j$. Since there is an analogue of (2.46) for the $F^s_{p,q}$ spaces, cf. [Fra86b], there is an estimate $\| u_k \,|F^{\frac{1}{p}+j}_{p,q}\| \leqq c' \, \| u \,|F^{\frac{1}{p}+j}_{p,q}\| \; \| w \,|L_r\| \, k^{\frac{1}{r}-1}$. Because $r > 1$, $u_k \to 0$ in $F^{\frac{1}{p}+j}_{p,q}$.

To obtain (2.45) for $z' = 0$ we take $f \in \mathscr{S}(\mathbb{R}^{n-1})$ and $g \in \mathscr{S}(\mathbb{R})$ satisfying

$$\text{(2.48)} \quad \begin{aligned} &\operatorname{supp} \hat{f} \subset \{|\xi'| \leqq \tfrac{1}{2}\}, \quad \hat{f}(0) = 1, \\ &\operatorname{supp} \hat{g} \subset \{|\xi_n| \leqq \tfrac{1}{2}\}, \quad \int \xi_n^j \hat{g}(\xi_n) \, d\xi_n = 2\pi, \end{aligned}$$

and let $v_k = \frac{1}{k}\sum_{l=k+1}^{2k} 2^{l(n-1-j)} f(2^l x') g(2^l x_n)$. Now $\tilde{\gamma}_j v_k = \frac{1}{k}\sum 2^{l(n-1)} f(2^l \cdot)$ and, by a modification of the usual proof of the fact that $2^{k(n-1)} f(2^k \cdot) * \cdot \to 1$ in the strong operator topology on $C(\mathbb{R}^{n-1})$, it is verified that $\tilde{\gamma}_j v_k * \cdot \to 1$ strongly on $C(\mathbb{R}^{n-1})$; in particular this implies $\tilde{\gamma}_j v_k(x') \to \delta_0(x')$ in $\mathscr{S}'(\mathbb{R}^{n-1})$.

Since $\operatorname{supp} \mathscr{F}(fg(2^l \cdot)) \subset \{|\xi| \leqq 2^l\}$ and since $p \leqq 1$, Theorem 2.4 can be applied to the sum defining v_k, which gives a constant c, independent of k, such that

$$\| v_k \,|\, B^{\frac{n}{p}-(n-1)+j}_{p,q} \| \leqq \frac{c}{k} \| \{ 2^{l\frac{n}{p}} \| fg(2^l \cdot) \,|\, L_p \| \}_{l=k+1}^{2k} \,|\, \ell_q \| = c \| fg \,|\, L_p \| \, k^{\frac{1}{q}-1}, \tag{2.49}$$

so for $z' = 0$ the properties of the v_k are proved. For $z' \neq 0$ one can simply translate.

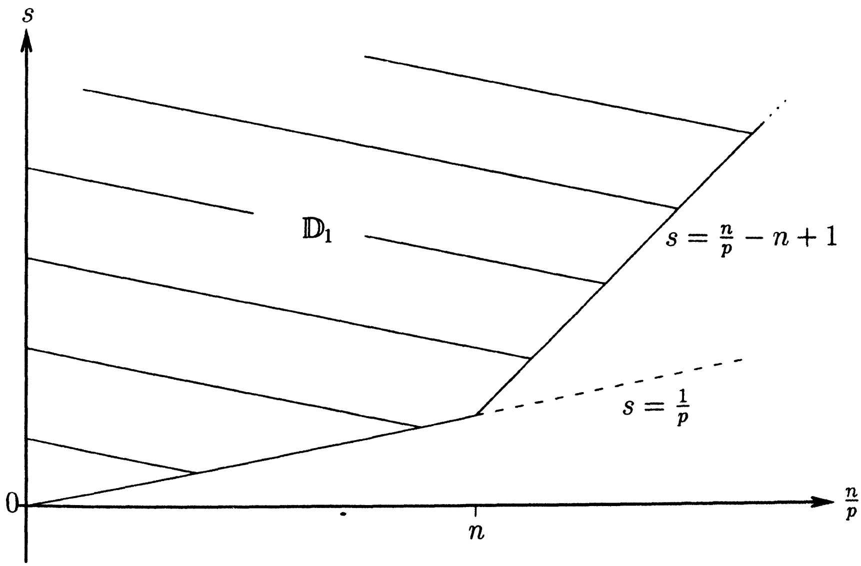

Fig. 1. The borderline cases for $\tilde{\gamma}_0$ (when $n = 5$)

REMARK 2.9. When applied to γ_j Lemma 2.8 gives a little more than stated in Lemma 2.7. In fact, if we for simplicity consider $\tilde{\gamma}_0$, for each Hausdorff topological space X, there does not exist any extension $\tilde{\gamma}_0 \colon B^s_{p,q}(\mathbb{R}^n) \to X$, that is continuous when $s = \frac{1}{p}$ and $q > 1$, or $s < \frac{1}{p}$, cf. (2.44). Moreover, there does not exist any continuous extension $\tilde{\gamma}_0\colon F^s_{p,q}(\mathsf{R}^n) \to X$ when $s < \frac{1}{p}$, since $B^{\frac{1}{p}}_{p,\infty} \hookrightarrow F^s_{p,q}$ then, and for $1 < p < \infty$ even $s = \frac{1}{p}$ is excluded by (2.44).

These counterexamples, that were inspired by [Gru90, Lemma 2.2], are sharper than previous ones obtained by H. Triebel, in that $X = \mathscr{S}'(\mathsf{R}^n)$ is not assumed and the borderline cases $s = \frac{1}{p}$ are included, cf. [Tri83, 2.7.2 Rem. 4].

Moreover, the counterexamples provided by the u_k for the Besov spaces are optimal for $p \in [1, \infty[$, since $\tilde{\gamma}_0(B^{\frac{1}{p}}_{p,q}(\mathsf{R}^n)) = L_p(\mathsf{R}^{n-1})$ for $q \leqq \min(1,p)$ when $p \in]0, \infty[$, cf. [FJ85]. For $p < 1$ it is shown in [FJ85] that one can not take $X = L_p + L_\infty$ when $q > p$. Contrary to this $\tilde{\gamma}_0(B^0_{\infty,1}(\mathsf{R}^n)) \subset C(\mathsf{R}^{n-1})$. In the cases with $p = \infty$ and $p < q \leqq 1$ a strengthening of the u_k-counterexample above would be appropriate.

For the Triebel-Lizorkin spaces it was obtained in [FJ90] that $\tilde{\gamma}_0(F^{\frac{1}{p}}_{p,q}(\mathsf{R}^n)) = L_p(\mathsf{R}^{n-1})$, independently of q when $p \leqq 1$, and that one can not take $X = \mathscr{S}'(\mathsf{R}^{n-1})$ for any $p \in]1, \infty[$ and $q \in]0, \infty]$. So for these spaces the u_k-example removes the restriction on the space X.

For $p < 1$ the v_k yield stronger conclusions in the sense that even $s \geqq \frac{n}{p} - (n-1)$ is necessary for (2.41) and (2.42). On the other hand, the conclusions are weaker in the sense that when $s < \frac{n}{p} - (n-1)$ any X with a continuous embedding $X \hookrightarrow \mathscr{D}'$ is impossible, while other choices of X might work. Indeed, from the results quoted above it is seen that one can take the non-locally convex space $X = L_p$ when $\frac{1}{p} \leqq s \leqq \frac{n}{p} - (n-1)$ (except when $q > p$ for $s = \frac{1}{p}$ in the Besov case). Furthermore, in the borderline case $s = \frac{n}{p} - (n-1)$ for $p < 1$, $\tilde{\gamma}_0(F^{\frac{n}{p}-(n-1)}_{p,q}(\mathsf{R}^n)) + \tilde{\gamma}_0(B^{\frac{n}{p}-(n-1)}_{p,q}(\mathsf{R}^n)) \subset \tilde{\gamma}_0(B^1_{1,1}(\mathsf{R}^n)) = L_1(\mathsf{R}^{n-1})$ are well defined subspaces of $\mathscr{S}'(\mathsf{R}^{n-1})$ when $q \leqq 1$ and $r \in]0, \infty]$; this shows the optimality of the v_k-counterexample in this case.

However, the range spaces for $\tilde{\gamma}_0$ considered on $B^0_{\infty,q}$, $B^{\frac{n}{p}-n+1}_{p,q}$ and $F^{\frac{n}{p}-n+1}_{p,q}$ with $q \leqq 1$ and $p < 1$ seems to be undetermined yet.

2.7. *Extension by zero.* For a function $f(x) \in L_2(\mathsf{R}^n_+)$ there is an extension by zero to the whole space, for example $e^+ f(x) = [x_n > 0] f(x', |x_n|)$.

The boundedness of e^+ $L_2(\mathsf{R}^n)$ and the properties that

(2.50) $$r^+e^+ = 1 \quad \text{and} \quad r^-e^+ = 0$$

extend to spaces with parameters (s, p, q) in a whole region around $(0, 2, 2)$.

In fact, when (s, p, q) satisfies

(2.51) $$\max\left(\frac{1}{p} - 1, \frac{n}{p} - n\right) < s < \frac{1}{p},$$

the operator e^+ can be given a sense as a bounded operator

(2.52) $$e^+ \colon B^s_{p,q}(\bar{\mathsf{R}}^n_+) \to B^s_{p,q}(\mathsf{R}^n), \qquad e^+ \colon F^s_{p,q}(\bar{\mathsf{R}}^n_+) \to F^s_{p,q}(\mathsf{R}^n)$$

(when $p < \infty$ in the F case), which has the properties in (2.50).

For this it is convenient to use the product $\pi(u, v)$ defined in [Joh95a] for u and $v \in \mathscr{S}'(\mathsf{R}^n)$ as

(2.53) $$\pi(u, v) = \lim_{k \to \infty} \mathscr{F}^{-1}(\psi(2^{-k}\cdot)\hat{u})\mathscr{F}^{-1}(\psi(2^{-k}\cdot)\hat{v})$$

when the limit exists in the w^*-topology on $\mathscr{D}'(\mathsf{R}^n)$ for each $\psi \in C_0^\infty(\mathsf{R}^n)$ that equals 1 on a neighbourhood of the origin. Here the limit is required to be independent of ψ.

Now, for u in $B^s_{p,q}(\bar{\mathsf{R}}^n_+)$ or $F^s_{p,q}(\bar{\mathsf{R}}^n_+)$ with (2.51) satisfied by (s, p, q) one can *define*, with $\chi(x) = [x_n > 0]$,

(2.54) $$e^+u = \pi(\chi, v) \quad \text{when} \quad r^+v = u$$

for $v \in B^s_{p,q}(\mathsf{R}^n)$ and $v \in F^s_{p,q}(\mathsf{R}^n)$ respectively.

Note that $v \in L_t(\mathsf{R}^n)$ when $\frac{n}{t} = \frac{n}{p} - s$ holds for $s > 0$ in addition to (2.51). Then [Joh95a, Prop. 3.8] gives that $\pi(\chi, v) = \chi v$, so e^+u has the usual meaning. Moreover, it was proved in [Tri83, Thm. 2.8.7] and [Fra86b, Cor. 3.4.6] that

(2.55) $$\pi(\chi, \cdot) \colon B^s_{p,q}(\mathsf{R}^n) \to B^s_{p,q}(\mathsf{R}^n) \quad \text{and} \quad \pi(\chi, \cdot) \colon F^s_{p,q}(\mathsf{R}^n) \to F^s_{p,q}(\mathsf{R}^n)$$

are bounded when (2.51) holds. By taking the infimum over v in (2.54) the boundedness of e^+ follows.

However, (2.54) needs to be justified. So, if v_1 belongs to the same space as v and $r^+v_1 = u$, then $\pi(\chi, v)$ and $\pi(\chi, v_1)$ coincide except at $\{x \mid x_n = 0\}$. For with $w = v - v_1$ at least one of the factors in $\pi(\chi, w)$ is 0 in R^n_- and in R^n_+, so that $r^\pm\pi(\chi, w) = 0$ by [Joh95a, Prop. 3.7].

For $s > 0$ or otherwise when $B^s_{p,q}$ and $F^s_{p,q}$ only countain functions, $\pi(\chi, w) = 0$ is necessary. When $s \leqq 0$ this conclusion is obtained from the inequality $s > \frac{1}{p} - 1$ by duality: from the definition of $(e^+)^* \colon B^{-s}_{p',q'}(\bar{\mathsf{R}}^n_+) \to B^{-s}_{p',q';0}(\bar{\mathsf{R}}^n_+)$

$\left(\text{where } e^+ \text{ refers to one of the already covered cases with } 0 \leqq s < \frac{1}{p}\right)$ it follows when $1 < q' < \infty$ that

$$\langle (e^+)^* r^+ \psi, \overline{r^+ f} \rangle = \int_{\mathsf{R}^n} \psi \chi \bar{f} = \langle \chi\psi, \overline{r^+ f} \rangle \tag{2.56}$$

when $\psi \in C^\infty(\mathsf{R}^n) \cap B^{-s}_{p',q'}$ and $f \in B^s_{p,q}(\mathsf{R}^n)$, so by closure the identity $(e^+)^* r^+ \psi = \pi(\chi, \psi)$ holds for every $\psi \in B^{-s}_{p',q'}(\mathsf{R}^n)$. This shows that $\pi(\chi, v)$ also in these cases only depends on $r^+ v$. The case with $q' = \infty$ are covered by simple embeddings.

Finally it should be mentioned that the second part of (2.50) is a direct consequence of [Joh95a, Prop. 3.7]. The first part also follows from this when combined with [Joh95a, Prop. 3.6]: $r^+ e^+ u = r^+ \pi(\chi, v) = r^+ \pi(1, v) = r^+ v$. Altogether the desired properties of e^+ as defined in (2.54) has been obtained.

REMARK 2.10. The extension operator e^+ has been defined with care above, albeit a definition as a self-adjoint operator by (2.56) is simpler. The present definition is more flexible, however, for it allows an analysis by means of paramultiplication, which is crucial for the proof of Theorem 4.5 below.

2.8. *Interpolation.* For the proof of Theorem 4.5 below it is necessary to have interpolation available. In addition to the real method, described in [Tri83, 2.3.2], properties similar to those of the complex method are needed.

Here it is on one hand well known that the usual complex interpolation method due to A. P. Calderón [Cal64] does not extend to quasi-Banach spaces. On the other hand, the so-called $+-$-method may serve as a substitute, as was pointed out to me by both W. Sickel and J. Marshall.

The $+-$-interpolation of two quasi-Banach spaces A_0 and A_1 – both lying inside some Hausdorff topological vector space X – is defined by J. Gustavsson and J. Peetre [GP77], and it is usually denoted by $\langle A_0, A_1, \rho \rangle$. Here the function ρ will be $t \mapsto t^\theta$ for some $\theta \in]0,1[$, and $(A_0, A_1)_{\pm,\theta} := \langle A_0, A_1, t^\theta \rangle$ in order to avoid confusion with dualities. In general $(A_0, A_1)_{\pm,\theta}$ is a quasi-Banach space.

Moreover, the interpolation property was proved in [GP77]. That is to say, when T is a linear operator defined on X (or a subspace) such that

$$T\colon A_j \to B_j \tag{2.57}$$

is bounded for $j = 0$ and 1, it is so also for $j = 2$, when $A_2 = (A_0, A_1)_{\pm,\theta}$ and $B_2 = (B_0, B_1)_{\pm,\theta}$. Here $\|T\|_2 \leqq \|T\|_0^{1-\theta} \|T\|_1^\theta$ holds for the operator quasi-norms.

For the $F^s_{p_0,q_0}$ scale it was shown by Frazier and Jawerth [FJ90] that

(2.58) $$(F^{s_0}_{p_0,q_0}(\mathsf{R}^n), F^{s_1}_{p_1,q_1}(\mathsf{R}^n))_{\pm,\theta} = F^{s_2}_{p_2,q_2}(\mathsf{R}^n)$$

for each $\theta \in]0, 1[$ and any admissible parameters provided

(2.59) $$s_2 = (1-\theta)s_0 + \theta s_1;$$
$$\frac{1}{p_2} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_2} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

This result is also valid for open sets $\Omega \subset \mathsf{R}^n$, when they have the extension property. This means that for each $N \in \mathsf{N}$ there exists an operator ℓ_Ω, which is bounded

(2.60) $$\ell_\Omega: B^s_{p,q}(\bar{\Omega}) \to B^s_{p,q}(\mathsf{R}^n), \quad \ell_\Omega: F^s_{p,q}(\bar{\Omega}) \to F^s_{p,q}(\mathsf{R}^n)$$

for $|s| < N$ and $\frac{1}{N} < p, q \leqq \infty$, and for which the composite

(2.61) $$F^s_{p,q}(\bar{\Omega}) \xrightarrow{\ell_\Omega} F^s_{p,q}(\mathsf{R}^n) \xrightarrow{r_\Omega} F^s_{p,q}(\bar{\Omega})$$

equals the identity, with a similar property for the Besov spaces.

The formulae (2.58) and (2.61) and the interpolation property now give

PROPOSITION 2.11. *Let $\Omega \subset \mathsf{R}^n$ be an open set with the extension property, let $\theta \in]0, 1[$ and let $s_j \in \mathsf{R}$, $p_j \in]0, \infty]$ and $q_j \in]0, \infty]$ for $j = 0$ and 1.*

When (s_2, p_2, q_2) satisfies (2.59), then

(2.62) $$F^{s_2}_{p_2,q_2}(\bar{\Omega}) = (F^{s_0}_{p_0,q_0}(\bar{\Omega}), F^{s_1}_{p_q,q_1}(\bar{\Omega}))_{\pm,\theta}$$

holds with equivalent quasi-norms.

That Ω has the extension property when it is bounded and C^∞ smooth or $\Omega = \mathsf{R}^n_+$ was proved in [Fra86b] (and with some restrictions for the F case also in [Tri83]). By the general result in [GP77, Prop. 6.1] the interpolation property holds for the $F^s_{p,q}(\bar{\Omega})$ spaces too.

REMARK 2.12. It deserves to be mentioned, that the $B^s_{p,q}(\mathsf{R}^n)$ and $F^s_{p,q}(\mathsf{R}^n)$ scales are invariant under a complex interpolation based on $\mathscr{S}'$-analytical functions, cf. [Tri83]. However, for this method the interpolation property has only been verified for $\max(p_2, q_2) < \infty$ in [Fra86a], and under the assumption that there is continuity from, say, $B^{s_2-\varepsilon}_{p_2,q_2}$ to $\mathscr{S}'(\mathsf{R}^n)$ the case $q_2 = \infty$ was included there too.

An overview of this is contained in [Joh93], even with a removal of the restriction to $p_2 < \infty$. Although this approach works equally well for the application in [Fra86a], and thus in the present paper too, the $+ -$-interpolation is preferred here because of the available references. Ultimately the proofs are also more structured and less technical, then.

3. Operators on $\bar{\mathbb{R}}^n_+$.

To begin with the operators are defined on the spaces $\mathscr{S}(\bar{\mathbb{R}}^n_+)$ and $\mathscr{S}(\mathbb{R}^{n-1})$. More general spaces are introduced afterwards in Section 4.

Since the inclusion of the $B^s_{\infty,q}$ spaces requires the Definitions 4.8 and 4.14 of the operators (because $\mathscr{S}$ is not dense there), the exposition in Sections 3 and 4 is intended to be fairly detailed.

In particular proofs are given for Propositions 4.1–4.13, albeit the contents are essentially known. However, none of the references apply directly, and at least the presented proofs should be of interest in view of their elementary nature.

For a general introduction to the Boutet de Monvel calculus the reader is referred to the exposition in [Gru91] and to Section 1.1 ff. in [Gru86].

3.1. *Review of the operators.* Recall that a truncated pseudo-differential operator P_+, a Poisson operator K, a trace operator T and a singular Green operator G, cf. (1.2) ff., act in the following way on $u \in \mathscr{S}(\bar{\mathbb{R}}^n_+)$ and $v \in \mathscr{S}(\mathbb{R}^{n-1})$ – when T and G are of *class zero*:

$$P_+u(x) = r^+(2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi) \widehat{e^+u}(\xi)\, d\xi, \tag{3.1}$$

$$Kv(x) = (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \tilde{k}(x', x_n, \xi') \hat{v}(\xi')\, d\xi', \tag{3.2}$$

$$Tu(x') = (2\pi)^{1-n} \int_{\mathbb{R}^n_+} e^{ix'\cdot\xi'} \tilde{t}(x', y_n, \xi') \acute{u}(\xi', y_n)\, dy_n d\xi', \tag{3.3}$$

$$Gu(x) = (2\pi)^{1-n} \int_{\mathbb{R}^n_+} e^{ix'\cdot\xi'} \tilde{g}(x', x_n, y_n, \xi') \acute{u}(\xi', y_n)\, dy_n d\xi', \tag{3.4}$$

The fifth kind of operators in the calculus are the pseudo-differential operators S acting on $v \in \mathscr{S}(\mathbb{R}^{n-1})$ in the usual way, cf. (3.13). The definition of class $r \in \mathbb{Z}$ of T, G and $P_+ + G$ is recalled in Subsections 4.3–4.5 below.

For P_+ the uniform two sided transmission condition will be employed to assure that P_+u belongs to $C^\infty(\bar{\mathbb{R}}^n_+)$ when $u \in \mathscr{S}(\bar{\mathbb{R}}^n_+)$, see [GK93] and [GH91] for a discussion of this condition.

The starting point is the uniform class $S^d_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ given with seminorms $\|p \mid S^d_{1,0}, \alpha, \beta\| := C_{\alpha,\beta}$ in (1.11) above. While the symbol of S is taken in $S^d_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, that of P is required to belong to $S^d_{1,0,\mathrm{uttr}}(\mathbb{R}^n \times \mathbb{R}^n)$:

DEFINITION 3.1. For $d \in \mathbb{R}$ the space $S^d_{1,0,\mathrm{uttr}}(\mathbb{R}^n \times \mathbb{R}^n)$ consists of the symbols $p(x,\xi) \in S^d_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying the *uniform two-sided* transmission condition (at $x_n = 0$), i.e., for every $\alpha, \beta \in \mathbb{N}^n_0$ and $l, m \in \mathbb{N}_0$ the condition

$$(3.5)\qquad C_{\alpha,\beta,l,m}(p) := \sup |z_n^l D_{z_n}^m \mathscr{F}^{-1}_{\xi_n \to z_n} D_\xi^\alpha D_x^\beta p(x', 0, \xi)| < \infty$$

holds for each ξ' when the supremum is taken over $(x', z_n) \in \mathsf{R}^n \setminus \{z_n = 0\}$.

