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VITALI SYSTEMS IN R^n WITH IRREGULAR SETS

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Abstract.

Vitali type theorems are results stating that out of a given family of sets one can select pairwise disjoint sets which fill out a "large" region. Usually one works with "regular" sets such as balls. We shall establish results with sets of a more complicated geometrical structure, e.g., Cantor-like sets are allowed. The results are related to a generalisation of the classical notion of a differentiation basis. They concern real n-space R^n and Lebesgue measure.

1. Introduction.

We shall work in real n-space R^n provided with a metric induced by a norm. The closed, resp. open, ball of centre x and radius r is denoted by $B[x,r]$, resp. $B(x,r)$. We more generally define $B[C,r] = \{x \in \mathsf{R}^n \mid \operatorname{dist}(x,C) \leqq r\}$, if $\emptyset \neq C \subset \mathsf{R}^n$. The diameter of a set $A \neq \emptyset$ is denoted by $\operatorname{diam}(A)$.

We use $|A|$ to indicate the (outer) Lebesgue measure of a set A.

Let Φ^n denote the class of (measurable) functions $\varphi\colon]0,+\infty[\, \to\,]0,+\infty[$ satisfying $\varphi(r) \leqq (2r)^n$ for all $r > 0$. In the applications, the functions $\varphi \in \Phi^n$ will be nondecreasing for geometrical reasons, but this assumption is in general not necessary for the proofs.

Let $\varphi \in \Phi^n$ and let $\mathscr{K}$ be a class of compact subsets of R^n. By $\mathscr{B}^n[\varphi,\mathscr{K}]$ we denote the differentiation basis (cf. also [1], [3]) with $\mathscr{B}_r(x) = \mathscr{B}_r^n(x;\varphi,\mathscr{K})$ defined by

$$\mathscr{B}_r^n(x;\varphi,\mathscr{K}) = \{K \in \mathscr{K} \mid K \subseteq B[x,r], |K| \geqq \varphi(r)\}.$$

The class $\mathscr{B}^n[\varphi,\mathscr{K}] = \{\mathscr{B}_r(x) \mid x \in \mathsf{R}^n, r > 0\}$ is nonempty for proper choices of φ and $\mathscr{K}$. Any element of the associated *Vitali system* $\mathscr{V}^n[\varphi,\mathscr{K}]$ is defined as a pair $(A,\mathscr{S})$, where $A \subseteq \mathsf{R}^n$ and $\mathscr{S}$ is a class of closed subsets of R^n such that

$$\forall x \in A \exists r_0 \forall r \leqq r_0 : \mathscr{S} \cap \mathscr{B}^n[\varphi,\mathscr{K}] \neq \emptyset.$$

For a formal introduction of Vitali systems, cf. [6].

We say that the Vitali system $\mathscr{V} = \mathscr{V}^n[\varphi,\mathscr{K}]$ has the *packing property*, if for

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every pair $(A,\mathscr{S})\in\mathscr{V}$ there exists a countable subclass $\mathscr{K}_0\subseteq\mathscr{S}$ consisting of disjointed sets, such that

$$|A\setminus\cup\{S\,|\,S\in\mathscr{S}_0\}|=0.$$

We shall consider special classes $\mathscr{K}$ of compact subsets. For given constants $\eta>0$, $\kappa>0$, let $\mathscr{K}^n_{\eta,\kappa}$ denote the class of compact sets K such that for any closed ball B, either $|B\cap K|\geqq\kappa|B|$, or else there exists an open ball $B_1\subset B\setminus K$ such that $\operatorname{diam}(B_1)\geqq\eta\cdot\operatorname{diam}(B)$.

The classes $\mathscr{K}^n_{\eta,\kappa}$ are crucial for our investigations. We shall study them closer in Section 2, where it will be proved that $\mathscr{K}^n_{\eta,\kappa}$ contains the class of closed balls, if and only if

(1) $$2\eta+\sqrt[n]{\kappa}\leqq 1.$$

Usually, the classes $\mathscr{K}^n_{\eta,\kappa}$ will contain sets which are much more "irregular" than balls. This fact will allow us to derive results going beyond previously known ones.