In formulae (3.2) and (3.4) the symbol-kernels $\tilde{k}$ and $\tilde{g}$ can belong to the uniform spaces $S_{1,0}^{d-1}(\mathsf{R}^{n-1} \times \mathsf{R}^{n-1}, \mathscr{S}(\bar{\mathsf{R}}_+))$ and $S_{1,0}^{d-1}(\mathsf{R}^{n-1} \times \mathsf{R}^{n-1}, \mathscr{S}(\bar{\mathsf{R}}^2_{++}))$ respectively. This means that for all indices α' and $\beta' \in \mathsf{N}_0^{n-1}$ and l, m, l' and $m' \in \mathsf{N}_0$ the following seminorms are finite:

$$(3.6)\quad \| \tilde{k} \,|S_{1,0}^{d-1}, \alpha', \beta', l, m\| := \sup \langle \xi' \rangle^{-(d-|\alpha'|-l+m)} |x_n^l D_{x_n}^m D_{\xi'}^{\alpha'} D_{x'}^{\beta'} \tilde{k}(x', x_n, \xi')|,$$

$$(3.7)\qquad \| \tilde{g} \,|S_{1,0}^{d-1}, \alpha', \beta', l, m, l'm'\| := \sup(\langle \xi' \rangle^{-(d+1-|\alpha'|-l+m-l'+m')} \times$$
$$|x_n^l D_{x_n}^m y_n^{l'} D_{y_n}^{m'} D_{\xi'}^{\alpha'} D_{x'}^{\beta'} \tilde{g}(x', x_n, y_n, \xi')|),$$

when the supremum is taken over $(x', x_n, \xi') \in \mathsf{R}^{n-1} \times \bar{\mathsf{R}}_+ \times \mathsf{R}^{n-1}$ respectively over (x', x_n, y_n, ξ') in $\mathsf{R}^{n-1} \times \bar{\mathsf{R}}_+ \times \bar{\mathsf{R}}_+ \times \mathsf{R}^{n-1}$. The symbol-kernel $\tilde{t}$ is usually taken in $S_{1,0}^{d}(\mathsf{R}^{n-1} \times \mathsf{R}^{n-1}, \mathscr{S}(\bar{\mathsf{R}}_+))$ (yet here the normal variable is integrated out, and hence denoted by y_n, cf. (3.15) below).

Occasionally we shall use the equivalent family of seminorms

$$(3.8)\qquad \| p \,|S_{1,0}^d, k\| = \max\{\| p \,|S_{1,0}^d, \alpha, \beta\| \mid |\alpha|, |\beta| \leqq k\}, \quad k \in \mathsf{N}_0$$

When the meaning is clear the symbol space is suppressed, i.e., $\| p \,|k\| := \| p \,|S_{1,0}^d, k\|$, and instead of $S_{1,0}^d(\mathsf{R}^n \times \mathsf{R}^n)$ we write $S_{1,0}^d$ and $S^{-\infty} := \cap_d S_{1,0}^d$. Similar abbreviations are used for the symbol-kernel spaces. Endowed with the topology of the introduced systems of seminorms, $S_{1,0}^d(\mathsf{R}^n \times \mathsf{R}^n)$, $S_{1,0}^d(\mathsf{R}^{n-1} \times \mathsf{R}^{n-1}, \mathscr{S}(\bar{\mathsf{R}}_+))$ and $S_{1,0}^d(\mathsf{R}^{n+1} \times \mathsf{R}^{n-1}, \mathscr{S}(\bar{\mathsf{R}}^2_{++}))$ are Fréchet spaces.

With the symbol-kernels belonging to the indicated spaces it is seen at once that the integrals in (3.2)–(3.4) above are convergent, and hence Kv, Tu and Gu are well defined:

PROPOSITION 3.2. *Let $\tilde{k} \in S_{1,0}^{d-1}(\mathscr{S}\bar{\mathsf{R}}_+))$ with $d \in \mathsf{R}$, and define for $v(x')$ in $\mathscr{S}(\mathsf{R}^{n-1})$ the function $Kv(x', x_n) = \mathrm{OPK}(\tilde{k})v$ by the formula* (3.2).

Then $(\tilde{k}, v) \mapsto Kv = \mathrm{OPK}(\tilde{k})v$ is continuous as a mapping

$$(3.9)\qquad S_{1,0}^{d-1}(\mathscr{S}(\bar{\mathsf{R}}_+)) \times \mathscr{S}(\bar{\mathsf{R}}^n_+) \xrightarrow{\mathrm{OPK}} \mathscr{S}(\bar{\mathsf{R}}^n_+).$$

Similarly the mappings

$$(3.10)\qquad S_{1,0}^{d}(\mathscr{S}(\bar{\mathsf{R}}_+)) \times \mathscr{S}(\bar{\mathsf{R}}^n_+) \xrightarrow{\mathrm{OPT}} \mathscr{S}(\bar{\mathsf{R}}^n_+)$$

$$(3.11)\qquad S_{1,0}^{d-1}(\mathscr{S}(\bar{\mathsf{R}}^2_{++})) \times \mathscr{S}(\bar{\mathsf{R}}^n_+) \xrightarrow{\mathrm{OPG}} \mathscr{S}(\bar{\mathsf{R}}^n_+)$$

defined by (3.3) *and* (3.4) *are continuous.*

PROOF. By use of (3.6), $\mathrm{OPK}(\tilde{k})v$ is in $\mathscr{S}(\bar{\mathsf{R}}^n_+)$: for any multiindices α and $\beta \in \mathsf{N}^n_0$ the seminorm $\sup\{x^\alpha D^\beta_x Kv \mid x \in \bar{\mathsf{R}}^n_+\}$ is finite and dominated by

$$(3.12)\quad (2\pi)^{1-n} \sum_{\omega' \leqq \alpha'} \sum_{\gamma' \leqq \beta'} \binom{\alpha'}{\omega'}\binom{\beta'}{\gamma'} \|\tilde{k} \mid S^{d-1}_{1,0}, \alpha' - \omega', \beta' - \gamma', \alpha_n, \beta_n\| \times$$
$$\|(1-\Delta)^N x^{\omega'} D^{\gamma'} v \mid L_1\| \int \langle \xi' \rangle^{d - |\alpha' - \omega'| - \alpha_n + \beta_n - 2N} \, d\xi',$$

when N is so large that $d - |\alpha' - \omega'| - \alpha_n + \beta_n - 2N < -(n-1)$.
OPT and OPG can be treated in a similar fashion.

Contrary to this, the formula (3.1) does not make sense for every $u \in \mathscr{S}(\bar{\mathsf{R}}^n_+)$ as it stands, so it should rather be read as $P_+ u = r^+ \, \mathrm{OP}(p) e^+ u$, where

$$(3.13)\qquad \mathrm{OP}(p(x,\xi))\psi = (2\pi)^{-n} \int e^{ix\cdot\xi} p(x,\xi)\hat{\psi}(\xi)\, d\xi, \quad \text{for} \quad \psi \in \mathscr{S}(\mathsf{R}^n).$$

Then $P_+ u$ is well defined in view of (1.10). More precisely:

When $P = \mathrm{OP}(p)$ for $p(x,\xi) \in S^d_{1,0}$, direct calculations show the continuity of $P\colon \mathscr{S}(\mathsf{R}^n) \to \mathscr{S}(\mathsf{R}^n)$. Since, by consideration of the sesqui-linear duality $\langle u, \bar{\varphi}\rangle$ for $u \in \mathscr{S}'(\mathsf{R}^n)$ and $\varphi \in \mathscr{S}(\mathsf{R}^n)$,

$$(3.14)\qquad \mathrm{OP}(p(x,\xi))^* = \mathrm{OP}(q(x,\xi)), \quad \text{with} \quad q(x,\xi) = e^{iD_x \cdot D_\xi} \bar{p}(x,\xi),$$

the continuity of $P\colon \mathscr{S}'(\mathsf{R}^n) \to \mathscr{S}'(\mathsf{R}^n)$ follows, cf. [Hör85, Sect. 18.1]. Here $e^{iD_x\cdot D_\xi}$ is a homeomorphism on $S^d_{1,0}(\mathsf{R}^n \times \mathsf{R}^n)$.

Recall that P has the boundedness properties in (1.10), which in particular apply when $p(x,\xi)$ belongs to the subclass $S^d_{1,0;\,\mathrm{uttr}}$.

For $p(x,\xi) \in S^d_{1,0}$ it follows that $P_+ = r^+ P e^+$ is bounded from $L_2(\mathsf{R}^n_+)$ to $F^{-d}_{2,2}(\bar{\mathsf{R}}^n_+)$, so in particular $P_+ u$ is defined for $u \in \mathscr{S}(\bar{\mathsf{R}}^n_+)$. When in addition $p \in S^d_{1,0,\,\mathrm{uttr}}$, one has $P_+ u \in \mathscr{S}(\bar{\mathsf{R}}^n_+)$ then cf. Proposition 4.6 below. (The result there supplements Proposition 3.2).

Moreover, letting $\mathrm{OP}(q(x', y_n, \xi))u := (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} q(x', y_n, \xi) u(y)\, dy d\xi$ for $u \in \mathscr{S}(\mathsf{R}^n)$, the technique in (3.14) shows that

$$(3.15)\qquad P = \mathrm{OP}(q(x', y_n, \xi)) \quad \text{for} \quad q(x', y_n, \xi) = e^{-iD_{x_n}\cdot D_{\xi_n}} p(x,\xi)|_{x_n = y_n},$$

and P is then said to be given in (x', y_n)-form. It is also known, cf. [Gru90], that $p \in S^d_{1,0,\,\mathrm{uttr}}$ implies that $q(x', y_n, \xi) \in S^d_{1,0,\,\mathrm{uttr}}$, i.e., (3.5) holds when $p(x', 0, \xi)$ is replaced by $q(x', 0, \xi)$.

For the symbol-kernel spaces one has results analogous to (3.14) above, and they follow from the pseudo-differential case by freezing x_n and y_n:

LEMMA 3.3. *Let* $\tilde{k} \in S^{d_1 - 1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+))$, $\tilde{t} \in S^{d_2}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+))$ *and* $\tilde{g} \in S^{d-1}_{1,0}(\bar{\mathsf{R}}^2_{++})$, *and let there be defined symbol-kernels by*

$$\tilde{k}^*(x', x_n, \xi') = e^{iD_{x'}\cdot D_{\xi'}}\bar{\tilde{k}}(x', x_n, \xi') \tag{3.16}$$

$$\tilde{g}^*(x', x_n, y_n, \xi') = e^{iD_{x'}\cdot D_{\xi'}}\bar{\tilde{g}}(x', y_n, x_n, \xi') \tag{3.17}$$

$$\tilde{k}\circ\tilde{t}(x', x_n, y_n, \xi') = e^{iD_{y'}\cdot D_{\eta'}}\tilde{k}(x', x_n, \eta')\tilde{t}(y', y_n, \xi')|_{y'=x', \eta'=\xi'} \tag{3.18}$$

The mappings $\tilde{k}\mapsto\tilde{k}^*$ and $\tilde{g}\mapsto\tilde{g}^*$ define homeomorphisms on $S^{d_1-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+))$ and $S^{d-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}^2_{++}))$ respectively, and the bilinear mapping given by $(\tilde{k},\tilde{t})\mapsto\tilde{k}\circ\tilde{t}$ is continuous from $S^{d_1-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+))\times S^{d_2}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+))$ to $S^{d_1+d_2-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}^2_{++}))$.

In particular, for each $j\in\mathsf{N}$ there is a constant c and a j' such that

$$\|\tilde{k}^*|S^{d_1-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_-)), j\| \leqq c\,\|\tilde{k}|S^{d_1-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+)), j'\| \tag{3.19}$$

$$\|\tilde{g}^*|S^{d-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}^2_{++})), j\| \leqq c\,\|\tilde{g}|S^{d-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}^2_{++})), j'\| \tag{3.20}$$

$$\|\tilde{k}\circ\tilde{t}|S^{d_1+d_2-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}^2_{++})), j\| \leqq c\,\|\tilde{k}|S^{d_1-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+)), j'\| \times \|\tilde{t}|S^{d_2}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+)), j'\| \tag{3.21}$$

hold for every $\tilde{k}$, $\tilde{t}$ and $\tilde{g}$ in the considered spaces.

PROOF. Obviously $x_n^l D^m_{x_n}\tilde{k}(\cdot, x_n, \cdot)\in S^{d_1-l+m}_{1,0}(\mathsf{R}^{n-1}\times\mathsf{R}^{n-1})$ for each x_n, and therefore $x_n^l D^m_{x_n}\tilde{k}^* = e^{iD_{x'}\cdot D_{\xi'}}\overline{x_n^l(-D_{x_n})^m\tilde{k}}$ belongs to $S^{d_1-l+m}_{1,0}$. Moreover, for each $|\alpha'|$, $|\beta'|$, l, and m there exist c and $N\geqq|\alpha'|, |\beta'|$ such that, with $N'=\max(N, l, m)$,

$$\begin{aligned}\langle\xi'\rangle^{-(d_1-l+m-|\alpha'|)}|D^{\beta'}_{x'}D^{\alpha'}_{\xi'}x_n^l D^m_{x_n}\tilde{k}^*| &\leqq c\,\|x_n^l D^m_{x_n}\tilde{k}(\cdot, x_n, \cdot)|S^{d_1-l+m}_{1,0}, N\| \\ &\leqq c\,\|\tilde{k}|S^{d_1-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_-)), N'\|.\end{aligned} \tag{3.22}$$

The statements on $\tilde{g}^*$ and $\tilde{k}\circ\tilde{t}$ carry over from the pseudo-differential case in the same manner.

3.2. *The transmission condition.* The requirement of the uniform two-sided transmission condition in (3.5) is not as innocent as it looks, with a seemingly arbitrary ξ' dependence of $C_{\alpha,\beta,l,m}$: Indeed, (3.5) is equivalent to a rather special ξ-dependence of $p(x,\xi)$, cf. (ii) in Proposition 3.4 below. Furthermore, there is also equivalence with the condition (iii) below, that implies a slowly increasing behaviour of $C_{\alpha,\beta,l,m}(\xi')$.

PROPOSITION 3.4. *When $p(x,\xi)\in S^d_{1,0}(\mathsf{R}^n\times\mathsf{R}^n)$ for $d\in\mathsf{R}$, the following conditions on $p(x,\xi)$ are equivalent:*

(i) $p\in S^d_{1,0,\mathrm{uttr}}(\mathsf{R}^n\times\mathsf{R}^n)$,

(ii) *For all α and $\beta\in\mathsf{N}^n_0$ there exist $s_{j,\alpha,\beta}(x',\xi')\in S^{d-j-|\alpha|}_{1,0}(\mathsf{R}^{n-1}\times\mathsf{R}^{n-1})$, for $j\in\mathsf{Z}$ with $j\leqq d-|\alpha|$, such that for every $m\in\mathsf{N}_0$*

$$|\xi_n^m D^\alpha_\xi D^\beta_x p(x', 0, \xi) - \sum_{-m\leqq j\leqq d-|\alpha|} s_{j,\alpha,\beta}(x',\xi')\xi_n^{j+m}| \leqq C\langle\xi'\rangle^{d+1-|\alpha|+m}\langle\xi\rangle^{-1} \tag{3.23}$$

holds with a constant C independent of $(x',\xi)\in \mathbb{R}^{n-1}\times \mathbb{R}^n$.

(iii) *For all $\alpha,\beta\in \mathbb{N}_0^n$ and $l,m\in \mathbb{N}_0$ the symbol $p(x,\xi)$ satisfies*

$$\text{(3.24)}\qquad \sup \langle \xi'\rangle^{-(d+1-|\alpha|-l+m)}\,|z_n^l D_{z_n}^m \mathscr{F}^{-1}_{\xi_n\to z_n} D_\xi^\alpha D_x^\beta p(x',0,\xi)| < \infty,$$

when the supremum is taken over x' and ξ' in $\mathbb{R}^{n-1}$ and $z_n \neq 0$.

In the affirmative case, the symbols $s_{j,\alpha,\beta}(x',\xi')$ are uniquely determined, and they are polynomials in $\xi'\in \mathbb{R}^{n-1}$ of degree $\leqq d-j-|\alpha|$.

Here and in the following C denotes a "global" constant (independent of variables like x and ξ), while c is a "local" constant (that might depend on x, say). The constants may differ on each occurrence, as usual.

In the rest of this section $e^+r^+ + e^-r^-$ is denoted $\dot{e}\dot{r}$, where $\dot{r}$ stands for restriction to the set $\mathbb{R}\setminus\{0\}$. One has $\dot{e}\dot{r} = \mathscr{F}^{-1}h_{-1}\mathscr{F}$ on $\dot{S}(\mathbb{R})$, when h_{-1} denotes the projection of $\mathscr{H}$ onto $\mathscr{H}_{-1}$. See [Gru86, Sect. 2.2] where this terminology, that is used in the following without further mention, is explained.

Proof. It is obvious that (iii) $\Rightarrow$ (i), and (ii) $\Rightarrow$ (iii) follows by use of the Parseval-Plancherel identity together with an application of the inequality

$$\text{(3.25)}\qquad \sup_t |f(t)| \leqq \sqrt{2}\,\|f\,|L_2\|^{\frac12}\,\|\dot{e}\dot{r}D_t f\,|L_2\|^{\frac12},$$

valid for functions $f\in e^+W_2^1(\bar{\mathbb{R}}_+) + e^-W_2^1(\bar{\mathbb{R}}_-)$, to the function defined for each (x',ξ') as $f(z_n) = \dot{e}\dot{r}z_n^l D_{z_n}^m \mathscr{F}^{-1}_{\xi_n\to z_n} D_\xi^\alpha D_x^\beta p(x',0,\xi)$. Indeed, for $\|f\,|L_2\|$ in (3.25) one finds

$$\text{(3.26)}\qquad \begin{aligned}\|f(z_n)\,|L_2(\mathbb{R})\| &= \|h_{-1}\bar{D}_{\xi_n}^l \xi_n^m D_\xi^\alpha D_x^\beta p(x',0,\xi)\,|L_2(\mathbb{R})\|\\ &\leqq C\langle\xi'\rangle^{d+1-|\alpha|-l+m}\,\|\langle\xi\rangle^{-1}\,|L_2(\mathbb{R})\| \leqq C'\langle\xi'\rangle^{d+\frac12-|\alpha|-l+m}\end{aligned}$$

when (3.23) is applied after Leibniz' rule. An estimate of $\dot{e}\dot{r}D_{z_n}f(z_n)$ can be derived from (3.26) with $m+1$ instead of m.

In the proof of (i) $\Rightarrow$ (ii) one observes first that $\mathscr{F}^{-1}_{\xi_n\to z_n} D_\xi^\alpha D_x^\beta p(x',0,\xi)$ for each (x',ξ') belongs to $\dot{\mathscr{S}}(\mathbb{R})$, since the only distributions supported by $\{z_n = 0\}$ are the finite linear combinations of derivatives of $\delta_0(z_n)$. Hence $D_\xi^\alpha D_x^\beta p(x',0,\xi)\in\mathscr{H}$, i.e., there exist numbers $s_{j,\alpha,\beta}$ for $j\in \mathbb{Z}$, such that for $|\xi_n|\geqq 1$ and $l,N\in\mathbb{N}_0$,

$$\text{(3.27)}\qquad |D_{\xi_n}^l(D_\xi^\alpha D_x^\beta p(x',0,\xi) - \sum_{d-|\alpha|-N<j\leqq d-|\alpha|} s_{j,\alpha,\beta}\xi_n^j)| \leqq c\,|\xi_n|^{\lfloor d\rfloor-|\alpha|-l-N}.$$

Such numbers are necessarily unique – and zero for $j > d-|\alpha|$ – hence *functions* $s_{j,\alpha,\beta}(x',\xi')$.

The construction of the $s_{j,\alpha,\beta}$ is completed, and it remains to be shown by a bootstrap-method that they are symbols with the desired properties.

From (i) and the well-known fact that (with $\gamma_0^\pm v = \lim_{z_n\to 0_\pm} v(z_n)$) one has

(3.28) $$s_{-1-k,\alpha,\beta}(x',\xi') = -i(\gamma_0^+ - \gamma_0^-)D^k_{z_n}\mathscr{F}^{-1}_{\xi_n\to z_n}D^\alpha_\xi D^\beta_x p(x',0,\xi)$$

for $k\in \mathbb{N}_0$, it follows that $s_{j,\alpha,\beta}(\cdot,\xi')\in C^\infty(\mathbb{R}^{n-1})$ for each ξ' when $j<0$.

The next step is to show that, with C independent of x' and ξ_n,

(3.29) $$|D^l_{\xi_n}(\xi_n^m D^\alpha_\xi D^\beta_x p(x',0,\xi) - \sum_{-m\le j\le d-|\alpha|} s_{j,\alpha,\beta}(x',\xi')\xi_n^{j+m})| \le C\langle\xi_n\rangle^{-1-l}.$$

Observe that the left hand side is equal to $D^l_{\xi_n}h_{-1}\xi_n^m D^\alpha_\xi D^\beta_x p(x',0,\xi)$, which is bounded in x' and ξ_n by (3.25) since, e.g., for the L_2 norm in ξ_n

(3.30) $$\begin{aligned}\| D^l_{\xi_n}h_{-1}\xi_n^m D^\alpha_\xi D^\beta_x p(x',0,\xi)\,|L_2\| &= \|\dot{e}\dot{r}z_n^l D^m_{z_n}\mathscr{F}^{-1}_{\xi_n\to z_n}D^\alpha_\xi D^\beta_x p(x',0,\xi)\,|L_2\|\\ &\le \|(1+|z_n|)^{-1}\,|L_2\| \sum_{k=l,l+1} C_{\alpha,\beta,k,m}.\end{aligned}$$

Moreover $\xi_n^{l+1}D^l_{\xi_n}h_{-1}\xi_n^m D^\alpha_\xi D^\beta_x p(x',0,\xi)$ is bounded with respect to x' and ξ_n, since

(3.31) $$\begin{aligned}\xi_n^{l+1}D^l_{\xi_n}h_{-1}\xi_n^m D^\alpha_\xi D^\beta_x p(x',0,\xi) &= (-1)^l l!\, s_{-m-1,\alpha,\beta}(x',\xi')\\ &\quad + h_{-1}\xi_n^{l+1}D^l_{\xi_n}\xi_n^m D^\alpha_\xi D^\beta_x p(x',0,\xi).\end{aligned}$$

Hence $(1+|\xi_n|^{l+1})D^l_{\xi_n}h_{-1}\xi_n^m D^\alpha_\xi D^\beta_x p(x',0,\xi)$ is bounded, so (3.29) is obtained.

A consequence of (3.29) is that $s_{j,\alpha,\beta}(\cdot,\xi')\in C^\infty(\mathbb{R}^{n-1})$ for $j\ge 0$. Indeed,

(3.32) $$s_{j,\alpha,\beta}(x',\xi') = \frac{1}{j!}\partial^j_{\xi_n}(D^\alpha_\xi D^\beta_x p(x',0,\xi) - h_{-1}D^\alpha_\xi D^\beta_x p(x',0,\xi))|_{\xi_n=0},$$

and here the fact that $p\in S^d_{1,0}$ can be applied together with (3.29).

The rest is similar to [Gru91, Thm. 1.9]: Only the case $\alpha=\beta=0$ will be considered since p and d can be replaced by $D^\alpha_\xi D^\beta_x p$ and $d-|\alpha|$ in the following. For $d<-m$ there is nothing to show in (3.23) so $d\ge -m$ is assumed. Let $\gamma' := (\gamma_1,\dots,\gamma_{n-1},0)$.

At this place the goal is to prove, for $j>-m$ when $m\in\mathbb{N}_0$, that with $N=\lceil d\rceil+1+m$

(3.33) $$s_{j,0,0}(x',\xi') = \sum_{|\gamma'|<N,\,|\gamma'|\le d-j} s_{j,\gamma',0}(x',0)\xi^{\gamma'}.$$

For every $j\le d$ the function $s_{j,0,0}(x',\xi')$ would then be a polynomial of degree $\lfloor d\rfloor - j$ in ξ' with coefficients in $C^\infty(\mathbb{R}^{n-1})$ – i.e. $s_{j,0,0}\in S^{d-j}_{1,0}$ – so in addition only (3.23) would still require a proof.

For (3.23) and (3.33) it suffices to show

(3.34) $$|\xi_n^m p(x',0,\xi) - \sum_{|\gamma'|<N}\ \sum_{-m\le j\le d-|\gamma'|} s_{j,\gamma,0}(x',0)\xi^{\gamma'}\xi_n^{j+m}| \le C\langle\xi'\rangle^{d+1+m}\langle\xi\rangle^{-1},$$

for on one hand (3.34) and (3.29) would imply that

$$(3.35)\quad \left|\sum_{j=-m}^{\lfloor d\rfloor-|\gamma'|} s_{j,\gamma',0}(x',\xi')\xi_n^{j+m} - \sum_{|\gamma'|<N}\sum_{j=-m}^{\lfloor d\rfloor-|\gamma'|} s_{j,\gamma',0}(x',0)\xi'^{\gamma'}\xi_n^{j+m}\right| \leqq C\langle\xi_n\rangle^{-1},$$

and here the ξ_n-polynomial on the left hand side is identical to zero precisely when (3.33) holds. On the other hand, (3.34) would then be the estimate required in (3.23).

When $\langle\xi_n\rangle \leqq |\xi'|$ both terms are $O(\langle\xi'\rangle^{d+1+m}\langle\xi\rangle^{-1})$ on the left hand side of (3.34), since $\langle\xi'\rangle \sim \langle\xi\rangle$ there. In the other region, $\langle\xi_n\rangle \geqq |\xi'|$, one shows by use of a Taylor expansion, cf. [Gru91], the uniform estimate

$$(3.36)\quad \left|\xi_n^m p(x',0,\xi) - \sum_{|\gamma'|<N} \partial_{\xi'}^{\gamma'} p(x',0,0,\xi_n)\frac{\xi^{\gamma'}}{\gamma'!}\xi_n^m\right|$$

$$\leqq \left(\sum_{|\gamma'|<N}\frac{N}{\gamma'!}\|p\,|\gamma',0\|\right)|\xi'|^N\langle\xi_n\rangle^{d-N+m} \leqq C\langle\xi'\rangle^{d+1+m}\langle\xi\rangle^{-1}\left(\frac{|\xi'|}{\langle\xi_n\rangle}\right)^{\lceil d\rceil-d}.$$

Now (3.36) and (3.29) applied to $\xi_n^m\partial_\xi^{\gamma'} p(x',0,0,\xi_n)$ lead to (3.34).

It was obtained during the course of the proof that $s_{j,\alpha,\beta}$ is uniquely determined and is a polynomial of degree $\leqq d-|\alpha|-j$ in ξ' as claimed. The proof is complete.

The contents of Proposition 3.4 is to some extent known. In fact the equivalence of (i) and (ii) was claimed but not proved in [GK93], so the proof of [Gru91, Thm. 1.9] has been modified into the one above with the appropriate uniform estimates.

Note that the essential thing is to show (3.29) and (3.34), since the proper x' and ξ' behaviour of the $s_{j,\alpha,\beta}$ is a gratis consequence, cf. (3.32) and (3.33).

The equivalence with (iii) fits in very naturally, so it seems reasonable to have the short proof of this available. Indeed, (iii) states that $r^+\mathscr{F}^{-1}_{\xi_n\to x_n}p(x',0,\xi)$ is the symbol-kernel of a Poisson operator of orders $d+1$, and this property is used in Proposition 4.1 below.

4. Continuity on $\bar{\mathrm{R}}^n_+$.

With the preparations made in the section above, the continuity properties of the operators introduced in (3.1)–(3.4) above shall now be described.

4.1. *Poisson operators.* The treatment of Poisson operators given here follows the line of thought in [Gru90]. Some observations are collected in the following proposition, where the proofs of (4.3) and (4.4) are intended to be more elementary than those of the corresponding facts in [Gru86] and [Gru90].

PROPOSITION 4.1. 1° *Let* $v\in\mathscr{S}(\mathrm{R}^{n-1})$ *and* $w\in\mathscr{S}(\mathrm{R})$ *satisfy* $v(0)=1$ *together with* $\int w(x_n)\,dx_n = 1$ *and* $\operatorname{supp} w \subset \{x_n \mid -1 \leqq x_n \leqq 0\}$.

Then it follows for every $k \in S^d_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+))$ *and* $d' > d$ *that*

$$v(\varepsilon\xi')w_\varepsilon *_n \tilde{k}(x', x_n, \xi') \in S^{-\infty}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+)), \tag{4.1}$$

$$v(\varepsilon\,\cdot)w_\varepsilon *_n \tilde{k} \to \tilde{k} \quad in \quad S^{d'}_{1,0}(\mathscr{S}(\bar{\mathsf{R}})), \tag{4.2}$$

when $w *_n \tilde{k}(x', x_n, \xi') = r^+ \int e^+\tilde{k}(x', x_n - y_n, \xi')w(y_n)\,dy_n$ *and* $w_\varepsilon(\cdot) = \dfrac{1}{\varepsilon}w\left(\dfrac{1}{\varepsilon}\cdot\right)$.

2° *When* $P = \mathrm{OP}(q(x', y_n, \xi))$ *is given in* (x', y_n)*-form with* $q \in S^d_{1,0,\mathrm{uttr}}$, *then* $\tilde{k} = r^+\mathscr{F}^{-1}_{\xi_n \to x_n} q(x', 0, \xi)$ *in* $S^d_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+))$ *and*

$$r^+P(u \otimes \delta_0) = \mathrm{OPK}(\tilde{k})u \quad holds\ for \quad u \in \mathscr{S}(\mathsf{R}^{n-1}). \tag{4.3}$$

3° *For each* $\tilde{k} \in S^{d-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+))$ *there exists a* $p(x', \xi) \in S^{d-1}_{1,0,\mathrm{uttr}}$ *such that*

$$Kv = r^+\,\mathrm{OP}(p)(v \otimes \delta_0), \quad for \quad v \in \mathscr{S}(\mathsf{R}^{n-1}). \tag{4.4}$$

PROOF. 1° The support condition on w implies that

$$w(y_n)r^+x_n^l D^m_{x_n}(e^+\tilde{k}(x', x_n - y_n, \xi')) = w(y_n)x_n^l D^m_{x_n}\tilde{k}(x', x_n - y_n, \xi'), \tag{4.5}$$

so one shows straightforwardly that $\|v(\varepsilon\,\cdot)w_\varepsilon *_n \tilde{k} \mid S^{-N}_{1,0}, \alpha', \beta', l, m\|$ is $< \infty$ for each $N \in \mathsf{N}$. Now $\|\tilde{k} - v(\varepsilon\,\cdot)\tilde{k} | S^{d'}_{1,0}, 0\| \leqq \|\tilde{k} | S^d_{1,0}, 0\| \sup_{\xi'}\langle\xi'\rangle^{d-d'}|1 - v(\varepsilon\xi')| \to 0$ for $\varepsilon \to 0$, and

$$\|v(\varepsilon\,\cdot)(\tilde{k} - w_\varepsilon *_n \tilde{k}) | S^{d'}_{1,0}, 0\| \leqq 2\,\|\tilde{k} | S^{d'}_{1,0}, 0\| \cdot \|v | L_\infty\| \int_{-\varepsilon}^0 |w|, \tag{4.6}$$

so $\|\tilde{k} - v(\varepsilon\,\cdot)w_\varepsilon *_n \tilde{k} | S^{d'}_{1,0}, 0\| \to 0$ for $\varepsilon \to 0$. The other seminorms can be handled in a manner similar to this; for $\alpha' \neq 0$ terms with $D^{\gamma'}(v(\varepsilon\,\cdot))$ obviously $\to 0$ for $\varepsilon \to 0$.

2° The formula (4.3) is first verified for $d = -\infty$, since Fubini's theorem then permits the following calculation, where $v \in C_0^\infty(\mathsf{R}^n_+)$, and $w_k \in C_0^\infty(\mathsf{R}^n)$ satisfy $w_k \to \delta_0$ in $\mathscr{S}'$,

$$\begin{aligned}\langle r^+P(u \otimes \delta_0), v\rangle &= \lim_{k\to\infty}\iiint e^{i(x-y)\cdot\xi}q(x', y_n, \xi) \\ &\qquad \times u(y')w_k(y_n)e^+v(x)\,dy\,đ\xi\,dx \\ &= \left\langle u(y'), \iint e^{i(x'-y')\cdot\xi'}e^{ix_n\xi_n}q(x', 0, \xi)e^+v\,dx\,đ\xi\right\rangle \\ &= \langle\mathrm{OPK}(r^+\mathscr{F}^{-1}_{\xi_n\to x_n}q(x', 0, \xi'))u, v\rangle;\end{aligned} \tag{4.7}$$

that $\tilde{\mathscr{K}}q := r^+\mathscr{F}^{-1}_{\xi_n\to x_n}q(x', 0, \xi')$ is in $S^d_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+))$ follows from (iii) in Proposition 3.4.

For $d \in \mathbb{R}$ the relation (4.3) follows from (4.7) by regularisation, since $P(u \otimes \delta_0)$ and Ku depend continuously on q and $\tilde{k}$, respectively.

More precisely, take v and w as in 1°, and define $q_\varepsilon = v(\varepsilon\xi')\hat{w}(\varepsilon\xi_n)q(x', y_n, \xi)$ in $S^{-\infty}_{1,0}$. Then $q_\varepsilon \to q$ in $S^{d'}_{1,0}$ when $d' > d$ and, as verified below, $q_\varepsilon \in S^{d'}_{1,0,\mathrm{uttr}}$ and $\tilde{\mathscr{K}} q_\varepsilon \to \tilde{k}$ in $S^{d'}_{1,0}(\mathscr{S}(\bar{\mathbb{R}}_+))$ for $\varepsilon \to 0$. Then (4.7) and (3.9) give, with limits taken in $\mathscr{D}'(\mathbb{R}^n_+)$,

$$(4.8)\qquad r^+ P(u \otimes \delta_0) = \lim_{\varepsilon \to 0} r^+ \operatorname{OP}(q_\varepsilon)(u \otimes \delta_0) = \lim_{\varepsilon \to 0} \operatorname{OPK}(\tilde{\mathscr{K}} q_\varepsilon) u = Ku.$$

To show $q_\varepsilon \in S^{d'}_{1,0,\mathrm{uttr}}$, one may write $\mathscr{F}^{-1}_{\xi_n \to x_n} q(x', 0, \xi')$ as $\tilde{q}(x', x_n, \xi') = s_{\lfloor d \rfloor} D^{\lfloor d \rfloor}\delta_0 + \cdots + s_0\delta_0 + \dot{e}\dot{r}\tilde{q}$ by (ii) in Proposition 3.4.

Then $r_\pm z_n^l D^m_{z_n} D^{\beta'}_{x'} D^{\alpha'}_{\xi'} \tilde{q}_\varepsilon(x', z_n, \xi')$ equals

$$(4.9)\qquad r_\pm z_n^l D^m_{z_n} D^{\beta'}_{x'} D^{\alpha'}_{\xi'} \sum_{0 \leq k \leq d} s_k(x', \xi') v(\varepsilon\xi') D^k_{z_n} w_\varepsilon(z_n)$$

$$+ r_\pm \sum_{\gamma' \leq \alpha'} \binom{\alpha'}{\gamma'} \int D^{\alpha' - \gamma'}_{\xi'}(v(\varepsilon\xi')) z^l D^m_{z_n}(w_\varepsilon(z_n - y_n)) \dot{e}\dot{r} D^{\beta'}_{x'} D^{\gamma'}_{\xi'} \tilde{q}(x', y_n, \xi')\, dy_n,$$

and using that $q \in S^d_{1,0,\mathrm{uttr}}$ majorisations global in (x', z_n) can be obtained.

It remains to show that $\tilde{\mathscr{K}} q_\varepsilon \to \tilde{k}$. But $\tilde{\mathscr{K}} q_\varepsilon = v(\varepsilon\xi') w_\varepsilon *_n \tilde{\mathscr{K}} q$, since $\operatorname{supp} w \subset [-1, 0]$, so 1° gives the rest.

3° To show the existence of $p(x', \xi)$ one can proceed as in [Gru90] by extending $\tilde{k}(x', x_n, \xi)$ for $x_n < 0$ to a function $\tilde{p}(x', x_n, \xi')$ by Seeley's method in [See64], and let $p(x', \xi) = \mathscr{F}_{x_n \to \xi_n} \tilde{p}$. It can be checked that $p \in S^{d-1}_{1,0,\mathrm{uttr}}$, where in particular the uniform two-sided transmission condition is satisfied since for each l and $m \in \mathbb{N}_0$ and α and $\beta \in \mathbb{N}^n_0$ the functions

$$(4.10)\qquad r_\pm z_n^l D^m_{z_n} \mathscr{F}^{-1}_{\xi_n \to z_n} D^\beta_x D^\alpha_\xi p(x', \xi) = r_\pm z_n^l D^m_{z_n} (-z_n)^{\alpha_n} D^\beta_x D^{\alpha'}_{\xi'} \tilde{p}(x', z_n, \xi')$$

are bounded on $\mathbb{R}^{n-1} \times \bar{\mathbb{R}}_\pm$ for each ξ' by the construction of $\tilde{p}$. (4.4) holds by use of 2° since $\tilde{\mathscr{K}} p = r^+ \mathscr{F}^{-1}_{\xi_n \to x_n} p(x', \xi) = r^+ \tilde{p}(x', x_n, \xi') = \tilde{k}$.

Since the composite $r^+ P(\cdot \otimes \delta_0)$ is continuous from $\mathscr{S}'(\mathbb{R}^{n-1})$ to $\mathscr{S}'(\bar{\mathbb{R}}^n_+)$ and $\mathscr{S}$ is dense in $\mathscr{S}'$ we can obviously make the following

DEFINITION 4.2. For $v \in \mathscr{S}'(\mathbb{R}^{n-1})$ the action of a Poisson operator K with symbol-kernel in $S^{d-1}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathscr{S}(\bar{\mathbb{R}}_+))$ is defined as $Kv = r^+ P(v \otimes \delta_0)$, where P is any pseudo-differential operator as in 3° in Proposition 4.1.

According to its definition K is a continuous operator

$$(4.11)\qquad K\colon \mathscr{S}'(\mathbb{R}^{n-1}) \to \mathscr{S}'(\bar{\mathbb{R}}^n_+).$$

To show that this extended definition of K has good continuity properties in the scales of Besov and Triebel-Lizorkin spaces also for $p < 1$ one can make use of Proposition 2.6 concerning the operator $f(x') \mapsto f(x') \otimes \delta_0(x_n)$:

THEOREM 4.3. *Let K be a Poisson operator of order $d \in \mathsf{R}$ and let $s \in \mathsf{R}$ and p and $q \in]0, \infty]$. Then the operator K is bounded*

$$K: B^s_{p,q}(\mathsf{R}^{n-1}) \to B^{s-d+\frac{1}{p}}_{p,q}(\bar{\mathsf{R}}^n_+), \tag{4.12}$$

$$K: F^s_{p,p}(\mathsf{R}^{n-1}) \to F^{s-d+\frac{1}{p}}_{p,q}(\bar{\mathsf{R}}^n_+), \tag{4.13}$$

when $p < \infty$ holds in (4.13).

PROOF. The symbol-kernel of K is denoted by $\tilde{k}(x', x_n, \xi') \in S^{d-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+))$ and Definition 4.2 is applied to write $K = r^+P(\cdot \otimes \delta_0)$ for some $P \in \mathrm{OP}(S^{d-1}_{1,0,\mathrm{uttr}})$.

1° For any $s < 0$, Proposition 2.6 and (1.10) give the boundedness of

$$B^s_{p,q}(\mathsf{R}^{n-1}) \xrightarrow{\cdot \otimes \delta_0} B^{s-1+\frac{1}{p}}_{p,q}(\mathsf{R}^n) \xrightarrow{P} B^{s-d+\frac{1}{p}}_{p,q}(\mathsf{R}^n), \tag{4.14}$$

$$B^s_{p,p}(\mathsf{R}^{n-1}) \xrightarrow{\cdot \otimes \delta_0} F^{s-1+\frac{1}{p}}_{p,q}(\mathsf{R}^n) \xrightarrow{P} F^{s-d+\frac{1}{p}}_{p,q}(\mathsf{R}^n) \quad (p < \infty). \tag{4.15}$$

Hence (4.12) and (4.13) follow for $s < 0$ for every Poisson operator K.

2° For a given $s \geqq 0$ it follows for any $m \in \mathsf{R}$ that on $B^s_{p,q}(\mathsf{R}^{n-1})$

$$K = r^+P((\Xi'^{(-m)} \cdot) \otimes \delta_0)\Xi'^m, \tag{4.16}$$

cf. (2.24). By 1°, if $m > s \geqq 0$ is fixed, it suffices for the conclusion of (4.12) and (4.13) to show that the operator $r^+P(\Xi'^{(-m)} \cdot \otimes \delta_0)$ acts on $B^{s-m}_{p,q}(\mathsf{R}^{n-1})$ as a Poisson operator K' of order $d - m$. However, first it is seen from (3.2) that for $v \in \mathscr{S}(\mathsf{R}^{n-1})$,

$$r^+P(\Xi'^{(-m)}v \otimes \delta_0) = K\Xi'^{(-m)}v = \mathrm{OPK}(\tilde{k}(x', x_n, \xi')\langle \xi' \rangle^{-m})v = K'v, \tag{4.17}$$

where $\tilde{k}(x, \xi')\langle \xi' \rangle^{-m} \in S^{d-m-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+))$. Secondly the formula (4.17) extends to every v in $\mathscr{S}'(\mathsf{R}^{n-1})$ by the denseness of $\mathscr{S}(\mathsf{R}^{n-1})$.

The proof above of Theorem 4.3 seems to be the first to cover the full scales of Besov and Triebel-Lizorkin spaces, since the (somewhat different) arguments in [Fra86a] rely on an article that has not appeared in Mathematische Nachrichten as announced. The proof is similar to the one in [Gru90], but in the present context it is an important point to show that (4.17) holds also when $\mathscr{S}$ is not dense in $B^s_{p,q}$.

Partly for this reason Definition 4.2 and Propositions 3.4 and 4.1 are stated explicitly. Another step in the above extension of the arguments in [Gru90] is to show (2.28) and (2.29), since it seems impossible to carry through the duality arguments from [Gru90] for $p < 1$ or $q < 1$.

For later reference an observation on the operator norms of K is included.

COROLLARY 4.4. *For a Poisson operator* $K = \mathrm{OPK}(\tilde{k})$ *of order* d *the operator norms in* (4.12) *and* (4.13) *satisfy the inequality*

$$\|K \mid \mathsf{L}(B^s_{p,q}, B^{s-d+\frac{1}{p}}_{p,q})\| + \|K \mid \mathsf{L}(F^s_{p,p}, F^{s-d+\frac{1}{p}}_{p,q})\| \leqq c\, \|\tilde{k} \mid S^{d-1}_{1,0}, j\| \tag{4.18}$$

for some (s, p, q)*-dependent* $c < \infty$ *and* $j \in \mathsf{N}$ (*when the* F*-term is omitted for* $p = \infty$).

PROOF. When $s < 0$ it is clear from (4.14) and (4.15) that $\|K\| \leqq c'(s, p, q)\, \|P\|$ holds for the operator norms. Here $\|P\| \leqq c''\, \|p\, |S^{d-1}_{1,0}, j'\|$ when j' is large enough (depending on s), see the formulation of (1.10) in [Yam86]. Since p is a Seeley extension of $\tilde{k}$, $\|p\, |S^{d-1}_{1,0}, j'\| \leqq \|\tilde{k}\, |S^{d-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+)), j\|$. Finally, when $s \geqq 0$ one has for K' in the proof above that $\|\tilde{k}\langle\xi'\rangle^{-m}\, |S^{d-m-1}_{1,0}, j\| \leqq c(j)\, \|\tilde{k}\, |S^{d-1}_{1,0}, j\|$, so it can be used that K acts as $K'\Xi'^m$.

4.2. *Truncated pseudo-differential operators,* P_+. The results for the P_+ operators are obtained for spaces with $p < 1$ by a combined application of interpolation and para-multiplication due to Franke.

Recall the extended definition of e^+ in Section 2.7. Since a truncated pseudo-differential operator is defined as $P_+ = r^+ P e^+$ it is clear that P_+ is defined for certain singular distributions $\left(\text{in spaces with } \frac{1}{p} - 1 < s \leqq 0\right)$.