Let us write $\mathscr{B}^n[\varphi;\eta,\kappa]$ and $\mathscr{V}^n[\varphi,\eta,\kappa]$ instead of $\mathscr{B}^n[\varphi,\mathscr{K}^n_{\eta,\kappa}]$ and $\mathscr{V}^n[\varphi,\mathscr{K}^n_{\eta,\kappa}]$.

The main result of this paper is the following

THEOREM 1. *Let* $\eta,\kappa>0$ *be constants, such that* $2\eta+\sqrt[n]{\kappa}\leqq 1$, i.e. (1) *holds, and let* $\varphi\in\Phi^n$. *Then* $\mathscr{V}^n[\varphi,\eta,\kappa]$ *has the packing property, if and only if*

(2) $$\int_0^{r_0}\frac{\varphi(r)}{r^n}\frac{dr}{r}=+\infty\quad\textit{for all}\quad r_0>0.$$

Theorem 1 generalises Theorem 2 of [4]. The necessity of (2) follows from that result. The sufficiency will be proved in Section 4.

It follows from Theorem 1 and Proposition 1 of Section 2 that if (2) holds, then $\mathscr{V}^n[\varphi,\mathscr{K}]$ has the packing property, where $\mathscr{K}$ is the class of compact convex sets.

By applying a standard argument it also follows from Theorem 1 that if (2) holds and if $(A,\mathscr{S})$ satisfies the condition

$$\forall x\in A\exists r_0,\eta,\kappa\forall r\leqq r_0:\mathscr{S}\cap\mathscr{B}^n_r(x;\varphi,\eta,\kappa)\neq\emptyset,$$

then $|A\setminus\cup\{S\,|\,S\in\mathscr{S}_0\}|=0$, where $\mathscr{S}_0$ is some countable disjointed subfamily of $\mathscr{S}$.

Finally, we note that from Theorem 1 and from the proof of the necessity of Theorem 2 of [4] it follows that if $\mathscr{V}^n[\varphi,\eta,\kappa]$ has the packing property, then every $\mathscr{V}^n[\varphi,\eta,\kappa]$*-meagre* set is a nullset. A set C is called $\mathscr{V}$*-meagre*, where $\mathscr{V}$ is a Vitali system, if $(C,\mathscr{S})\in\mathscr{V}$, where $\mathscr{S}$ is the class of all closed sets disjoint from C, cf. e.g. [7].

The result above on meagre sets depends on the special nature of the Vitali systems under consideration. It is actually easy to construct Vitali systems which only involve sets from the classes $\mathscr{K}^n_{\eta,\kappa}$ such that every meagre set is a nullset, while the packing property does not hold. In fact, it suffices to work with sets which are unions of just two balls and so that only the empty set is meagre.

2. The classes $\mathscr{K}^n_{\eta,\kappa}$

For a bounded subset A of R^n we define $\ker(A)$, the *kernel* of A, as a closed ball B of maximal radius such that $\operatorname{int}(B) \subseteq A$. There may of course be several such balls, so we assume that $\ker(A)$ selects one of these according to some fixed rule. When no such ball B exists, we define $\ker(A) = \emptyset$. With this notation, a compact set $K \subseteq \mathsf{R}^n$ lies in the class $\mathscr{K}^n_{\eta,\kappa}$, where $\eta, \kappa \in]0,1]$, if and only if for every ball B either

$$|B \cap K| \geqq \kappa |B|, \tag{3}$$

or

$$\operatorname{diam}(\ker(B \setminus K)) \geqq \eta \cdot \operatorname{diam}(B). \tag{4}$$

Clearly, the classes $\mathscr{K}^n_{\eta,\kappa}$ increase with decreasing η and with decreasing κ.

PROPOSITION 1. *For positive constants η and κ the following conditions are equivalent*:

(i) $\mathscr{K}^n_{\eta,\kappa}$ *contains the class of all closed balls in* R^n,

(ii) $\mathscr{K}^n_{\eta,\kappa}$ *contains the class of all compact convex sets in* R^n,

(iii) $2\eta + \sqrt[n]{\kappa} \leqq 1$.

PROOF. (iii)$\Rightarrow$(ii). Let K be a compact and convex set. Assume that $B[x,r] \cap K \neq \emptyset$ and $|B[x,r] \cap K| < \kappa |B[x,r]|$, i.e. (3) does not hold for $B[x,r]$. Then $B[x, \sqrt[n]{\kappa} \cdot r] \setminus K \neq \emptyset$, so by the Hahn-Banach theorem we can find y and y^*, $\|y^*\| = 1$, such that

$$\|y - x\| = \langle y - x, y^* \rangle = \sqrt[n]{\kappa} \cdot r,$$

and

$$\langle z - x, y^* \rangle \leqq \sqrt[n]{\kappa} \cdot r \quad \text{for every } z \in K.$$

Let

$$B = B\left(x + \frac{1 + \sqrt[n]{\kappa}}{2}(y - x), \frac{1 - \sqrt[n]{\kappa}}{2} r\right).$$

As $\langle z - x, y^* \rangle > \sqrt[n]{\kappa} \cdot r$ for all $z \in K$, we get $K \cap B = \emptyset$, and as $\sqrt[n]{\kappa} < 1$, we also have $B \subset B[x,r]$. Hence,

$$\operatorname{diam}(\ker(B[x,r]\backslash K)) \geqq \operatorname{diam}(B) = \frac{1-\sqrt[n]{\kappa}}{2}\operatorname{diam}(B[x,r]) \geqq \eta\cdot\operatorname{diam}(B[x,r]),$$

and we have proved (4).

(ii) $\Rightarrow$ (i). Trivial.

(i) $\Rightarrow$ (iii). Let $K = B[0,1]$ and $B = B[0,r]$ with $r > 1/\sqrt[n]{\kappa}$. Then (3) fails, hence (4) holds. It follows that $r - 1 \geqq \eta\cdot 2r$. As this is true for all $r > 1/\sqrt[n]{\kappa}$, we get (iii).

In Section 4 and Section 6 we shall work with a generalisation of the classes $\mathscr{K}^n_{\eta,\kappa}$, in which η is replaced by a function. Let $\eta\colon]0,+\infty[\,\to\,]0,1]$ be a function and $\kappa\in]0,1]$ a constant. Then $\mathscr{K}^n_{\eta,\kappa}$ denotes the class of compact sets $K\subset \mathsf{R}^n$ such that for every closed ball B either (3) holds, or else

$$\operatorname{diam}(\ker(B\backslash K)) \geqq \eta(\operatorname{diam}(B))\cdot\operatorname{diam}(B). \tag{5}$$

Introducing the function

$$\eta^*(d) = d\cdot\eta(d), \tag{6}$$

the inequality (5) may be written in the form

$$\operatorname{diam}(\ker(B\backslash K)) \geqq \eta^*(\operatorname{diam}(B)). \tag{7}$$

Obviously, $\mathscr{K}^n_{\eta,\kappa}$ contains the class of all closed balls in R^n, if

$$2\sup\eta(\mathrm{d}) + \sqrt[n]{\kappa} \leqq 1,$$

cf. Proposition 1.

3. The classes $\mathscr{K}^1_{\eta,\kappa}$.

In the one-dimensional case we may generalise Proposition 1 as follows: To each $n\in\mathsf{N}$, the family $\mathscr{K}^1_{\eta,\kappa}$ contain the class of all sets which can be written as a union of at most n intervals, if and only if

$$(n+1)\eta + \kappa \leqq 1. \tag{8}$$

The simple proof is left to the reader. This result indicates that for η and κ sufficiently small, the classes $\mathscr{K}^1_{\eta,\kappa}$ contain sets which are far more "irregular" than convex sets. As an example of an "irregular" set in these classes we mention that the usual Cantor set belongs to $\mathscr{K}^1_{\eta,\kappa}$ if (and only if) $\eta < 1/5$, but this set is of measure zero and therefore not so interesting for our purposes.