THEOREM 4.5. *Let* $p(x, \xi) \in S^d_{1,0,\mathrm{uttr}}(\mathsf{R}^n \times \mathsf{R}^n)$ *for some* $d \in \mathsf{R}$, *and let* p *and* $q \in]0, \infty]$. *If* $s > \max\left(\frac{1}{p} - 1, \frac{n}{p} - n\right)$ *the operator* $P_+ = r^+\, \mathrm{OP}(p) e^+$ *is bounded*

$$P_+\colon B^s_{p,q}(\bar{\mathsf{R}}^n_+) \to B^{s-d}_{p,q}(\bar{\mathsf{R}}^n_+), \tag{4.19}$$

$$P_+\colon F^s_{p,q}(\bar{\mathsf{R}}^n_+) \to F^{s-d}_{p,q}(\bar{\mathsf{R}}^n_+), \tag{4.20}$$

where in addition $p < \infty$ *is assumed in* (4.20).

PROOF. The case $1 \leqq p \leqq \infty$ are covered first. When $\frac{1}{p} - 1 < s < \frac{1}{p}$, (4.19) and (4.20) follow from (1.10) and (2.52). For $s > \frac{1}{p}$ the induction argument as presented in [Gru90] can be used to cover the Besov as well as the Triebel-Lizorkin cases with $s - \frac{1}{p} \notin \mathsf{N}_0$ when one uses Proposition 4.1 2°. Here the equivalent norms for these spaces given in [Tri83, 3.3.5] are needed; the unnecessary restriction in [Tri83, 3.3.5/2] is removable by [Fra86b, Thm. 4.1.1].

The cases $s - \frac{1}{p} \in \mathbb{N}_0$ are then covered by use of real interpolation, cf. Theorem 2.4.2 and Proposition 2.4.1 in [Tri83].

It remains to consider the case $0 < p < 1$, where it by real interpolation suffices to prove (4.20). Let $u \in F^s_{p,q}(\bar{\mathbb{R}}^n_+)$ be given and take $v \in F^s_{p,q}(\mathbb{R}^n)$ such that $r^+ v = u$. Then v is an L^t function (for some $t > 1$) and $e^+ u = \chi v$ as seen above (2.55).

The product $\chi v = \pi(\chi, v)$ may be analysed by means of the para-multiplication operators $\pi_j(\cdot,\cdot)$ with $j = 1$, 2 and 3 (in the sense of [Yam86]), provided these make sense on χ and v. In fact it is obtained then, cf. [Joh95a, (3.6)], that

$$(4.21) \qquad r^+ P\pi(\chi, v) = r^+ P\pi_1(\chi, v) + r^+ P\pi_2(\chi, v) + r^+ P\pi_3(\chi, v).$$

From (1.10) and the results for the $\pi_j(\cdot,\cdot)$ it follows that the operators

$$(4.22) \qquad r^+ P\pi_j(\chi, \cdot)\colon F^s_{p,q}(\mathbb{R}^n) \to F^{s-d}_{p,q}(\bar{\mathbb{R}}^n_+), \quad \text{with} \quad j = 1,2,3,$$

are bounded when $s \in \mathbb{R}$, $s > \max\left(0, \frac{n}{p} - n\right)$ and $s < 0$, respectively: for $j = 1$ [Joh95a, (5.1)] applies, since $\chi \in L_\infty$; for $j = 2$ and $q \geqq p$ formula (5.10) there is easily modified to give a version for $B^0_{\infty,\infty} \oplus F^s_{p,q}$, and generally the proof of (2.55) in [Fra86b, Thm. 3.4.2] show the property; for $j = 3$ a variant of [Joh95a, (5.9)] may be used.

By Proposition 2.11 it would be enough to show that $r^+ P\pi_3(\chi, \cdot)$ is bounded between the spaces in (4.22) for, say, $s \geqq 0$ and $p = q = 2$. Indeed, in this case it would follow by $+-$-interpolation that $r^+ P\pi_3(\chi, \cdot)$ is bounded between the spaces in (4.22) for any s, p and q, and then, by (4.21), boundedness would hold for $r^+ P\pi(\chi, v)$ for $p < 1$ when $s > \frac{n}{p} - n$. Clearly (4.20) follows from this by taking the infimum over v.

Therefore we shall derive the continuity of $r^+ P\pi_3(\chi, \cdot)$ in (4.22) for $s > \frac{1}{p} - 1$ and $1 \leqq p < \infty$ from the fact that (4.20) holds for $1 \leqq p < \infty$. First note that (4.20) implies that the operator $r^+ P\pi(\chi, \cdot)$ is bounded between the spaces in (4.22) when $s > \frac{1}{p} - 1$ for some $1 \leqq p < \infty$. From (4.21) and (4.22) it then follows that $r^+ P\pi_3(\chi, \cdot)$ has the desired property.

The theorem above contains an improvement over [Fra86a], in that for $p = \infty$ it is not assumed that the operators are properly supported.

From Theorem 4.5 it follows that $P_+(\mathscr{S}(\bar{\mathbb{R}}^n_+)) \subset C^\infty(\bar{\mathbb{R}}^n_+)$, and we even have

PROPOSITION 4.6. *Let P be a pseudo-differential operator with symbol $p(x, \xi)$ in $S^d_{1,0,\mathrm{uttr}}$. Then $P_+\colon \mathscr{S}(\bar{\mathbb{R}}^n_+) \to \mathscr{S}(\bar{\mathbb{R}}^n_+)$ is continuous.*

PROOF. Recall the commutator identities $[D_x^\alpha, P] = \mathrm{OP}(D_x^\alpha p)$ and $[x^\alpha, P] = \mathrm{OP}(D_\xi^\alpha p)$ valid on $\mathscr{S}'(\mathsf{R}^n)$ and $[D_x^\alpha, e^+]u = -[\alpha_n = 1]i\gamma_0 u(x') \otimes \delta_0(x_n)$ valid for $u \in \mathscr{S}(\bar{\mathsf{R}}^n_+)$ when $|\alpha| = 1$. By use of these it is seen that $x^\alpha D^\beta P_+ u$ is a sum of terms either of the form $Q_+ x^\gamma D^\omega u$, with Q in $\mathrm{OP}(S^d_{1,0,\mathrm{uttr}})$, $\gamma \leqq \alpha$ and $\omega \leqq \beta$, or of the form $K\gamma_0(\xi^{\gamma'} D^{\omega'} u)$, where $K \in \mathrm{OPK}(S^{d+\alpha_n}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}^n_+)))$, $\gamma' \leqq \alpha'$ and $\omega' \leqq \beta'$. Hence $x^\alpha D^\beta P_+ u \in C(\bar{\mathsf{R}}^n_+)$ with $\| x^\alpha D^\beta P_+ u \,|L_\infty\| \leqq C \| u \,|\mathscr{S}(\bar{\mathsf{R}}^n_+), N\|$ for appropriate constants C and N.

4.3. *Trace operators.* A trace operator of *class* $r \in \mathsf{Z}$ and order $d \in \mathsf{R}$ is of the form

$$Tu(x') = \sum_{0 \leqq j < r_+} S_j \gamma_j u(x') + T_0 u(x'), \quad \text{for} \quad u \in \mathscr{S}(\bar{\mathsf{R}}^n_+), \tag{4.23}$$

where each $S_j = \mathrm{OP}'(s_j)$ is a pseudo-differential operator on R^{n-1}, with symbol $s_j(x', \xi')$ in $S^{d-j}_{1,0}(\mathsf{R}^{n-1} \times \mathsf{R}^{n-1})$, and the sum is void when $r < 1$. $T_0 = \mathrm{OPT}(\tilde{t}_0)$ given as in (3.3) is the part of class $\leqq 0$ with $\tilde{t}_0 \in S^d_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+))$.

T is of class $r < 0$ when (the sum is void and) one of the equivalent conditions in Proposition 4.7 3° below is satisfied. To prepare for these, let

$$\mathscr{S}_m(\bar{\mathsf{R}}_+) = \{f \in \mathscr{S}(\bar{\mathsf{R}}_+) \,|\, \gamma_0 f = \cdots = \gamma_{m-1} f = 0\}, \tag{4.24}$$

where the index $m \in \mathsf{N}$ counts the number of traces required to vanish. (This should not be confounded with $\mathscr{S}_0(\bar{\mathsf{R}}^n_+)$ that consists of functions on R^n supported by $\bar{\mathsf{R}}^n_+$.) The conditions in 3° below for negative class have been introduced by Franke and Grubb, cf. [Fra85, Fra86a], [Gru91] and [GK93].

The analysis of the trace operators departs from a description of the standard traces γ_j that enter in (4.23) above. See Section 2.6 for a definitions and the basic results.

Recall in particular the D_k-notation, cf. Figure 2. It is chosen as a reminder of the fact that D_k is a domain consisting of numbers (rather than of vectors). Observe that Theorem 4.5 states that P_+ satisfies (4.19) and (4.20) when $(s, p, q) \in \mathsf{D}_0$.

The aim in the following is to show that when, say, a trace operator T is of class $r \in \mathsf{Z}$ then it is bounded from spaces with parameters (s, p, q) in D_r.

Recall also that the dual of $\mathscr{S}(\bar{\mathsf{R}}^n_+)$ is $\mathscr{S}'_0(\bar{\mathsf{R}}^n_+)$, with $\langle u, \varphi \rangle = \langle u, \psi \rangle$ when $\varphi = r^+\psi$ for $\psi \in \mathscr{S}(\mathsf{R}^n)$ and $u \in \mathscr{S}'_0(\bar{\mathsf{R}}^n_+)$. Similarly $\mathscr{S}'(\bar{\mathsf{R}}^n_+)' = \mathscr{S}_0(\bar{\mathsf{R}}^n_+)$.

Among the statements in Proposition 4.7 below fairly elementary proofs of 1° and 2° are given (until now simple explanations are available in the x'-independent case).

Observe that in 1° below, $r^+ \bar{\mathscr{F}}^{-1}_{\xi_n \to y_n} q(x', \xi) = r^- \mathscr{F}^{-1}_{\xi_n \to z_n} q(x', \xi)|_{y_n = -z_n}$ belongs

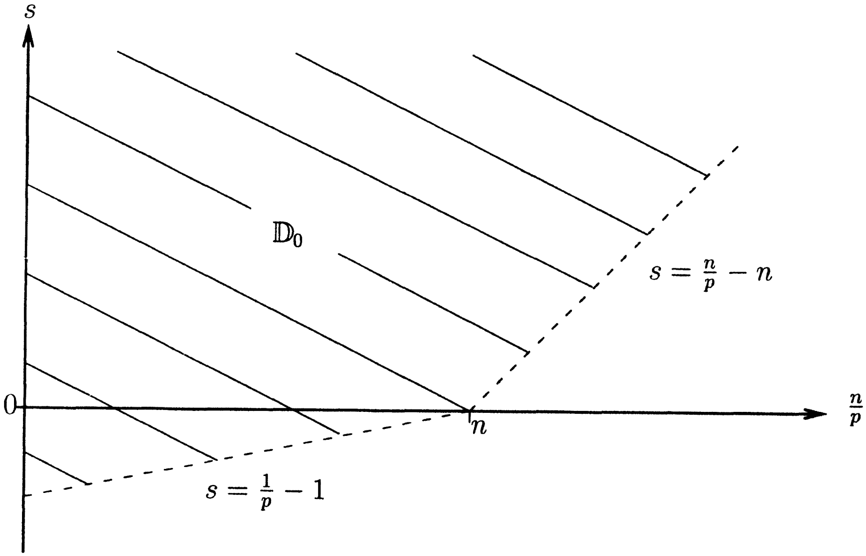

Fig. 2. The definition of D_0 (for $n = 5$).

to $\mathscr{S}(\bar{\mathsf{R}}_+)$ as a function of y_n, because the transmission condition is two-sided, i.e., in (3.5) the supremum is also taken over $z_n \in \mathsf{R}^n_-$.

PROPOSITION 4.7. 1° *Every* $q(x', \xi) \in S^d_{1,0,\mathrm{uttr}}(\mathsf{R}^n \times \mathsf{R}^n)$ *satisfies the relation, for* $u \in \mathscr{S}(\bar{\mathsf{R}}^n_+)$,

$$\gamma_0 \operatorname{OP}(q)_+ u = \sum_{0 \leqq j \leqq d} \operatorname{OP}'(s_j)\gamma_j u + \operatorname{OPT}(r^+ \bar{\mathscr{F}}^{-1}_{\xi_n \to y_n} q)u, \tag{4.25}$$

when the symbols $s_j(x', \xi') \in S^{d-j}_{1,0}(\mathsf{R}^{n-1} \times \mathsf{R}^{n-1})$ *are determined from* q *by Proposition* 3.4. *Hence* $\gamma_0 \operatorname{OP}(q)_+$ *is a trace operator of class* $\leqq (d+1)_+$.

2° *For each symbol-kernel* $\tilde{t} \in S^d_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+))$ *there exists a* $p(x', \xi) \in S^d_{1,0,\mathrm{uttr}}$ *and a Poisson operator* K *such that*

$$Tu = \operatorname{OPT}(\tilde{t})u = K^* e^+ u = \gamma_0 \operatorname{OP}(p)_+ u \quad \textit{holds for} \quad u \in \mathscr{S}(\bar{\mathsf{R}}^n_+). \tag{4.26}$$

Here $K = \operatorname{OPK}(e^{iD_{x'} \cdot D_{\xi'}} \bar{\tilde{t}})$ *and* $r^+ \bar{\mathscr{F}}^{-1}_{\xi_n \to y_n} p(x', \xi) = \tilde{t}(x', y_n, \xi')$. *Moreover, the continuous operator* K^*: $\mathscr{S}'_0(\bar{\mathsf{R}}^n_+) \to \mathscr{S}'(\mathsf{R}^{n-1})$ *is uniquely determined by* (4.26).

3° *Let* $S^{(k)}_T = \operatorname{OP}'(i\bar{D}^k_{y_n} \tilde{t}(x', 0, \xi'))$ *for* $k \in \mathsf{N}_0$, *whenever* T *is a class* 0 *trace operator with symbol-kernel* $\tilde{t}(x', y_n, \xi') \in S^d_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+))$. *Then, for each* $m \in \mathsf{N}$, *the following conditions are equivalent:*

(i) $\tilde{t}(x', \cdot, \xi') \in \mathscr{S}_m(\bar{\mathsf{R}}_+)$ *for each* x' *and* ξ'.

(ii) $t(x',\xi',\xi_n) := \bar{\mathscr{F}}_{y_n\to\xi_n} e^+_{y_n}\tilde{t}(x',y_n,\xi') \in \mathscr{H}^-_{-1-m}$ *as a function of* ξ_n, *for each* (x',ξ').

(iii) $S_T^{(0)} = \cdots = S_T^{(m-1)} = 0$.

(iv) TD^α *is a trace operator of class* 0 *for each* $|\alpha| \leqq m$.

In the affirmative case, T is said to be of class $-m$, *and when this holds for every* $m \in \mathsf{N}$, *the class of T is said to be* $-\infty$.

PROOF. 1° According to Proposition 3.4 there is a decomposition

$$q(x',\xi) = \sum_{0\leqq j\leqq d} s_j(x',\xi')\xi_n^j + h_{-1,\xi_n}q(x',\xi). \tag{4.27}$$

Here $\gamma_0 \operatorname{OP}(\sum s_j\xi_n^j)_+ u$ equals $\sum \operatorname{OP}'(s_j)\gamma_j u$, because $s_j(x',\xi')\xi_n^j$ is a polynomial in ξ (so $\operatorname{OP}(s_j\xi_n^j)$ acts on $\mathscr{S}'(\mathsf{R}^n)$ as a differential operator).

Thus $q = h_{-1}q$ can be assumed. Any $\eta \in C^\infty(\mathsf{R})$ with $\eta(t) = 0$ for $t < \frac{1}{2}$ and $\eta = 1$ for $t > 1$ can now be used to approximate e^+u in $\mathscr{S}'(\mathsf{R}^n)$ by $\eta(kx_n)u \in \mathscr{S}(\mathsf{R}^n)$ for $k \in \mathsf{N}$, so

$$\langle \operatorname{OP}(q)e^+u, \psi\rangle = \lim_{k\to\infty} \iint e^{ix\cdot\xi} q(x',\xi)\mathscr{F}(\eta(k\cdot)e^+u)(\xi)\psi(x)\, đ\xi\, dx, \tag{4.28}$$

when $\psi \in \mathscr{S}(\mathsf{R}^n)$. By Fubini's theorem the ξ_n variable can be integrated first, and since

$$\left\langle \frac{e^{ix_n\xi_n}}{2\pi} q(\xi_n), \mathscr{F}_{y_n\to\xi_n}(\eta(ky_n)e^+\acute{u}(y_n))\right\rangle = \langle (\bar{\mathscr{F}}^{-1}_{\xi_n\to y_n}q)(\cdot - x_n), \eta(k\cdot)e^+\acute{u}(\cdot)\rangle \tag{4.29}$$

for each x' and ξ' when $\langle\cdot,\cdot\rangle$ denotes the duality between $\mathscr{S}'(\mathsf{R})$ and $\mathscr{S}(\mathsf{R})$, it is found that

$$\begin{aligned} \langle \operatorname{OP}(q)e^+u, \psi\rangle &= \lim_k \iiint e^{ix'\cdot\xi'}(\bar{\mathscr{F}}^{-1}_{\xi_n\to y_n}q)(x', y_n - x_n, \zeta') \\ &\qquad \times \eta(ky_n)e^+\acute{u}(\xi',y_n)\psi(x)\, dy_n\, đ\xi'\, dx \\ &= \left\langle \iint e^{ix'\cdot\xi'}(\bar{\mathscr{F}}^{-1}_{\xi_n\to y_n}q)(x', y_n - x_n, \xi')e^+\acute{u}(\xi',y_n)\, dy_n\, đ\xi', \psi\right\rangle. \end{aligned} \tag{4.30}$$

Indeed, the limit is calculated by a majorisation using that

$$\begin{aligned} |\bar{\mathscr{F}}^{-1}_{\xi_n\to y_n}q|(x',z_n,\xi') &\leqq C\langle\xi'\rangle^{d+1} \leqq C\langle\xi'\rangle^{2|d+1|}, \\ |\langle\xi'\rangle^l\langle y_n\rangle^2\acute{u}(\xi',y_n)| &\leqq \sup_{y_n} \int |(1-\Delta_{y'})^{n+|d+1|}(1+y_n^2)u(y',y_n)|\, dy' \\ &\leqq C\, \| u \,|\mathscr{S}(\bar{\mathsf{R}}^n_+), l\| \int \langle y'\rangle^{-2n}\, dy'. \end{aligned} \tag{4.31}$$

when $l = 2(n + |d + 1|)$. This procedure shows, in fact, that

$$(4.32)\qquad L_\infty(\mathbb{R}^n) \ni \iint e^{ix'\cdot\xi'}(\bar{\mathscr{F}}^{-1}_{\xi_n\to y_n}q)(x', y_n - x_n, \xi')e^+\acute{u}(\xi', y_n)\, dy_n đ\xi',$$

which justifies the last relation in (4.30). Hence this function equals $\mathrm{OP}(q)e^+u$.

However, by application of (4.31) it is seen that $r^+\,\mathrm{OP}(q)e^+u$ is continuous in $x \in \bar{\mathbb{R}}^n_+$ (since the integrand in (4.32) converges a.e. when x belongs to a convergent sequence). Thus $r^+\,\mathrm{OP}(q)e^+u$ may be restricted to a hyperplane $\{x_n = a\}$, $a > 0$, and the limit for $x_n \to 0_+$ of this continuous function of x' is calculated for $\varphi \in C_0^\infty(\mathbb{R}^{n-1})$ by majorised convergence (using (4.31)):

$$(4.33)\qquad \langle \mathrm{OP}(q)e^+u(\cdot, x_n), \varphi\rangle = \iiint e^{ix'\cdot\xi'}(\bar{\mathscr{F}}^{-1}_{\xi_n\to y_n}q)(x', y_n - x_n, \xi') \times e^+\acute{u}(\xi', y_n)\varphi(x')\, dy_n đ\xi' dx' \xrightarrow[x_n\to 0_+]{} \langle \mathrm{OPT}(r^+\bar{\mathscr{F}}^{-1}_{\xi_n\to y_n}q(x', \xi)u, \varphi\rangle.$$

It follows that $\gamma_0\,\mathrm{OP}(q)_+u = \mathrm{OPT}(r^+\bar{\mathscr{F}}^{-1}_{\xi_n\to y_n}q)u$.

2° The symbol $p(x', \xi)$ can be taken as $\bar{\mathscr{F}}_{y_n\to\xi_n}\tilde{p}(x', y_n, \xi')$, where $\tilde{p}$ denotes a Seeley extension to $y_n < 0$ of $\tilde{t}(x', y_n, \xi')$, cf. Proposition 4.1 3° or [Gru90]. Observe that the smoothness of $\tilde{p}$ in y_n implies that $h_{-1,\xi_n}p = p$, so that $\gamma_0\,\mathrm{OP}(p)_+u = \mathrm{OPT}(\tilde{t})u$ by (4.25).

Let K denote the Poisson operator with the symbol-kernel $\tilde{k} = \tilde{t}^*$ as in the proposition. For $u \in \mathscr{S}(\bar{\mathbb{R}}^n_+)$ and $\psi \in \mathscr{S}(\mathbb{R}^{n-1})$ we have

$$(4.34)\qquad \langle Tu, \bar{\psi}\rangle = \left\langle \int_0^\infty \mathrm{OP}'(\tilde{t}(\cdot, y_n, \cdot))\acute{u}(\cdot, y_n)\, dy_n, \bar{\psi}\right\rangle$$

since the majorisation

$$(4.35)\qquad |\tilde{t}(x', y_n, \xi')\acute{u}(\xi', y_n)| \leqq 2\, \|\tilde{t}\,|S^d_{1,0}, 2\|\, \langle\xi'\rangle^{2|d+1|}\langle y_n\rangle^{-2}\, |\acute{u}(\xi', y_n)|$$
$$\leqq 2\, \|\tilde{t}\,|S^d_{1,0}, 2\| \int_{\mathbb{R}^n_+} (1 - \Delta_{y'})^{n+|d+1|}u(y)\, dy \cdot \langle\xi'\rangle^{-2n}\langle y_n\rangle^{-2}$$

allows a change in the order of integration in the definition of Tu. Then

$$(4.36)\qquad \langle Tu, \bar{\psi}\rangle = \int_0^\infty \langle \mathrm{OP}'(\tilde{t}(\cdot, y_n, \cdot))\acute{u}(\cdot, y_n), \bar{\psi}\rangle\, dy_n,$$

for $(\mathrm{OP}'(\tilde{t}(\cdot, y_n, \cdot))u(\cdot, y_n))(x')\bar{\psi}(x')$ has by (4.35) a majorant that is integrable with respect to (x', y_n). However, with $(\cdot, \cdot) = \langle \cdot, \bar{\cdot}\rangle$ it is found from (4.36) that

$$\langle Tu, \bar{\psi}\rangle = \int_0^\infty (u(\cdot, y_n), \mathrm{OP}'(e^{iD_{x'}\cdot D_{\xi'}}\bar{\tilde{t}}(\cdot, y_n, \cdot))\psi)\,dy_n \tag{4.37}$$
$$= \int_{\mathsf{R}^n_+} u(x)\overline{K\psi(x)}\,dx = \langle e^+u, \overline{K\psi}\rangle,$$

and this shows that $Tu = K^*e^+u$ for $u \in \mathscr{S}(\bar{\mathsf{R}}^n_+)$.

Moreover, since $e^+C_0^\infty(\mathsf{R}^n_+)$ is dense in $\mathscr{S}'_0(\bar{\mathsf{R}}^n_+)$ it follows from this relation that the continuous operator $K^*\colon \mathscr{S}'_0(\bar{\mathsf{R}}^n_+) \to \mathscr{S}'(\mathsf{R}^{n-1})$ is uniquely determined.

3° That (i) $\Leftrightarrow$ (ii) is clear from the $\mathscr{H}$-theory, for $\bar{\mathscr{F}}\colon e^+\mathscr{S}(\bar{\mathsf{R}}_+) \to \mathscr{H}^-_{-1}$ is a bijection with the property that s_{-1-k} of $\bar{\mathscr{F}}e^+u$ equals $-\gamma_k u$, when $u \in \mathscr{S}(\mathsf{R}_+)$.

(i) $\Rightarrow$ (iii) is trivial, and when $S_T^{(k)} = 0$ one has $i\bar{D}^k_{y_n}\tilde{t}(x', 0, \xi') \equiv 0$, since in the uniform calculus there is a bijective correspondence between operators and symbols, cf. [Hör85, 18.1]. Hence (iii) $\Rightarrow$ (i). (iii) $\Leftrightarrow$ (iv) since [Gru91, Prop. 2.6] is valid also in the uniform case.

The restriction to x_n-independent symbols $q(x', \xi)$ in 1° above was made partly because this generality is sufficient for the application in the proof of 2°; and partly because it requires extra techniques to handle symbols $q(x, \xi)$, since a decomposition like that in (4.27) holds only for $x_n = 0$, then.

It is an important result in Proposition 4.7 2° that for each trace operator T of class 0

$$Tu = K^*e^+u, \quad \text{for} \quad u \in \mathscr{S}(\bar{\mathsf{R}}^n_+). \tag{4.38}$$

Definition 4.8. Let u belong to $B^s_{p,q}(\bar{\mathsf{R}}^n_+)$ or to $F^s_{p,q}(\bar{\mathsf{R}}^n_+)$ for some (s, p, q) in D_0 (with $p < \infty$ in the Triebel-Lizorkin case), and let the trace operator T have class 0 and symbol-kernel $\tilde{t} \in S^d_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+))$.

Then the action of T on u is defined as $Tu = K^*e^+u$, where $K = \mathrm{OPK}(\tilde{t}^*) = \mathrm{OPK}(e^{iD_{x'}\cdot D_{\xi'}}\bar{\tilde{t}})$.

The justification is, of course, that the action of K^* is determined by T. The definition is natural when compared to a pseudo-differential operator $P\colon \mathscr{S}(\mathsf{R}^n) \to \mathscr{S}(\mathsf{R}^n)$ that is extended to $\mathscr{S}'(\mathsf{R}^n)$ as $P = \mathrm{OP}(e^{iD_x\cdot D_\xi}\bar{p})^*$, cf. (3.14).

In the following we shall for (s, p, q) in D_0 derive the continuity in (4.45) and (4.46) below for $r = 0$.

The idea is to show that K^*e^+ for $d \leqq -1$ acts as $\gamma_0 P_+$, when $P = \mathrm{OP}(p)$ is chosen according to 2° in Proposition 4.7. This is useful because Lemma 2.7 and Theorem 4.5 give the boundedness of

$$B^s_{p,q}(\bar{\mathsf{R}}^n_+) \xrightarrow{P_+} B^{s-d}_{p,q}(\bar{\mathsf{R}}^n_+) \xrightarrow{\gamma_0} B^{s-d-\frac{1}{p}}_{p,q}(\mathsf{R}^{n-1}) \tag{4.39}$$

for every $(s, p, q) \in \mathsf{D}_0$, and similarly for the $F^s_{p,q}$ spaces.

For $(s_1, p_1, q_1) \in \mathsf{D}_0$ there is always an embedding $B^{s_1}_{p_1,q_1} + F^{s_1}_{p_1,q_1} \to B^s_{p,q}$ where p and $q \in]1,\infty]$ and $\frac{1}{p} - 1 < s < \frac{1}{p}$. Thus it suffices to check that $K^* e^+ u = \gamma_0 P_+ u$ for $u \in B^s_{p,q}(\bar{\mathsf{R}}^n_+)$ when (s,p,q) satisfies these requirements. Moreover, since $e^+ \colon B^s_{p,q}(\bar{\mathsf{R}}^n_+) \to B^s_{p,q;0}(\bar{\mathsf{R}}^n_+)$ is bounded then, it will be enough to check that $K^* = \gamma_0 r^+ P$ holds on $B^s_{p,q;0}(\bar{\mathsf{R}}^n_+)$ for the specified (s,p,q).

To carry out this programme one can show that for $(s,p,q) \in \mathsf{D}_0$ with p and $q \in]1,\infty]$ there is a commutative diagram

(4.40)
$$\begin{array}{ccc} B^s_{p,q;0}(\bar{\mathsf{R}}^n_+) & \xrightarrow{\ \iota\ } & \mathscr{S}'_0(\bar{\mathsf{R}}^n_+) \\ {\scriptstyle \gamma_0 r^+ P}\downarrow & & \downarrow {\scriptstyle K^*} \\ B^{s-d-\frac{1}{p}}_{p,q}(\mathsf{R}^{n+1}) & \xrightarrow{\ \iota\ } & \mathscr{S}'(\mathsf{R}^{n-1}) \end{array}$$

However, when $e^+ C^\infty_0(\mathsf{R}^n_+) \subset B^s_{p,q;0}(\bar{\mathsf{R}}^n_+)$ is not dense, this is not trivial.

Note that P^* given in (x', y_n)-form has the symbol

(4.41)
$$q(x', y_n, \xi') = e^{-iD_{x_n} D_{\xi_n}} e^{iD_x \cdot D_\xi} \bar{p}(x', \xi) = e^{iD_{x'} \cdot D_{\xi'}} \bar{p}(x', \xi),$$

which is y_n-independent. Hence the Poisson operator $r^+ p^*(\cdot \otimes \delta_0)$ has the symbol-kernel.

(4.42)
$$r^+ \mathscr{F}^{-1}_{\xi_n \to x_n} q(x', \xi) = e^{iD_{x'} \cdot D_{\xi'}} \overline{r^+ \mathscr{F}^{-1}_{\xi_n \to x_n} p(x', \xi)} = e^{iD_{x'} \cdot D_{\xi'}} \bar{f} = \tilde{k}.$$

One has that $B^s_{p,q;0}(\bar{\mathsf{R}}^n_+) = (B^{-s}_{p',q'}(\bar{\mathsf{R}}^n_+))'$, for $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$, so (4.40) is obtained from the commutative diagram

(4.43)
$$\begin{array}{ccc} B^{-s}_{p',q'}(\bar{\mathsf{R}}^n_+) & \xleftarrow{\ \iota\ } & \mathscr{S}(\bar{\mathsf{R}}^n_+) \\ {\scriptstyle r^+ P(\cdot \otimes \delta_0)}\uparrow & & \uparrow {\scriptstyle K} \\ B^{-s+d+\frac{1}{p}}_{p',q'}(\mathsf{R}^{n-1}) & \xleftarrow{\ \iota\ } & \mathscr{S}(\mathsf{R}^{n-1}) \end{array}$$

by taking adjoints. Indeed, for $v \in B^s_{p,q;0}(\bar{\mathsf{R}}^n_+)$ and $w \in B^{-s+d+\frac{1}{p}}_{p',q'}(\mathsf{R}^{n-1})$

(4.44)
$$\langle r^+ P^*(w \otimes \delta_0), \bar{v} \rangle = \langle w \otimes \delta_0, \overline{Pv} \rangle = \langle w, \overline{\tilde{\gamma}_0 P v} \rangle;$$

here the last relation is obtained by closure from the case with $w \in C^\infty_0(\mathsf{R}^{n-1})$ and $v \in C^\infty(\bar{\mathsf{R}}^n_+) \cap B^s_{p,q;0}$, for $q < \infty$ suffices and $\tilde{\gamma}_0$ makes sense on $P(B^s_{p,q;0})$ when $(s-d, p, q) \in \mathsf{D}_1$. By definition $\gamma_0 r^+ P = \tilde{\gamma}_0 P$.

For $d \leqq -1$ this shows (4.45) and (4.46). When $d > -1$ note that $T = \Xi'^{(d+1)} K_1^* e^+$ with $K_1 := \mathrm{OPK}(\langle \xi' \rangle^{-d-1} \tilde{k}) = K \Xi'^{(-d-1)}$, simply because $K_1^* = \Xi'^{(-d-1)} K^*$. Since $K_1^* e^+$ acts as a trace operator of order -1, it follows that the formulae hold also in this case (but still for $r = 0$).

We shall now lift these considerations to a much stronger result, that in special cases can be found in [Gru90]. By and large we modify the proofs there.

THEOREM 4.9. *A trace operator T of order $d \in \mathbb{R}$ and class $r \in \mathbb{Z}$ is continuous*

$$T\colon B^s_{p,q}(\bar{\mathbb{R}}^n_+) \to B^{s-d-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}), \quad \text{for} \quad (s,p,q) \in \mathsf{D}_r, \tag{4.45}$$

$$T\colon F^s_{p,q}(\bar{\mathbb{R}}^n_+) \to F^{s-d-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}), \quad \text{for} \quad (s,p,q) \in \mathsf{D}_r, \tag{4.46}$$

when in (4.46) *also $p < \infty$ holds.*

Moreover, if T is continuous from either $B^s_{p,q}(\bar{\mathbb{R}}^n_+)$ or $F^s_{p,q}(\bar{\mathbb{R}}^n_+)$ to $\mathscr{D}'(\mathbb{R}^{n-1})$ for some $(s,p,q) \notin \bar{\mathsf{D}}_r$, then the class of T is $\leqq r-1$.

T is continuous from $\mathscr{S}'(\bar{\mathbb{R}}^n_+)$ to $\mathscr{S}'(\mathbb{R}^{n-1})$ if and only if T has class $-\infty$.

PROOF. For $r = 0$ the proof of the first part has been conducted above, and for $r > 0$ one can treat the sum in (4.23) by use of Lemma 2.7 and (1.10). Operators of negative class $r = -m$, where $m \in \mathbb{N}$, can be handled with Proposition 4.7 3° (iv) as the point of departure: for u in $B^{s_1}_{p_1,q_1}(\bar{\mathbb{R}}^n_+)$ respectively $F^{s_1}_{p_1,q_1}(\bar{\mathbb{R}}^n_+)$ and (s_1,p_1,q_1) arbitrary there is a decomposition,

$$u = \sum_{|\alpha| \leqq m} D^\alpha v_\alpha, \quad \text{where each} \quad v_\alpha \in B^{s_1+m}_{p_1,q_1}(\bar{\mathbb{R}}^n_+) \quad \text{resp.} \quad F^{s_1+m}_{p_1,q_1}(\bar{\mathbb{R}}^n_+), \tag{4.47}$$

in such a way that each operator $u \mapsto v_\alpha$ is bounded from $B^{s_1}_{p_1,q_1}$ to $B^{s_1+m}_{p_1,q_1}$ and from $F^{s_1}_{p_1,q_1}$ to $F^{s_1+m}_{p_1,q_1}$. (As usual this can be seen by expansion of the identity $1 = (1 + \xi_1^2 + \cdots + \xi_n^2)^m \langle \xi \rangle^{-2m}$.) Since $(s+m,p,q) \in \mathsf{D}_0$ and TD^α is of class 0, it follows that $u \mapsto \sum_{|\alpha| \leqq m} TD^\alpha v_\alpha$ has the boundedness properties in (4.45) and (4.46). For $(s_1,p_1,q_1) \in \mathsf{D}_0$ this operator equals T on $B^{s_1}_{p_1,q_1}$ and $F^{s_1}_{p_1,q_1}$ that are dense in $B^s_{p,q}$ and $F^s_{p,q}$ when $q < \infty$. Hence this extension is unique.

On the other hand, let T be continuous on $B^s_{p,q}(\bar{\mathbb{R}}^n_+)$ for some $(s,p,q) \in \mathsf{D}_{r_1} \setminus \bar{\mathsf{D}}_r$ such that $(s,p,q) \notin \bar{\mathsf{D}}_{r_1+1}$ (the argument is the same in the Triebel-Lizorkin case). If $r > r_1 \geqq 0$ the operator T has the form in (4.23), so $S_{r-1}\gamma_{r-1} = T - \sum_{0 \leqq j < r-1} S_j \gamma_j - T_0$. Since the case $p < 1$ is a novelty we begin with this.

Obviously $T\colon B^{\frac{n}{p}-n+r}_{p,2} \to \mathscr{D}'(\mathbb{R}^{n-1})$ is continuous, hence $S_{r-1}\gamma_{r-1}$ has the same property and we shall deduce from this fact that $S_{r-1} = 0$. According to Lemma 2.8 there exists for each $z' \in \mathbb{R}^{n-1}$ a sequence $v_k \in \mathscr{S}(\bar{\mathbb{R}}^n_+)$ such that $v_k \to 0$ in $B^{\frac{n}{p}-n+r}_{p,2}(\bar{\mathbb{R}}^n_+)$ while $\gamma_{r-1} v_k \to \delta_{z'}$ in $\mathscr{S}'(\mathbb{R}^{n-1})$. Because of the continuity of $S_{r-1}\gamma_{r-1}$ and of S_{r-1} this implies $S_{r-1}\delta_{z'} = 0$. Since $z' \in \mathbb{R}^{n-1}$ is arbitrary, the identity

$$0 = \langle S_{r-1}\delta_{z'}, \bar{\psi} \rangle = \langle \delta_{z'}, \overline{S^*_{r-1}\psi} \rangle, \quad \text{where} \quad \psi \in \mathscr{S}(\mathbb{R}^{n-1}), \tag{4.48}$$

gives that both S^*_{r-1} and S_{r-1} are equal to 0. In the case $1 \leqq p \leqq \infty$ one concludes that the operator $S_{r-1}\gamma_{r-1}\colon B^{\frac{1}{p}+r-1-\varepsilon}_{p,1} \to \mathscr{D}'(\mathbb{R}^{n-1})$ is continuous for some $\varepsilon > 0$, and then it is inferred from Lemma 2.8 that S_{r-1} is 0.

This procedure can be repeated until all the terms in (4.23) with $j \geqq r_1$ are shown to be 0; then T is of class r_1. If $r_1 = -m < 0$, $0 \leqq |\alpha| \leqq m$, the operators TD^α are trace operators, cf. [Gru91], that are continuous $TD^\alpha\colon B^{s+m}_{p,q}(\bar{\mathsf{R}}^n_+) \to B^{s-d-\frac{1}{p}}_{p,q}(\mathsf{R}^{n-1})$, and since $(s+m, p, q) \in \mathsf{D}_0$, each TD^α is of class zero according to the preceding argument. By use of Proposition 4.7 3° (iv) this implies that T is of class $-m = r_1$.

If $T = K^* e^+$ has class $-\infty$, the symbol-kernel of K vanishes of infinite order at $x_n = 0$, implying that e^+K is bounded $\mathscr{S}(\mathsf{R}^{n-1}) \to \mathscr{S}_0(\bar{\mathsf{R}}^n_+)$. For $u \in C^\infty(\bar{\mathsf{R}}^n_+)$ and $v \in \mathscr{S}(\mathsf{R}^{n-1})$ we see that $\langle u, \overline{e^+Kv}\rangle = \int u\overline{Kv} = \langle e^+u, \overline{Kv}\ \rangle = \langle Tu, \bar{v}\rangle$. Because $C^\infty(\bar{\mathsf{R}}^n_+)$ is dense in $B^{-N}_{\infty,1}(\bar{\mathsf{R}}^n_+)$ for any $N > 0$ and in $\mathscr{S}'(\bar{\mathsf{R}}^n_+)$ it follows that $T \subset (e^+K)^*\colon \mathscr{S}'(\bar{\mathsf{R}}^n_+) \to \mathscr{S}'(\mathsf{R}^{n-1})$ with uniqueness of the extension. Conversely, if T is continuous from $\mathscr{S}'(\bar{\mathsf{R}}^n_+)$, it is so from $F^{-N}_{2,2}$, hence of class $-N$, for any N.

The statement above is somewhat more general than the corresponding one in [Fra86a]. First of all because Definition 4.8 allowed the inclusion of the case $p = \infty$ without assuming that the operator is properly supported. Secondly the result for operators of class $-\infty$ seems to be new. Moreover, there are the limitations on the (s, p, q) parameters in terms of T's class (that except for the sharpness generalise the corresponding ones in [Gru90]).

Like for the Poisson operators the operator norms are estimated.

COROLLARY 4.10. *For each trace operator* $T = \mathrm{OPT}(\tilde{t})$ *of class* 0 *and order* d *the operator norms in* (4.45) *and* (4.46) *satisfy the inequality*

$$(4.49)\qquad \| T\,|\,\mathsf{L}(B^s_{p,q}, B^{s-d-\frac{1}{p}}_{p,q})\| + \| T\,|\,\mathsf{L}(F^s_{p,q}, F^{s-d-\frac{1}{p}}_{p,p})\| \leqq c\,\|\tilde{t}\,|S^d_{1,0}, j\|$$

for some (s,p,q)*-dependent* $c < \infty$ *and* $j \in \mathsf{N}$ *(when the* F*-term is omitted for* $p = \infty$*).*

PROOF. By means of the closed graph theorem, which is applicable by Remark 2.2, it is easy to show that, say, $S^d_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+)) \to \mathsf{L}(B^s_{p,q}, B^{s-d-\frac{1}{p}}_{p,q})$ given by $\tilde{t} \mapsto \mathrm{OPT}(\tilde{t})$ is continuous. Indeed, when $\tilde{t}_\nu \to \tilde{t}$ and $\mathrm{OPT}(\tilde{t}_\nu) =: T_\nu \to T$ we let $\tilde{k}_\nu = e^{iD_{x'}\cdot D_{\xi'}}\bar{\tilde{t}}_\nu$ and $K_\nu = \mathrm{OPK}(\tilde{k}_\nu)$; $\tilde{k}$ and K are defined similarly. Then $\tilde{k}_\nu \to \tilde{k}$ in $S^d_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+))$ by (3.19). For $u \in B^s_{p,q}(\bar{\mathsf{R}}^n_+)$ and $\psi \in \mathscr{S}(\mathsf{R}^{n-1})$ one has

$$(4.50)\qquad \langle Tu, \psi\rangle = \lim_\nu \langle e^+u, K_\nu\psi\rangle = \langle \mathrm{OPT}(\tilde{t})u, \psi\rangle,$$

when the limit is calculated using (3.9). This shows that $T = \mathrm{OPT}(\tilde{t})$.

REMARK. 4.11. It should be emphasised that K^* in the formula (4.38) is the adjoint of $K\colon \mathscr{S}(\mathsf{R}^{n-1}) \to \mathscr{S}(\bar{\mathsf{R}}^n_+)$, and as such it is continuous $K^*\colon \mathscr{S}'_0(\bar{\mathsf{R}}^n_+) \to \mathscr{S}'(\mathsf{R}^{n-1})$. However, it is a result in the calculus that the trace operators of order d and class 0 constitute precisely the adjoints of the Poisson operators of order $d+1$. Seemingly this contradicts the fact that K^* acts on $\mathscr{S}'_0(\bar{\mathsf{R}}^n_+)$ whereas T acts on spaces over R^n_+.

But it is understood in the cited result that $K\colon F^{d+\frac{1}{2}}_{2,2}(\mathsf{R}^{n-1}) \to L_2(\mathsf{R}^n_+)$ is the operator that has a trace operator $T\colon L_2(\mathsf{R}^n_+) \to F^{-d-\frac{1}{2}}_{2,2}(\mathsf{R}^{n-1})$ as adjoint cf. [Gru86, (1.2.34)].

More generally, one can restrict the distributional adjoint K^* to a bounded operator from $F^s_{p,q;0}(\bar{\mathsf{R}}^n_+)$ to $F^{s-d-\frac{1}{p}}_{p,p}(\bar{\mathsf{R}}^{n-1})$ when $(s,p,q)\in \mathsf{D}_0\setminus\bar{\mathsf{D}}_1$ with $1<p$, $q \leqq \infty$. Then e^+ is a bijection $F^s_{p,q}(\bar{\mathsf{R}}^n_+) \to F^s_{p,q;0}(\bar{\mathsf{R}}^n_+)$, and by Definition 4.8 the *composite* K^*e^+ is a bounded trace operator $T\colon F^s_{p,q}(\bar{\mathsf{R}}^n_+) \to F^{s-d-\frac{1}{p}}_{p,p}(\mathsf{R}^{n-1})$. Identifying $F^s_{p,q;0}(\bar{\mathsf{R}}^n_+)$ with $F^s_{p,q}(\bar{\mathsf{R}}^n_+)$ corresponds to omitting e^+ (as above where $L_2(\mathsf{R}^n_+)' = L_2(\mathsf{R}^n_+)$), that is, to identify K^* with T.

With this explanation the formula $Tu = K^*e^+u$ will be used throughout since the e^+ there is a reminder of the fact that T need not be defined for every $u\in\mathscr{S}'(\bar{\mathsf{R}}^n_+)$.

REMARK 4.12. Contrary to the treatment of the subscales $B^s_{p,p}$ and $F^s_{p,2}$ with $1<p<\infty$ in [Gru90], the borderline cases $s = r + \max\left(\frac{1}{p}-1, \frac{n}{p}-n\right)$ for a trace operator of class r are left open here.

It is quite tempting, though, to treat these cases beginning with the analysis of γ_0 in Remark 2.9. However, since $\gamma_0(B^{\frac{1}{p}}_{p,q}) = L_p$ when $1 \leqq p < \infty$, it is not even sufficient to use the scales $B^s_{p,q}$ and $F^s_{p,q}$ simultaneously (which is required since $F^0_{p,2} = L_p$ for $1<p<\infty$); for $p=1$ it is necessary to go outside of these scales since $\gamma_0\colon B^1_{1,1}(\bar{\mathsf{R}}^n_+) \to L_1(\mathsf{R}^{n-1})$ is surjective.

Besides this, an investigation of the borderline cases for truncated pseudo-differential operators P_+ would also have to be carried out. Altogether the exposition would be heavily burdened by consideration of these borderline cases, so this topic is left for the future.

4.4. *Singular Green operators.* A singular Green operator – abbreviated as a s.g.o. in the sequel – of *class* $r\in\mathsf{Z}$ and order $d\in\mathsf{R}$ is of the form

$$Gu(x') = \sum_{0\leqq j<r_+} K_j\gamma_j u(x') + G_0u(x'), \quad \text{for} \quad u\in\mathscr{S}(\bar{\mathsf{R}}^n_+), \tag{4.51}$$

where each K_j is a Poisson operator of order $d-j$. Here $G_0 = \mathrm{OPG}(\tilde{g}_0)$ is the part of class $\leqq 0$ with $\tilde{g}_0 \in S^{d-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}^2_{++}))$, given as in (3.4).

G is of class $r<0$ when (the sum is void and) one of the equivalent conditions in Proposition 4.13 3° below is satisfied.

Below some known basic results on s.g.o.s, including a certain Laguerre expansion, shall be modified. Concerning the conditions for negative class the reader is referred to the same sources as in Subsection 4.3.

PROPOSITION 4.13. 1° *If K_ν and T_ν have symbol-kernels in $S^{d_1-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+))$ respectively $S^{d_2}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+))$ that for each $N \in \mathsf{N}_0$ satisfy*

$$\|\tilde{k}_\nu \circ \tilde{t}_\nu \,|\, S^{d_1+d_2-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+)), j\| = \mathcal{O}(\nu^{-N}) \quad \text{for} \quad \nu \to \infty, \tag{4.52}$$

then the series $\sum_{\nu=0}^{\infty} \tilde{k}_\nu \circ \tilde{t}_\nu$ is rapidly convergent in $S^{d_1+d_2-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}^2_{++}))$ to a limit $\tilde{g}(x', x_n, y_n, \xi')$, i.e.,

$$\left\| \tilde{g} - \sum_{\nu=0}^{l} \tilde{k}_\nu \circ \tilde{t}_\nu \,\middle|\, S^{d_1+d_2-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}^2_{++})), j \right\| = \mathcal{O}(l^{-N}) \quad \text{for} \quad l \to \infty, \tag{4.53}$$

for each j and N in N_0.

2° *For $\tilde{g}(x', x_n, y_n, \xi') \in S^{d-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}^2_{++}))$ there exist sequences $(\tilde{k}_\nu(x', x_n, \xi'))$ in $S^{d-\frac{1}{2}}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+))$ and $(\tilde{t}_\nu(x', y_n, \xi'))$ in $S^{-\frac{1}{2}}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+))$ such that* (4.52) *and* (4.53) *hold.*

Moreover, for such sequences $(\tilde{k}_\nu)$ and $(\tilde{t}_\nu)$ one has, when $G = \mathrm{OPG}(\tilde{g})$, $K_\nu = \mathrm{OPK}(\tilde{k}_\nu)$ and $T_\nu = \mathrm{OPT}(\tilde{t}_\nu) = L^*_\nu e^+$ (cf. (4.38)), *that*

$$Gu = \sum_{\nu=0}^{\infty} K_\nu T_\nu u = \sum_{\nu=0}^{\infty} K_\nu L^*_\nu e^+ u, \quad \text{for} \quad u \in \mathscr{S}(\bar{\mathsf{R}}^n_+), \tag{4.54}$$

with convergence of the series in $\mathscr{S}(\bar{\mathsf{R}}^n_+)$.

Furthermore, $\sum_0^\infty K_\nu L^*_\nu$ converges weakly on $\mathscr{S}'_0(\bar{\mathsf{R}}^n_+)$ to $r^+ G^*_1 \colon \mathscr{S}'_0(\bar{\mathsf{R}}^n_+) \to \mathscr{S}'(\bar{\mathsf{R}}^n_+)$, where the s.g.o. *$G_1$ has symbol-kernel $\tilde{g}_1 = e^{iD_{x'} \cdot D_{\xi'}} \bar{\tilde{g}}(x', y_n, x_n, \xi')$.*

3° *Let $K^{(k)}_G = \mathrm{OPK}(i\bar{D}^k_{y_n} \tilde{g}(x', x_n, 0, \xi'))$ for $k \in \mathsf{N}_0$, whenever G is a class* 0 s.g.o. *with symbol-kernel $\tilde{g}(x', x_n, y_n, \xi') \in S^{d-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}^2_{++}))$. Then, for each $m \in \mathsf{N}_0$, the following conditions are equivalent:*

(i) *$\tilde{g}(x', x_n, \cdot, \xi') \in \mathscr{S}_m(\bar{\mathsf{R}}_+)$ for each x', x_n and ξ'.*

(ii) *$g(x', \xi', \xi_n, \eta_n) := \bar{\mathscr{F}}_{y_n \to \eta_n} \mathscr{F}_{x_n \to \xi_n} e_{\mathsf{R}^2_{++}} \tilde{g}(x', x_n, y_n, \xi') \in \mathscr{H}^+ \hat{\otimes} \mathscr{H}^-_{-1-m}$ as a function of (ξ_n, η_n), for each (x', ξ').*

(iii) *$K^{(0)}_G = \cdots = K^{(m-1)}_G = 0$.*

(iv) *GD^α is a* s.g.o. *of class* 0 *for each $|\alpha| \leq m$.*

In the affirmative case, G is said to be of class $-m$, and when this holds for every $m \in \mathsf{N}$, the class of G is said to be $-\infty$.

PROOF. 1° Each composite $K_\nu T_\nu$ maps $\mathscr{S}(\bar{\mathsf{R}}^n_+)$ into itself by Proposition 3.2, and $K_\nu T_\nu u$ equals

$$\int e^{ix' \cdot \xi'} \tilde{k}_\nu(x', x_n, \xi') \iiint e^{iy' \cdot (\eta' - \xi')} \tilde{t}_\nu(y', y_n, \eta') \acute{u}(\eta', y_n) \, dy_n \, đ\eta' \, dy' \, đ\xi', \tag{4.55}$$

for each $u \in \mathscr{S}(\bar{\mathsf{R}}^n_+)$. First it is inferred that, with $\tilde{k}_\nu \circ \tilde{t}_\nu$ given as in (3.18),

$$K_\nu T_\nu u = \iint e^{ix' \cdot \xi'} \tilde{k}_\nu \circ \tilde{t}_\nu(x', x_n, y_n, \xi') \acute{u}(\xi', y_n) \, dy_n \, đ\xi'. \tag{4.56}$$

The idea is to let the y_n-integration be the last one in (4.55), and then apply the result for composition of two pseudo-differential operators on R^{n-1}; in this case for each parameter value x_n and y_n. Then (4.56) is obtained, for there one can integrate in any order.

In (4.55) a change of integration order needs a justification, that can be obtained by inserting x_n^l in front of $\tilde{k}_\nu$ for $l = d_+ + 2$ and a convergence factor $\chi(\varepsilon y')$, with $\chi \in C_0^\infty$, $\chi(0) = 1$, in front of $\tilde{t}_\nu$: evidently $K_\nu T_\nu u = x_n^{-l} \lim_{\varepsilon \to 0} x_n^l K_\nu(\chi(\varepsilon\,\cdot) T_\nu u)$ in $\mathscr{S}(\bar{\mathsf{R}}^n_+)$. For each x_n and y_n the method used in the proof of [Hör85, Thm. 18.1.8] gives that

$$(4.57)\quad \lim_{\varepsilon \to 0} \operatorname{OP}'(\tilde{k}_\nu(\cdot, x_n, \cdot) \circ \tilde{t}_{\nu,\varepsilon}(\cdot, y_n, \cdot)) u(\cdot, y_n) = \operatorname{OP}'(\tilde{k}_\nu(\cdot, x_n, \cdot) \circ \tilde{t}_\nu(\cdot, y_n, \cdot)) u(\cdot, y_n),$$

when $\tilde{t}_{\nu,\varepsilon} = \chi(\varepsilon y')\tilde{t}_\nu$. Since $\langle y_n \rangle^2 |\operatorname{OP}'(\tilde{k}_\nu \circ \tilde{t}_{\nu,\varepsilon}) u(x', y_n)| < C$ for some constant C independent of (x', y_n) and of ε for $0 < \varepsilon \leqq 1$, we infer that

$$(4.58)\quad \begin{aligned} K_\nu T_\nu u &= \lim_{\varepsilon \to 0} \int_0^\infty \operatorname{OP}'(\tilde{k}_\nu(\cdot, x_n, \cdot) \circ \tilde{t}_{\nu,\varepsilon}(\cdot, y_n, \cdot)) u(\cdot, y_n)\, dy_n \\ &= \int_0^\infty \operatorname{OP}'(\tilde{k}_\nu(\cdot, x_n, \cdot) \circ \tilde{t}_\nu(\cdot, y_n, \cdot)) u(\cdot, y_n)\, dy_n, \end{aligned}$$

from where (4.56) is obtained by application of Fubini's theorem.

From (4.52) it is seen that $(\sum_{\nu=0}^{l} \tilde{k}_\nu \circ \tilde{t}_\nu)_{l \in \mathsf{N}}$ is a Cauchy-sequence in the Fréchet space $S^{d_1+d_2-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}^2_{++}))$. Hence the series converges to a limit $\tilde{g}$ as claimed, even rapidly since (4.52) gives that $\|\sum_{\nu=l}^{\infty} \tilde{k}_\nu \circ \tilde{t}_\nu \,|\, j\|$ is $\mathcal{O}(l^{-N})$ for any N and j.

2° To define $\tilde{k}_\nu$ and $\tilde{t}_\nu$ one can use that $L_2(\mathsf{R}_+)$ has an orthonormal basis consisting of (certain unconventional) Laguerre functions $(2\pi)^{-1}\varphi_\nu(y_n, \sigma)$ for $\nu \in \mathsf{N}_0$, cf. [Gru91, (1.27) ff.]. Using this with the parameter $\sigma = \langle \xi' \rangle$ one has

$$(4.59)\quad \tilde{g}(x', x_n, y_n, \xi') = \sum_{\nu=0}^{\infty} b_\nu(x', x_n, \xi') \varphi_\nu(y_n, \langle \xi' \rangle)$$

for each x', x_n and ξ' when

$$(4.60)\quad b_\nu(x', x_n, \xi') = (2\pi)^{-1} \int \tilde{g}(x', x_n, y_n, \xi') \varphi_\nu(y_n, \langle \xi' \rangle)\, dy_n.$$

It remains to be verified that one can let $\tilde{k}_\nu = b_\nu$ and $\tilde{t}_\nu = \varphi_\nu(\cdot, \langle \cdot \rangle)$.

To see that $\tilde{k}_\nu$ and $\tilde{t}_\nu$ satisfy (4.52) we use the inequality (3.21). Concerning $\tilde{t}_\nu$ an application of [Gru86, (2.2.20)] for $\alpha' = 0$ leads to

$$(4.61)\quad \| x_n^m D_{x_n}^l \varphi_\nu(x_n, \langle \xi' \rangle) \,|\, L_{2,x_n} \| \leqq C(1 + \nu)^{(1+\varepsilon)l - m} \langle \xi' \rangle^{l-m},$$

when $\varepsilon > 0$. (Here and in the following the equivalent seminorms based on the L_2 norm on $\mathscr{S}(\bar{\mathsf{R}}_+)$ are used.) For $\alpha' \neq 0$ the identity

$$(4.62)\quad D_{\xi_j} \varphi_\nu = (\nu \varphi_{\nu-1} - (\nu + 1)\varphi_{\nu+1})(2\langle \xi' \rangle)^{-1} D_j \langle \xi' \rangle$$

can be used successively, and when combined with (4.61) it is seen that the worst resulting term contains $(\nu + 1)\dots(\nu + |\alpha'|)\varphi_{\nu+|\alpha'|}(2\langle\xi'\rangle)^{-|\alpha'|}\prod(D_j\langle\xi'\rangle)^{|\alpha'_j|}$. However, this is estimated by $\nu^{|\alpha'|+(1+\varepsilon)l-m}\langle\xi'\rangle^{-|\alpha'|+l-m}$, and the other terms, of which there is a fixed α'-dependent number, have similar estimates. Hence $\|\tilde{t}_\nu \mid S^{-\frac{1}{2}}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+)), j\| = \mathcal{O}(\nu^{3j})$.

To treat the $\tilde{k}_\nu$ the proof in [Gru86, p. 169] is modified. Using [Gru86, (2.2.15)] it is found that

$$
\begin{aligned}
(4.63)\qquad & \|\{b_\nu(x',x_n,\xi')\}_{\nu=0}^{\infty} \mid \ell_{2,N}\| \\
&:= \|\{b_\nu(x',x_n,\xi')(1+\nu)^N\}_{\nu=0}^{\infty} \mid \ell_2\| \\
&= \frac{1}{2^N}\left\|\left(\frac{1}{\langle\xi'\rangle}\partial_{y_n}y_n\partial_{y_n} + \langle\xi'\rangle y_n + 1\right)^N \tilde{g}(x',x_n,\cdot,\xi')\middle| L_{2,y_n}\right\| \\
&\leqq C_N \sum_{j+k\leqq N}\langle\xi'\rangle^{k-l-\frac{1}{2}} \sup_{x_n,y_n,x'} |(1+\langle\xi'\rangle y_n)(\partial_{y_n}y_n\partial_{y_n})^j y_n^k\tilde{g}| \\
&\leqq C'_N\langle\xi'\rangle^{d+\frac{1}{2}}.
\end{aligned}
$$

In a similar manner one finds for the "coefficient sequence" $x_n^m D^l_{x_n} D^{\beta'}_{x'} b_\nu$ that its $\ell_{2,N}$-norm is $\mathcal{O}(\langle\xi'\rangle^{d+\frac{1}{2}+l-m})$; $D^{\alpha'}_{\xi'}$ can be applied to the defnition of b_ν and using Leibniz' formula and (4.62) one reduces to the case where $\alpha' = 0$. This implies that $\|\tilde{k}_\nu \mid S^{d-\frac{1}{2}}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+))\| = \mathcal{O}(\nu^{-N})$ for any N, so altogether (4.55) is verified.

After the completion of this construction (4.54) is proved by application of Proposition 3.2 and 1°.

With G_1 given as in the proposition we shall now prove that $G \subset r^+G_1^*e^+$. By use of Fubini's theorem

$$
\begin{aligned}
(4.64)\qquad \langle e^+Gu, \bar{v}\rangle &= \left\langle \int_0^\infty \mathrm{OP}'(\tilde{g}(\cdot,x_n,y_n,\cdot))u(\cdot,y_n)\,dy_n, \bar{v}\right\rangle \\
&= \int_{\bar{\mathsf{R}}^2_{++}} (u(\cdot,y_n), \mathrm{OP}'(\tilde{g}_1(\cdot,x_n,y_n,\cdot))v(\cdot,x_n))\,dy_n dx_n \\
&= \langle e^+u, \overline{G_1 v}\rangle = \langle G_1^* e^+ u, \bar{v}\rangle
\end{aligned}
$$

for each u and v in $\mathscr{S}(\bar{\mathsf{R}}^n_+)$. The inclusion of G into $r^+G_1^*e^+$ follows from this.

Each composite $K_\nu L_\nu^*$ has the adjoint $L_\nu K_\nu^* e^+$, where $L_\nu = \mathrm{OPK}(\tilde{t}_\nu^*)$ and $K_\nu^* e^+ = \mathrm{OPT}(\tilde{k}_\nu^*)$. Moreover, from Lemma 3.3 one finds that

$$
(4.65)\quad \|\tilde{t}_\nu^* \circ \tilde{k}_\nu^* \mid S^{d-1}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}^2_{++})), j\| \leqq c\,\|\tilde{t}_\nu \mid S^{-\frac{1}{2}}_{1,0}(\mathscr{S}(\mathsf{R}_+)), j'\|\,\|\tilde{k}_\nu \mid S^{d-\frac{1}{2}}_{1,0}(\mathscr{S}(\bar{\mathsf{R}}_+)), j'\|,
$$

so the asymptotic properties of $\tilde{k}_\nu$ and $\tilde{t}_\nu$ shown above imply that (4.52) is satisfied by $\tilde{t}_\nu^*$ and $\tilde{k}_\nu^*$. Then (3.11) leads to convergence of $G_2 = \sum L_\nu K_\nu^* e^+$ on $\mathscr{S}(\bar{\mathsf{R}}^n_+)$, and so

(4.66) $\lim_{l\to\infty} \langle \sum_{\nu=0}^{l} K_\nu L_\nu^* u, \bar{v}\rangle = \langle u, \overline{G_2 r^+ v}\rangle = \langle G_2^* u, \overline{r^+ v}\rangle = \langle r^+ G_2^* u, \bar{v}\rangle$

for each $v \in \mathscr{S}_0(\bar{\mathrm{R}}^n_+)$ and $u \in \mathscr{S}'_0(\bar{\mathrm{R}}^n_+)$. Hence $\sum K_\nu L_\nu^*$ converges weakly to $r^+ G_2^*$ as operators $\mathscr{S}'_0(\bar{\mathrm{R}}^n_+) \to \mathscr{S}'(\bar{\mathrm{R}}^n_+)$; for $u \in e^+ C_0^\infty(\bar{\mathrm{R}}^n_+)$ it is seen that $r^+ G_2^* u = r^+ G_1^* u$ in $\mathscr{S}'(\bar{\mathrm{R}}^n_+)$.

3° is shown by a modification of the corresponding part of Proposition 4.7. Observe that when a Poisson operator $K = \mathrm{OPK}(\tilde{k}) \equiv 0$, then $K\psi = 0$ for $\psi \in \mathscr{S}(\mathrm{R}^{n-1})$ in particular. Therefore the pseudo-differential operator $\mathrm{OP}'(\tilde{k}(\cdot, x_n, \cdot))$ equals 0 on $\mathscr{S}(\mathrm{R}^{n-1})$ for each $x_n > 0$, that is to say, $\tilde{k}(x', x_n, \xi') \equiv 0$ for each x_n. Hence $\tilde{k} = 0$.

Since the action of G_1^* is determined by $\tilde{g}$ we can use 2° in Proposition 4.13 to extend the definition of s.g.o.s of class 0 to the spaces where e^+ is defined.

Definition 4.14. When u belongs to $B^s_{p,q}(\bar{\mathrm{R}}^n_+)$ or $F^s_{p,q}(\bar{\mathrm{R}}^n_+)$ for some $(s,p,q) \in \mathrm{D}_0$ (with $p < \infty$ in the F-case) then the action of a s.g.o. G of class 0 on u is defined as $Gu = r^+ G_1^* e^+ u$, where $G_1 = \mathrm{OPG}(e^{iD_{x'} \cdot D_{\xi'}} \tilde{g}(x', y_n, x_n, \xi'))$.

Having made this definition, 2° in Proposition 4.13 may be applied to the sequences $(K, 0, \ldots)$ and $(T, 0, \ldots)$ whereby the usual composition rule is obtained. More precisely, the identity $Gu := \mathrm{OPG}(\tilde{k} \circ \tilde{t})u = KTu$, valid for $u \in \mathscr{S}(\bar{\mathrm{R}}^n_+)$ by (4.54), extends to the situation where $e^+ u$ makes sense, for KTu may be written $KL^* e^+ u$ then, and here $KL^* = r^+ G_1^*$ according to 2°.

Generally the s.g.o.s have the following continuity properties:

Theorem 4.15. *A s.g.o. G of order $d \in \mathrm{R}$ and class $r \in \mathrm{Z}$ is continuous*

(4.67) $G\colon B^s_{p,q}(\bar{\mathrm{R}}^n_+) \to B^{s-d}_{p,q}(\bar{\mathrm{R}}^n_+)$, *for* $(s,p,q) \in \mathrm{D}_r$,

(4.68) $G\colon F^s_{p,q}(\bar{\mathrm{R}}^n_+) \to F^{s-d}_{p,o}(\bar{\mathrm{R}}^n_+)$, *for* $(s,p,q) \in \mathrm{D}_r$, $o \in]0,\infty]$,

when in (4.68) *also $p < \infty$ holds.*

Moreover, if G is continuous from either $B^s_{p,q}(\bar{\mathrm{R}}^n_+)$ or $F^s_{p,q}(\bar{\mathrm{R}}^n_+)$ to $\mathscr{D}'(\mathrm{R}^n_+)$ for some $(s,p,q) \notin \bar{\mathrm{D}}_r$, then the class of G is $\leqq r-1$.

G is continuous from $\mathscr{S}'(\bar{\mathrm{R}}^n_+)$ to $\mathscr{S}'(\bar{\mathrm{R}}^n_+)$ if and only if G has class $-\infty$.

Proof. Suppose first that $r \geqq 0$. For each of the terms $K_j\gamma_j$ in (4.51) it is clear from Lemma 2.7 and Theorem 4.3 that it is continuous as in (4.67) and (4.68). When G_0 in (4.51) is written $G_0 = \sum K_\nu T_\nu$ as in Proposition 4.13 2° it is also clear that each $K_\nu T_\nu$ has the stated continuity properties.

For a given $(s,p,q) \in \mathrm{D}_r$ and $u \in B^s_{p,q}(\bar{\mathrm{R}}^n_+)$ there is for j large an estimate

(4.69) $$\left(\sum_{m=0}^{\infty} \| K_\nu T_\nu u \,|B^{s-d}_{p,q}\|^r\right)^{\frac{1}{r}} \leqq c\left(\sum_{m=0}^{\infty} \| \tilde{k}_\nu \,|S^{d-\frac{1}{2}}_{1,0}, j\|^r \, \| \tilde{t}_\nu \,|S^{-\frac{1}{2}}_{1,0}, j\|^r\right)^{\frac{1}{r}} \| u \,|B^s_{p,q}\|$$

when $r = \min(1, p, q)$, according to the Corollaries 4.4 and 4.10. From the estimates of $\tilde{k}_\nu$ and $\tilde{t}_\nu$ in the proof of 2° in Proposition 4.13 above it follows that the sum on the right hand side is finite. Hence $\sum K_\nu T_\nu$ converges strongly to an operator in $\mathsf{L}(B^s_{p,q}, B^{s-d}_{p,q})$; this operator is G_0, since $\sum K_\nu T_\nu u$ converges in $\mathscr{D}'(\mathsf{R}^n_+)$ to $G_0 u$. For similar reasons G_0 is also in $\mathsf{L}(F^s_{p,q}, F^{s-d}_{p,o})$.

When $r = -m < 0$ and in any case when $(s,p,q) \in \mathsf{D}_{r_1} \setminus \bar{\mathsf{D}}_r$ is given one can simply carry over the proof of Theorem 4.9. E.g., in (4.48) one can replace ψ by $e^+\varphi \in e^+ C_0^\infty(\mathsf{R}^n_+)$ and S_{r-1} by K_{r-1}; then the denseness of $e^+ C_0^\infty(\mathsf{R}^n_+) \subset \mathscr{S}'_0(\bar{\mathsf{R}}^n_+)$ shows that $K^*_{r-1} = 0$.

In the case G has class $-\infty$, one can let $P = 0$ in Section 4.5 below.

The proof above of the properties of G in the $B^s_{p,q}$ and $F^s_{p,q}$ scales was inspired by the one in [Gru90]. There it was shown that $B^s_{p,p} \cup F^s_{p,2}$ is mapped into $B^{s-d}_{p,p} \cap F^{s-d}_{p,2}$ (when $1 < p < \infty$) by a s.g.o.; this also follows from (4.68) and the fact that $B^s_{p,p} = F^s_{p,p}$, but the property does not hold for the full scales here. For the trace and Poisson operators similar remarks can be made.

Concerning the adjoints of s.g.o.s of class 0 the situation is analogous to the one for trace operators, cf. Remark 4.11.

REMARK 4.16. The extension of class $-\infty$ operators T and G to $\mathscr{S}'(\bar{\mathsf{R}}^n_+)$ is a natural consequence of the formulae $T = K^* e^+$ and $G = r^+ G_1^* e^+$, cf. Definitions 4.8 and 4.14. Moreover, it was shown in [GK93, Cor. 4.3] that the restrictions of operators T and G (of any class) to $r^+\mathscr{S}_0(\bar{\mathsf{R}}^n_+)$ have continuous extensions to the spaces $B^s_{p,q;0}(\bar{\mathsf{R}}^n_+)$ and $F^s_{p,q;0}(\bar{\mathsf{R}}^n_+)$ for any $s \in \mathsf{R}$ (when $q = p \in]1, \infty[$ resp. $q = 2$ and $1 < p < \infty$). Consistent with the present level of pedantry, the restrictions to $r^+\mathscr{S}_0(\bar{\mathsf{R}}^n_+)$ extend (factor) *through* e^+ to bounded operators, that evidently are equal to K^* and $r^+ G_1^*$, respectively, for formally T and G cannot act on (subspaces of) $\mathscr{S}'_0(\bar{\mathsf{R}}^n_+)$.

4.5. *Operators $P_+ + G$ of negative class.* Because of the properties (4.19) and (4.20) a truncated ps.d.o. P_+ is in general of class 0, but if it is differential the class is said to be $-\infty$ since it has the mentioned properties for any s.

However, a sum $P_+ + G$ may be continuous also for $(s,p,q) \notin \bar{\mathsf{D}}_0$, simply because two (or more) contributions cancel each other. A non-trivial example is given in [Gru90, Ex. 3.15]. Results on these phenomena are included here – the underlying analysis is that of [Gru90], where such operators $P_+ + G$, that are said to be of negative class, were studied first.

A criterion for $P_+ + G$ to be of class $-m$, for $m \in \mathsf{N}$, is that $(P_+ + G)D^j_{x_n} = (PD^j_{x_n})_+ + G^{(j)}$ holds for some s.g.o. $G^{(j)}$ of class 0 for each $j \in \{1, \dots, m\}$. (See [Gru90] for the general formula for $(P_+ + G)D^j_{x_n}$.) This is equivalent to the fulfilment of $K^{(j)}_P + K^{(j)}_G = 0$ for $j \in \{1, \dots, m\}$, when $K^{(j)}_G$ refer to Proposition 4.13 and

$$K_P^{(k)} v = r^+ i P D_{x_n}^k (v \otimes \delta_0) \quad \text{for} \quad v \in \mathscr{S}(\mathsf{R}^{n-1}); \tag{4.70}$$

the reader can refer to [Gru91, (3.43) ff.] for this. In addition $P_+ + G$ is said to be of class $r \in \mathsf{N}_0$ when G is so, and to have class $-\infty$ when it is of class r for each $r \in \mathsf{Z}$.

THEOREM 4.17. *Let P_+ be a truncated pseudo-differential operator and G a* s.g.o., *both of order $d \in \mathsf{R}$, and suppose that $P_+ + G$ is of class $r \in \mathsf{Z}$. Then*

$$P_+ + G \colon B^s_{p,q}(\bar{\mathsf{R}}^n_+) \to B^{s-d}_{p,q}(\bar{\mathsf{R}}^n_+) \quad \textit{for} \quad (s,p,q) \in \mathsf{D}_r \tag{4.71}$$

$$P_+ + G \colon F^s_{p,q}(\bar{\mathsf{R}}^n_+) \to F^{s-d}_{p,q}(\bar{\mathsf{R}}^n_+) \quad \textit{for} \quad (s,p,q) \in \mathsf{D}_r \tag{4.72}$$

are continuous operators (*when $p < \infty$ in* (4.72)).

Moreover, if $P_+ + G$ is continuous from either $B^s_{p,q}(\bar{\mathsf{R}}^n_+)$ or $F^s_{p,q}(\bar{\mathsf{R}}^n_+)$ to $\mathscr{D}'(\mathsf{R}^n_+)$ for some $(s,p,q) \notin \bar{\mathsf{D}}_r$, then $P_+ + G$ is of class $\leq r-1$; and $P_+ + G$ is continuous $\mathscr{S}'(\bar{\mathsf{R}}^n_+) \to \mathscr{S}'(\bar{\mathsf{R}}^n_+)$ if and only if it has class $-\infty$.

PROOF. The continuity follows from Theorems 4.5 and 4.15; for $r < 0$ as in (4.47) ff. In case $(s,p,q) \notin \bar{\mathsf{D}}_r$ the techniques used for trace and s.g.o.s can be adapted, cf. [Gru91]. When $P_+ + G$ has class $-\infty$, it is used that

$$\langle (P_+ + G)u, \overline{e^+ w} \rangle = \int_{\mathsf{R}^n_+} u \cdot \overline{(r^+ P_1 e^+ w + G_1 w)} \tag{4.73}$$

when $u \in C^\infty(\bar{\mathsf{R}}^n_+)$, $w \in \cap_{m>0} \mathscr{S}_m(\bar{\mathsf{R}}^n_+)$ while $P_1 = \mathrm{OP}(e^{iD_x \cdot D_\xi} \bar{p})$ and $G = r^+ G_1^* e^+$. Given that $P_{1+} + G_1 \colon \mathscr{S}(\bar{\mathsf{R}}^n_+) \to \mathscr{S}(\bar{\mathsf{R}}^n_+)$ maps $\cap \mathscr{S}_m$ into itself, then $e^+(r^+ P_1 + G_1 r^+)$ is continuous $\mathscr{S}_0(\bar{\mathsf{R}}^n_+) \to \mathscr{S}_0(\bar{\mathsf{R}}^n_+)$, since $\mathscr{S}_0 = e^+ (\cap \mathscr{S}_m)$, hence $P_+ + G$ is contained in $(e^+(r^+ P_1 + G_1 r^+))^* \colon \mathscr{S}'(\bar{\mathsf{R}}^n_+) \to \mathscr{S}'(\bar{\mathsf{R}}^n_+)$ by (4.73).

Thus it remains to see that $\gamma_k(P_{1+} + G_1)w = 0$ for each $w \in \cap \mathscr{S}_m(\bar{\mathsf{R}}^n_+)$. By (3.4), $\gamma_0 D^k_{x_n} G_1 w$ equals $\mathrm{OPT}(D^k_{x_n} \tilde{g}_1(x', 0, y_n, \xi'))w$, that is $i K_G^{(k)*} e^+ w$, and

$$\langle v, \overline{\gamma_k P_{1+} w} \rangle = \langle P D^k_n (v \otimes \delta_0), \overline{e^+ w} \rangle = \langle v, \overline{i K_p^{(k)*} e^+ w} \rangle$$

for $v \in \mathscr{S}'(\mathsf{R}^{n-1})$, by (4.70). Therefore, $\gamma_k(P_{1+} + G_1) = i(K_P^{(k)} + K_G^{(k)})^* e^+ \equiv 0$ on $\cap \mathscr{S}_m(\bar{\mathsf{R}}^n_+)$ when $P_+ + G$ has class $-\infty$.

5. Green operators.

In full generality the results for elliptic operators on manifolds shall now be presented. The main goal is to obtain Theorem 5.2 below, that is a generalisation of [Gru90, Corollary 5.5]. Since there are not any substantial changes from the usual texts on the calculus, a brief explanation will suffice.

To begin with it should be made clear that only *bounded*, open C^∞ smooth sets $\Omega \subset \mathsf{R}^n$ will be considered in this section. And for operators of class $k \in \mathsf{Z}$ only spaces $B^s_{p,q}$ and $F^s_{p,q}$ satisfying $(s, p, q) \in \mathsf{D}_k$ are treated, cf. Section 2.6 and Figure 2. The difficulties connected to the unbounded manifolds and to the borderline cases $s = k + \max\left(\frac{1}{p} - 1, \frac{n}{p} - n\right)$ mentioned in Remark 4.12 are thus left open here, with the convenience of doing so illuminated by

REMARK 5.1. In this paper, none of the spaces are larger than those considered in [Gru90]: Even for $(s, p, q) \in \mathsf{D}_k$ with $p \leqq 1$ or $p = \infty$ there is an embedding $B^s_{p,q} \hookrightarrow B^{s'}_{p',p'}$ with $(s', p', p') \in \mathsf{D}_k$ and $1 < p' < \infty$, for a simple embedding can be combined with a Sobolev embedding or an embedding as in (2.23). (These facts do not hold for the borderline cases, and (2.23) cannot be extended to the case of unbounded manifolds.) See Figure 3, which for $k = 0$ also illustrates that one can even embed into spaces in $\mathsf{D}_k \setminus \bar{\mathsf{D}}_{k+1}$.

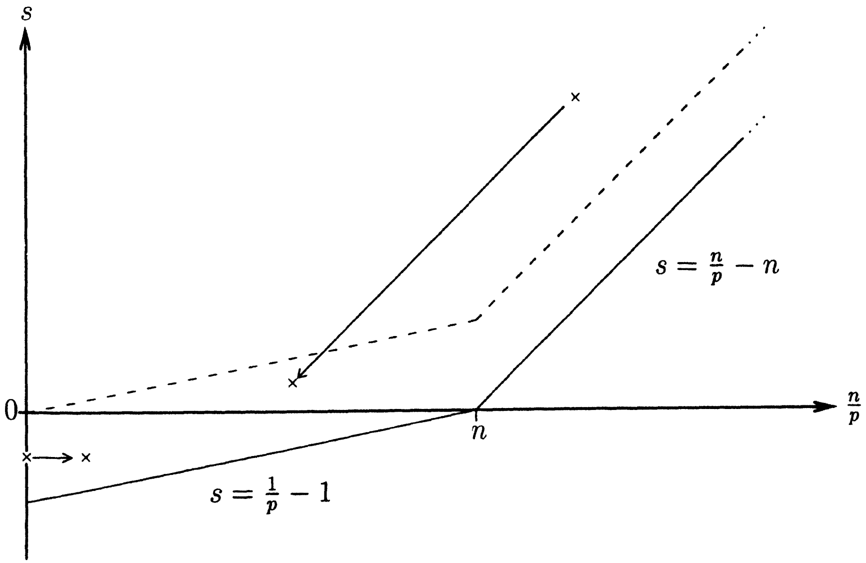

Fig. 3. Embedding into classical spaces in $\mathsf{D}_0 \setminus \bar{\mathsf{D}}_1$.

For convenience the spaces with $1 < p, q < \infty$ are referred to as classical spaces. In other words, only some norms and quasi-norms not included in [Gru90] are introduced, the spaces being subspaces of the classical spaces in focus there.

In addition it is remarked that, when Ω is bounded, the uniformly estimated operators considered here coincide with the locally estimated operators in, e.g., [Gru90].

First the operators are generalised to act on sections of vector bundles E over smooth open bounded subsets $\Omega \subset \mathsf{R}^n$, respectively on vector bundles F over $\Gamma = \partial\Omega$ (all C^∞ and hermitian). See for example [GK93] where it is shown that one can do so invariantly. In particular the uniform two-sided transmission condition and the class concept is invariantly defined on (such) manifolds. However, to make sense of the transmission condition the pseudo-differential operator P should be given on an extending bundle E_1, that is, a bundle with a boundaryless base manifold $\Omega_1 \supset \Omega$ for which $E_1|_\Omega = E$. This will be a tacit assumption on P in the following. For further explanations of the vector bundle set-up, see [Gru90] or [Gru86, App. A. 5]. The space of C^∞ sections of say, the bundle E is written $C^\infty(E)$. (Since Ω is an intrinsic part of E, $C^\infty(E)$ instead of the more tedious $C^\infty(\Omega, E)$ should not course confusion.)

Then, when P_Ω and G send sections of the same bundle, E, into sections of another bundle E' etc., the Green operator $\mathscr{A} = \begin{pmatrix} P_\Omega + G & K \\ T & S \end{pmatrix}$ sends $C^\infty(E) \oplus C^\infty(F)$ into $C^\infty(E') \oplus C^\infty(F')$. In general the fibre dimensions of E and F are denoted by $N > 0$ and $M \geqq 0$, and similarly for the primed bundles. Then (1.2) is a case with trivial bundles.

The spaces of sections $B^s_{p,q}(E), \dots, F^s_{p,q}(F')$ are defined in the standard way by use of local trivialisations, and it is verified that Theorems 4.3, 4.5, 4.9 and 4.15 as well as Theorem 4.17 remain valid if $\bar{\mathsf{R}}^n_+$ is replaced by bundles E and E' over Ω whereas R^{n-1} is replaced by bundles F and F' over Γ.

Consequently Theorem 1.1 is extended thus: for each $(s, p, q) \in \mathsf{D}_r$, there is boundedness of

$$\text{(5.1)} \qquad \mathscr{A}\colon \begin{matrix} B^s_{p,q}(E) \\ \oplus \\ B^{s-\frac{1}{p}}_{p,q}(F) \end{matrix} \to \begin{matrix} B^{s-d}_{p,q}(E') \\ \oplus \\ B^{s-d-\frac{1}{p}}_{p,q}(F') \end{matrix},$$

$$\text{(5.2)} \qquad \mathscr{A}\colon \begin{matrix} F^s_{p,q}(E) \\ \oplus \\ F^{s-\frac{1}{p}}_{p,q}(F) \end{matrix} \to \begin{matrix} F^{s-d}_{p,q}(E') \\ \oplus \\ F^{s-d-\frac{1}{p}}_{p,q}(F') \end{matrix}, \quad \text{if } p < \infty,$$

when each entry in $\mathscr{A}$ has order $d \in \mathsf{R}$ and class $r \in \mathsf{Z}$. In addition, $\mathscr{A}$ cannot be continuous from any of the spaces on the left hand side in (5.1) and (5.2) to $\mathscr{D}'(E') \times \mathscr{D}'(F')$ when $(s, p, q) \notin \bar{\mathsf{D}}_r$ without the class of each entry being $\leqq r - 1$.

Secondly, when $\mathscr{A}' = \begin{pmatrix} P'_\Omega + G' & K' \\ T' & S' \end{pmatrix}$ is a Green operator defined on $C^\infty(E') \oplus C^\infty(F')$, so that $\mathscr{A}'\mathscr{A}$ makes sense on $C^\infty(E) \oplus C^\infty(F)$, then the composition rules simply express that this composite $\mathscr{A}'\mathscr{A}$ is equal to yet another Green operator $\mathscr{A}''$, cf. [GK93, Cor. 5.5]. The identity $\mathscr{A}'' = \mathscr{A}'\mathscr{A}$ holds also when

$\mathcal{A}'\mathcal{A}$ is considered on (the larger) Besov and Triebel-Lizorkin spaces of sections. Indeed, by Remark 5.1 the composition rules shown for the $B^s_{p,p}$ spaces with $1 < p < \infty$ also hold in the spaces treated here.

A main case of interest is the one in which there exists a parametrix, $\tilde{\mathcal{A}}$, of $\mathcal{A}$, that is, another Green operator such that the operator identities

$$\tilde{\mathcal{A}}\mathcal{A} = 1 - \mathcal{R} \tag{5.3}$$

$$\mathcal{A}\tilde{\mathcal{A}} = 1 - \mathcal{R}' \tag{5.4}$$

hold on the spaces on the left and right hand sides of (5.1) and (5.2), respectively, for *negligible* operators $\mathcal{R}$ and $\mathcal{R}'$. This means that they have order $-\infty$, so necessarily $\mathcal{R}$ has range in $C^\infty(E) \oplus C^\infty(F)$ and $\mathcal{R}'$ in $C^\infty(E') \oplus C^\infty(F')$.

When the order of $\mathcal{A}$ is an integer $d \in \mathbf{Z}$ and each entry in $\mathcal{A}$ is *polyhomogeneous* (explained in [Gru86, Sect. 1.2], e.g.), there is a well-known ellipticity condition assuring the existence of $\tilde{\mathcal{A}}$. If the principal symbols are denoted by $p^0(x, \xi)$, $g^0(x, \eta_n, \xi)$ etc., *ellipticity* means that the following two conditions (which are expressed in local coordinates) are fulfilled:

(I) The principal symbol of P is for each $|\xi| \geqq 1$ a bijection

$$p^0(x, \xi)\colon \mathbf{C}^N \to \mathbf{C}^{N'}. \tag{5.5}$$

(II) The principal boundary symbol operator

$$a^0(x', \xi', D_n) = \begin{pmatrix} p^0(x', 0, \xi', D_n)_+ + g^0(x', \xi, D_n) & k^0(x', \xi', D_n) \\ t^0(x', \xi', D_n) & s^0(x', \xi') \end{pmatrix} \tag{5.6}$$

is a bijection

$$\mathcal{S}(\bar{\mathbf{R}}_+)^N \times \mathbf{C}^M \xrightarrow{a^0(D_n)} \mathcal{S}(\bar{\mathbf{R}}_+)^{N'} \times \mathbf{C}^{M'} \tag{5.7}$$

for each $x' \in \Gamma$ and each $|\xi'| \geqq 1$.

It was shown in [Gru90, Thm. 5.4] that if $\mathcal{A}$ is elliptic, then there exists a parametrix $\tilde{\mathcal{A}}$ of order $-d$ and class $r - d$. In this case (5.3) becomes an operator identity valid on the spacees $B^s_{p,q}(E) \oplus B^{s-\frac{1}{p}}_{p,q}(F)$ and $F^s_{p,q}(E) \oplus F^{s-\frac{1}{p}}_{p,p}(F)$, and (5.4) holds on $B^{s-d}_{p,q}(E') \oplus B^{s-d-\frac{1}{p}}_{p,q}(F')$ and $F^{s-d}_{p,q}(E') \oplus F^{s-d-\frac{1}{p}}_{p,p}(F')$ – in both cases for each $(s, p, q) \in \mathbf{D}_r$. Observe that $\mathcal{R}$ is then necessarily of class r while the class of $\mathcal{R}'$ must be $r - d$.

Injective and *surjective* ellipticity of $\mathcal{A}$ means that (I) and (II) above hold only with "bijection" replaced by, respectively, "injection" and "surjection". In the affirmative case there exists an $\tilde{\mathcal{A}}$ satisfying (5.3) and (5.4), respectively, and it is termed a left respectively a right parametrix.

5.1. *Fredholm properties.* Already when $\mathcal{A}$ is either injectively or surjectively elliptic one can deduce various properties for its kernel and range. Instead of generalising the Fredholm theory to the category of quasi-Banach spaces, one can proceed as in [Gru90]. Basically this is possible because, as seen in Remark 5.1, the spaces considered here are contained in the Banach spaces treated there.

This will be explained in the following, where a version of Theorem 1.3 for vector bundles will be proved. First it will be convenient to introduce the vector bundles $V = E \oplus F$ and $V' = E' \oplus F'$ and use them to borrow the spaces $B^{s+\mathbf{a}}_{p,q}(V)$ and $B^{s-\mathbf{b}}_{p,q}(V')$, respectively $F^{s+\mathbf{a}}_{p,q}(V)$ and $F^{s-\mathbf{b}}_{p,q}(V')$, from (5.18) ff. below, where they are introduced systematically. The vectors **a** and **b** indicate that there is a space for each column and row in $\mathscr{A}$; in the present case they are equal to zero.

The injectively elliptic case is quite simple: for each $(s,p,q)\in \mathsf{D}_r$ it is seen from the embedding relations and (5.3) that

$$\{u\in B^{s+\mathbf{a}}_{p,q}(V)\mid \mathscr{A}u=0\}=\{u\in C^\infty(V)\mid \mathscr{A}u=0\}. \tag{5.8}$$

A similar argument works in the Triebel-Lizorkin case, and thus, since $F^s_{2,2} = B^s_{2,2}$, the kernel of $\mathscr{A}$, written $\ker\mathscr{A}$, is independent of $(s,p,q)\in \mathsf{D}_r$ as well as of whether we consider (5.1) or (5.2). Hence $\ker\mathscr{A}$ equals the space in (5.8) – and it has finite dimension by [Gru90]. Moreover, the image $\mathscr{A}(B^{s+\mathbf{a}}_{p,q}(V))$ is closed in $B^{s-d-\mathbf{b}}_{p,q}(V')$ and similarly for the operator in (5.2). The latter fact was proved in [Gru90] for the spaces in consideration there, and if $\mathscr{A}u_m \to v$ in $B^{s-d-\mathbf{b}}_{p,q}(V')$ we determine (s',p',p') as in Remark 5.1 and conclude from [Gru90] that $v=\mathscr{A}w$ for some $w\in B^{s'+\mathbf{a}}_{p',p'}(V)$; here $w=\tilde{\mathscr{A}}v+\mathscr{R}w$ according to (5.3), so $w\in B^{s+\mathbf{a}}_{p,q}(V)$. In a similar way the analogous statement for the Triebel-Lizorkin spaces carry over from [Gru90]. Hereby 1° of Theorem 1.3 is proved.

As a preparation for the surjectively elliptic case we shall first treat the case where $\mathscr{A}$ is elliptic; evidently the arguments above for the injectively elliptic case apply to $\mathscr{A}$ then. Concerning the range of $\mathscr{A}$ we use an embedding of $B^s_{p,q}$ into a classical space $B^{s'}_{p',p'}$. For the classical spaces it was shown in [Gru90] that there exists a finite dimensional subspace $\mathscr{N}\subset C^\infty(V')$ which is a complement of $\operatorname{im}\mathscr{A}$, that is,

$$B^{s'-d-\mathbf{b}}_{p',p'}(V')=\mathscr{N}\oplus\mathscr{A}(B^{s'+\mathbf{a}}_{p',p'}(V)) \tag{5.9}$$

for every $(s',p',p')\in\mathsf{D}_r$ with $1<p'<\infty$. This implies that

$$B^{s-d-\mathbf{b}}_{p,q}(V')=\mathscr{N}\oplus\mathscr{A}(B^{s+\mathbf{a}}_{p,q}(V)) \tag{5.10}$$

for when (5.9) is applied to an element of the subspace $B^{s-d-\mathbf{b}}_{p,q}(V')$ it follows from (5.3) that the component in the range of $\mathscr{A}$ belongs to $\mathscr{A}(B^{s+\mathbf{a}}_{p,q}(V))$. That the sum is direct is seen already from (5.9). From (5.10) we conclude that $\mathscr{N}$ is a complement of $\operatorname{im}\mathscr{A}$ also in the non-classical cases, and by the construction it is independent of (s,p,q) and of finite dimension. The F case is covered by a similar argument.

When $\mathscr{A}$ is surjectively elliptic the study of *ranges* of $\mathscr{A}$ that is found in [Gru90, (5.21) ff.] is easily modified, and we sketch this in the B case when $d=r=0$; the

F case is completely analogous. The tools in [Gru90] consist of some remarks on the Banach space cases with (s, p, q) in $\mathsf{D}_0 \setminus \bar{\mathsf{D}}_1$, and these need not be changed at all. So recall from [Gru90] that

$$(5.11) \qquad \mathscr{A}(B^{s'+\mathbf{a}}_{p',q'}) = \{f \in B^{s'-\mathbf{b}}_{p',q'}(V') \mid \langle f, \bar{g} \rangle = 0 \quad \text{for} \quad g \in \ker \mathscr{A}^*\}$$

when (s', p', q') is a parameter in $\mathsf{D}_0 \setminus \bar{\mathsf{D}}_1$ with $1 < p', q' < \infty$. In addition there is an argument which by Remark 5.1 easily gives that $\rho_0 \leqq \nu(s, p, q) \leqq \rho_1$ for *any* $(s, p, q) \in \mathsf{D}_0$; here $\rho_0 = \dim \ker \mathscr{A}^*$, $\rho_1 = \operatorname{codim} \mathscr{A}\mathscr{A}^*(B^{s+\mathbf{a}}_{p,q})$ and $\nu(s, p, q)$ denotes the codimension of $\mathscr{A}(B^{s+\mathbf{a}}_{p,q})$ in $B^{s-\mathbf{b}}_{p,q}(V')$. In virtue of the injectively and two-sided ellitic cases treated above the numbers ρ_0 and ρ_1 are (s, p, q)-independent. Consequently the conclusion from [Gru90] that $\rho_0 = \rho_1$ yields the independence of ν from (s, p, q).

Now we take $g_1, \dots, g_\nu$ in $C^\infty(V')$ as a basis for $\ker \mathscr{A}^*$, and we may assume that $\langle g_j, \overline{g_k} \rangle = [j = k]$. From (5.11) it is seen that $\mathscr{A}(B^{s+\mathbf{a}}_{p,q})$ and $\ker \mathscr{A}^*$ are linearly independent, and for this reason $g_1, \dots, g_\nu$ are linearly independent in the quotient $B^{s-\mathbf{b}}_{p,q} / \mathscr{A}(B^{s+\mathbf{a}}_{p,q})$. Hence $(g_1, \dots, g_\nu)$ is a basis for the quotient, so $\ker \mathscr{A}^* + \mathscr{A}(B^{s+\mathbf{a}}_{p,q})$ is equal to $B^{s-\mathbf{b}}_{p,q}$. Altogether this shows that

$$(5.12) \qquad B^{s-\mathbf{b}}_{p,q}(V') = \ker \mathscr{A}^* \oplus \mathscr{A}(B^{s+\mathbf{a}}_{p,q}),$$

that is, $\mathscr{A}(B^{s+\mathbf{a}}_{p,q})$ has the finite dimensional (s, p, q)-independent complement $\ker \mathscr{A}^*$. When $\mathscr{A}u_m \to v$ in $B^{s-\mathbf{b}}_{p,q}$ we write $v = \mathscr{A}w + \lambda_1 g_1 + \cdots + \lambda_\nu g_\nu$ by use of (5.12). Then (5.11) gives that $0 = \langle v, \overline{g_j} \rangle = \lambda_j$, so that $v = \mathscr{A}w$. Hence the range of $\mathscr{A}$ is closed.

More generally one can reduce to the case where $d = r = 0$, see [Gru90]. This reduction uses order-reducing operators, written as $\Lambda^m_{-,E}$ for $m \in \mathbf{Z}$, that are chosen so that they for all admissible parameters (s, p, q) have the following group and continuity properties:

$$(5.13) \qquad \Lambda^k_{-,E} \Lambda^m_{-,E} = \Lambda^{k+m}_{-,E}, \quad \Lambda^0_{-,E} = 1,$$

$$(5.14) \qquad \Lambda^m_{-,E} \colon B^s_{p,q}(E) \xrightarrow{\sim} B^{s-m}_{p,q}(E), \quad \Lambda^m_{-,E} \colon F^s_{p,q}(E) \xrightarrow{\sim} F^{s-m}_{p,q}(E).$$

Such operators were constructed in [Gru90, Thm. 5.1 3°] (but called $\Xi^m_{-,E}$ there, see also [Gru91, Ex. 2.10] for a brief review) and in [Fra86a]. Their continuity properties are a consequence of Section 4, since they in general are of the form $P^{(m)}_\Omega + G^{(m)}$, and the group property, valid by the earlier remark on composition rules, implies the bijectivity.

One should observe that in the reduction procedure (5.12) easily carries over to a similar statement for the more general surjectively elliptic Green operators, except that $\ker \mathscr{A}^*$ is replaced by another fixed finite dimensional space of smooth sections. Altogether this proves 2° of Theorem 1.3.

To show 3° there, suppose that (the vector bundle version of) (1.6) holds for some $(s_1, p_1, q_1) \in \mathsf{D}_r$ for a subspace $\mathscr{N} \subset C^\infty(V')$. By 2° there is a parameter independent range complement $\mathscr{M} \subset C^\infty(V')$. If $(g_1, \dots, g_k)$ is a linearly independent tupel in $\mathscr{N}$, its image is so in $B^{s_1-d-\mathbf{b}}_{p_1,q_1}(V')/\mathscr{A}(B^{s_1+\mathbf{a}}_{p_1,q_1})$ by (1.6). Hence $\dim \mathscr{N} \leqq \dim \mathscr{M}$. In addition the quotient is isomorphic as a vector space to both $\mathscr{N}$ and $\mathscr{M}$, so there is equality.

For an arbitrary $(s, p, q) \in \mathsf{D}_r$ the identity (1.6) now holds if and only if a basis $(g_1, \dots, g_k)$ for $\mathscr{N}$ still gives a basis in $B^{s-d-\mathbf{b}}_{p,q}(V')/\mathscr{A}(B^{s+\mathbf{a}}_{p,q})$. Because $k = \dim \mathscr{M}$ was seen above, it suffices to see that $(Qg_1, \dots, Qg_k)$ is linearly independent, when Q denotes the quotient operator. Let $0 = \lambda_1 Qg_1 + \dots \lambda_k Qg_k$. Then

$$\lambda_1 g_1 + \cdots + \lambda_k g_k = \mathscr{A}u \tag{5.15}$$

for a unique $\mathscr{A}u$ in $B^{s-d-\mathbf{b}}_{p,q}(V')$. But

$$\lambda_1 g_1 + \cdots + \lambda_k g_k = w + \mathscr{A}v \tag{5.16}$$

for uniquely determined $w \in \mathscr{M}$ and $\mathscr{A}v \in B^{s_1-d-\mathbf{b}}_{p_1,q_1}(V')$. It suffices now to verify that $w = 0$, for then (5.16) implies that

$$0 = \lambda_1 Q_1 g_1 + \dots \lambda_k Q_1 g_k \quad \text{in} \quad B^{s_1-d-\mathbf{b}}_{p_1,q_1}(V')/\mathscr{A}(B^{s_1+\mathbf{a}}_{p_1,q_1}), \tag{5.17}$$

when Q_1 denotes the quotient operator for (s_1, p_1, q_1). That $w = 0$ is evident when $B^{s_1-d-\mathbf{b}}_{p_1,q_1} \hookrightarrow B^{s-d-\mathbf{b}}_{p,q}$, for then (5.16) is also a decomposition in $B^{s-d-\mathbf{b}}_{p,q}$. Similarly $w = 0$ holds when the reverse embedding does so. Hereby (1.6) has been established for $(s_1 + 1, \infty, 1)$, so by repeating the argument any $(s, p, q) \in \mathsf{D}_r$ with $s < s_1$ is covered, and then in a last application *any* $(s, p, q) \in \mathsf{D}_r$ is so. Thus 3° is proved.

In Corollary 1.4 any (f, φ) in $B^{s_1-d-\mathbf{b}}_{p_1,q_1}(V')$ or $F^{s_1-d-\mathbf{b}}_{p_1,q_1}(V')$ gives a functional $\langle f, \cdot \rangle + \langle \varphi, \cdot \rangle$ on $\mathscr{N}$: one can take a larger classical space $B^{s'-d-\mathbf{b}}_{p',q'}(V')$ with $(s', p', q') \in \mathsf{D}_r \setminus \bar{\mathsf{D}}_{r+1}$, cf. Remark 5.1; with $s = -s' + d$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$ and $\dfrac{1}{q} + \dfrac{1}{q'} = 1$, this space is dual to $B^s_{p,q;0}(E) \oplus B^{s+1-\frac{1}{p}}_{p,q}(F)$ and $e_\Omega g \in B^s_{p,q;0}(E)$, for by construction $(s, p, q) \in \mathsf{D}_{d-r} \setminus \bar{\mathsf{D}}_{d-r+1}$ so it suffices with $\gamma_j g = 0$ for $j < d - r$ as assumed. Moreover, any $\varphi \in \mathscr{D}'(F')$ gives a functional on $C^\infty(F')$.

Using this it is elementary to verify that validity of (1.8) or (1.9) at (s_1, p_1, q_1) implies (1.6) and (1.7), respectively. For every (s, p, q) in D_r, 3° then gives that $\mathscr{N}$ is a range complement, and the inclusions into $\mathscr{N}^\perp$ in (1.8) and (1.9) are clear since $\mathscr{A}(C^\infty(V)) \subset \mathscr{N}^\perp$ is seen by consideration of (s_1, p_1, q_1). The other inclusions follow from the ones proved, for when an element in, say, $B^{s-d-\mathbf{b}}_{p,q}(V') \cap \mathscr{N}^\perp$ is decomposed as in 2°, then the $\mathscr{N}$ component is trivial.

5.2. *Multi-order operators.* Using order-reducing operators one can also reduce multi-order Green operators to the case of order and class 0 and carry through the preceding considerations, cf. [Gru90]. Instead of going into details this section is concluded with a precise summary, which contains the previous results as well as those in Section 1 on single order operators as special cases.

In the following, $\Omega \subset \mathbf{R}^n$ is a smooth open bounded set with $\partial\Omega = \Gamma$, and $\mathcal{A}$ denotes a multiorder Green operator, i.e., $\mathcal{A} = \begin{pmatrix} P_\Omega + G & K \\ T & S \end{pmatrix}$, where $P = (P_{ij})$ and $G = (G_{ij})$, $K = (K_{ij})$, $T = (T_{ij})$ and $S = (S_{ij})$. Here $1 \leqq i \leqq i_1$ and $i_1 < i \leqq i_2$, respectively, in the two rows of the block matrix $\mathcal{A}$, and $1 \leqq j \leqq j_1$ respectively $j_1 < j \leqq j_2$ holds in the two columns of $\mathcal{A}$; that is, $\mathcal{A}$ is an $i_2 \times j_2$ matrix operator.

Each P_{ij}, G_{ij}, K_{ij}, T_{ij} respectively S_{ij} belongs to the poly-homogeneous calculus, i.e., they are a pseudo-differential operator satisfying the uniform two-sided transmission condition (at Γ), a singular Green, a Poisson and a trace operator, resp. a pseudo-differential operator on Γ. The orders of the operators are taken to be $d + b_i + a_j$, where $d \in \mathbf{Z}$, $\mathbf{a} = (a_j) \in \mathbf{Z}^{j_2}$ and $\mathbf{b} = (b_i) \in \mathbf{Z}^{i_2}$, and the class of $P_{ij,\Omega} + G_{ij}$ and of T_{ij} is supposed to be $r + a_j$ for some $r \in \mathbf{Z}$ (*).

The operators are supposed to act on sections of vector bundles E_j over Ω and F_j over Γ, with values in other bundles E'_i and F'_i. Letting $V = (E_1 \oplus \cdots \oplus E_{j_1}) \cup (F_{j_1+1} \oplus \cdots \oplus F_{j_2})$, while $V' = (E'_1 \oplus \cdots \oplus E'_{i_1}) \cup (F'_{i_1+1} \oplus \cdots \oplus F'_{i_2})$, the Green operator $\mathcal{A}$ sends $C^\infty(V)$ to $C^\infty(V')$. Here one can either regard $C^\infty(V)$ as an abbreviation for $C^\infty(E_1) \oplus \cdots \oplus C^\infty(F_{j_2})$, or verify that V is a vector bundle with base manifold $\Omega \cup \Gamma$, cf. the definition in [Lan72]. Observe that hereby the dimension of the base manifold as well as of the fibres over its points x depend on whether $x \in \Omega$ or $x \in \Gamma$. Similar remarks apply to V'.

To have a convenient notation we shall now introduce spaces that are adapted to the order and class of each entry in $\mathcal{A}$, namely (with $p < \infty$ in the Triebel-Lizorkin spaces)

$$B^{s+\mathbf{a}}_{p,q}(V) = \Big(\bigoplus_{j \leqq j_1} B^{s+a_j}_{p,q}(E_j)\Big) \oplus \Big(\bigoplus_{j_1 < j} B^{s+a_j-\frac{1}{p}}_{p,q}(F_j)\Big) \tag{5.18}$$

$$B^{s-\mathbf{b}}_{p,q}(V') = \Big(\bigoplus_{i \leqq i_1} B^{s-b_i}_{p,q}(E'_i)\Big) \oplus \Big(\bigoplus_{i_1 < i} B^{s-b_j-\frac{1}{p}}_{p,q}(F'_i)\Big), \tag{5.19}$$

$$F^{s+\mathbf{a}}_{p,q}(V) = \Big(\bigoplus_{j \leqq j_1} F^{s+a_j}_{p,q}(E_j)\Big) \oplus \Big(\bigoplus_{j_1 < j} F^{s+a_j-\frac{1}{p}}_{p,p}(F_j)\Big) \tag{5.20}$$

$$F^{s-\mathbf{b}}_{p,q}(V') = \Big(\bigoplus_{i \leqq i_1} F^{s-b_i}_{p,q}(E'_i)\Big) \oplus \Big(\bigoplus_{i_1 < i} F^{s-b_j-\frac{1}{p}}_{p,p}(F'_i)\Big), \tag{5.21}$$

(*) For short $\mathcal{A}$ is then said to have order d and class r.

(As usual $F^s_{p,p}(F_j) = B^s_{p,p}(F_j)$ etc.) It is convenient to take the quasi-norms to be $\| v\,|B^{s+\mathbf{a}}_{p,q}\| = (\| v_1\,|B^{s+a_1}_{p,q}(E_1)\|^q + \cdots + \| v_{j_2}\,|B^{s+a_{j_2}-\frac{1}{p}}_{p,q}(F_{j_2})\|^q)^{\frac{1}{q}}$, and $\| v\,|F^{s+\mathbf{a}}_{p,q}\| = (\| v_1\,|F^{s+a_1}_{p,q}(E_1)\|^p + \cdots + \| v_{j_2}\,|F^{s+a_{j_2}-\frac{1}{p}}_{p,p}(F_{j_2})\|^p)^{\frac{1}{p}}$, with similar conventions for $B^{s-\mathbf{b}}_{p,q}$ and $F^{s-\mathbf{b}}_{p,q}$.

The ellipticity concept for multi-order Green operators is like the one for single-order operators, except that $p^0(x,\xi)$ is a matrix with p^0_{ij} equal to the principal symbol of P_{ij} *relative* to the order $d + b_i + a_j$ of P_{ij}; and similarly for $a^0(x',\xi',D_n)$.

THEOREM 5.2. *Let $\mathscr{A}$ denote a multi-order Green operator going from V to V' as described above. Then $\mathscr{A}$ is* continuous

$$\mathscr{A}\colon B^{s+\mathbf{a}}_{p,q}(V) \to B^{s-d-\mathbf{b}}_{p,q}(V'), \quad \mathscr{A}\colon F^{s+\mathbf{a}}_{p,q}(V) \to F^{s-d-\mathbf{b}}_{p,q}(V'), \tag{5.22}$$

for each $(s,p,q) \in \mathrm{D}_r$, when $p < \infty$ in the Triebel-Lizorkin spaces.

If $\mathscr{A}$ is injectively elliptic, surjectively elliptic, respectively elliptic, then $\mathscr{A}$ has a left, right respectively two-sided parametrix *$\tilde{\mathscr{A}}$ in the calculus; it can be taken of order $-d$ and class $r - d$, and then $\tilde{\mathscr{A}}$ is continuous in the opposite direction in* (5.22) *for all the parameters (s,p,q) mentioned above.*

When $\mathscr{A}$ is continuous $B^{s+a}_{p,q}(V) \to \mathscr{D}'(V')$ or $\tilde{\mathscr{A}}$ is so from $B^{s-d-\mathbf{b}}_{p,q}(V')$ to $\mathscr{D}'(V)$ for some $(s,p,q) \notin \bar{\mathrm{D}}_r$, then the class *of $\mathscr{A}$ is $\leqq r - 1$ and $\tilde{\mathscr{A}}$ has class $\leqq r - 1 - d$, respectively. A similar conclusion holds for $F^{s+\mathbf{a}}_{p,q}(V)$ and $F^{s-d-\mathbf{b}}_{p,q}(V')$.*

Furthermore, when $\mathscr{A}$ is injectively elliptic, the inverse regularity *properties in Corollary* 1.2 *carry over to the operators in* (5.22). *Moreover,* 1° *of Theorem* 1.3 *is valid mutatis mutandem for $\mathscr{A}$, and the ranges are closed.*

When $\mathscr{A}$ is surjectively elliptic, analogues of 2° *and* 3° *of Theorem* 1.3 *as well as of Corollary* 1.4 *hold for $\mathscr{A}$ (when $\gamma_j g_i = 0$ for $j < d + b_i - r$).*

In the elliptic case, all these properties hold for $\mathscr{A}$, and the parametrices are two-sided.

On the basis of the single-order case described above, Theorem 5.2 is obtained by a straightforward extension of the proof of [Gru90, Cor. 5.5].

It should be observed explicitly, that the (s,p,q)-independence of $\ker \mathscr{A}$ implies that it is the same space regardless of whether $\mathscr{A}$ is considered on $B^{s+\mathbf{a}}_{p,q}(V)$ or on $F^{s+\mathbf{a}}_{p,q}(V)$; this follows since $B^{s+\mathbf{a}}_{2,2} = F^{s+\mathbf{a}}_{2,2}$. A similar argument shows that one can take a space $\mathscr{N}$ that is a complement of $\mathscr{A}(B^{s+\mathbf{a}}_{p,q})$ in $B^{s-d-\mathbf{b}}_{p,q}(V')$ as well as of $\mathscr{A}(F^{s+\mathbf{a}}_{p,q})$ in $F^{s-d-\mathbf{b}}_{p,q}(V')$.

Also it should be observed that the inverse regularity properties as in Corollary 1.2 follow by application of (5.3) to (1.5) (or its analogue).

The theorem above is a generalisation of [Gru90, Cor. 5.5] to the scales of $B^s_{p,q}$ and $F^s_{p,q}$ spaces with $0 < p,q \leqq \infty$ (with $p < \infty$ for the F spaces) with a rather more detailed Fredholm theory characterising the ranges of $\mathscr{A}$. Moreover, the

Hölder-Zygmund spaces $C^s = B^s_{\infty,\infty}$, $s > 0$, are included, and unlike [RS82] it is unnecessary to assume that $s - d > 0$ when applying operators of order d to C^s: the space $B^{s-d}_{\infty,\infty}$ can receive in any case. For further historical remarks see the beginning and end of [Gru90].

6. Applications.

As a first example it is clear that Theorem 5.2 above applies to such boundary problems as those in (1.1) above,

Secondly the various orthogonal decompositions into divergence-free and gradient subspaces in [Gru90, Ex. 3.14] are generalised by Theorem 5.2 to the spaces $B^s_{p,q}(\bar{\Omega})^n$ and $F^s_{p,q}(\bar{\Omega})^n$ for $(s, p, q) \in \mathrm{D}_0$.

Thirdly the inverse regularity properties in Corollary 1.2 carry over to *semilinear* perturbations of $\mathscr{A}$, as long as the non-linear term is "better behaved" than $\mathscr{A}$.This is proved for the stationary Navier-Stokes equations (considered with boundary conditions of class 1 and 2) in [Joh93, Thm. 5.5.3]. A paper on this application to more general semi-linear problems is being worked out; results and methods are sketched in [Joh95b].

For the stationary Navier-Stokes equations the mentioned inverse regularity results have led to an extension of the weak L_2 solvability theory for the Dirichlét problem, cf. [Tem84], to existence of solutions in the $B^s_{p,q}$ and $F^s_{p,q}$ spaces, in rough terms when $s \geqq \max\left(1, \frac{n}{p} - \frac{n}{2} + 1\right)$. See [Joh93, Thm. 5.5.5] for details.

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