We shall show that if the inequality of Proposition 1 fails in the one-dimensional case, i.e. if $2\eta + \kappa > 1$, then $\mathscr{K}^1_{\eta,\kappa}$ only contains Lebesgue nullsets. We may therefore in the one-dimensional case add to the equivalent conditions of Proposition 1 the condition that $\mathscr{K}^1_{\eta,\kappa}$ contains at least one set of positive measure. It is convenient first to introduce some notions.

Let $K \subset \mathsf{R}$ and $r > 1$. An interval I is said to be an *r-component* of K if the endpoints of I belong to K and $K \cap I_r \subseteq I$, where I_r denotes the interval of the same centre as I such that $|I_r| = r|I|$. If k is an isolated point of K, then $\{k\}$ is a *trivial component* of K. We shall say that K *dissolves into r-components* if each nontrivial r-component I of K contains two disjoint (possibly trivial) r-components of K, say I_1 and I_2 such that $I \cap K \subseteq I_1 \cup I_2$. Note that if I_1 and I_2 with the stated properties exist, then $|I_1 \cup I_2| \leqq \alpha |I|$ must hold for some $\alpha < 1$. Indeed, $\alpha = 4/(3 + r)$ will do. This fact leads to the observation that *every set which dissolves into r-components is a nullset* (start with the convex hull of K and construct successively more and more r-components). The indicated argument also shows that a set which dissolves into r-components is homeomorphic to a subset of the Cantor set. The Cantor set itself dissolves into 3-components.

Proposition 2. *For $2\eta + \kappa > 1$, the class $\mathscr{K}^1_{\eta,\kappa}$ only contains nullsets.*

Proof. Assume $2\eta + \kappa > 1$ and choose $r > 1$, such that $1 - 2\eta < r^{-1} < \kappa$. Let $K \in \mathscr{K}^1_{\eta,\kappa}$ and let I be an r-component of K. As $|I_r \cap K| \leqq |I| < \kappa |I_r|$, we must have $|J| \geqq \eta \cdot |I_r|$, where $J = \ker(I_r \backslash K)$. Each of the two intervals making up $I_r \backslash I$ have length $\frac{1}{2}(1 - r^{-1})|I_r|$, which is less than $|J|$. Hence, $J \subseteq I$. By enlarging J if necessary, we may assume that the endpoints of J lie in K (here J is taken to be open). It is now straightforward to check that the two disjoint intervals making up $I \backslash J$ are both r-components of K. This argument shows that K dissolves into r-components, hence $|K| = 0$.

Note that for $2\eta + \kappa > 1$, the classes $\mathscr{K}^1_{\eta,\kappa}$ only depend on η, and they are hereditary. If $\eta > \frac{1}{2}$, then $\mathscr{K}^1_{\eta,\kappa} = \{\emptyset\}$, and if $\frac{1}{3} < \eta \leqq \frac{1}{2}$, then $\mathscr{K}^1_{\eta,\kappa}$ only contains $\emptyset$ and singletons (still assuming that $2\eta + \kappa > 1$). In general, the classes $\mathscr{K}^1_{\eta,\kappa}$ have little stability. Indeed, they need not be closed under unions or under intersections. Some further considerations show that the class of nullsets which are members of at least one $\mathscr{K}^1_{\eta,\kappa}$-class coincides with the class of sets which dissolves into r-components for some $r > 1$.

In general, the classes $\mathscr{K}^n_{\eta,\kappa}$ also contain Cantor-like sets of positive measure. This is the content of the following proposition.

Proposition 3. *For every $\alpha \in]0, 1[$ there exists a subset of the unit cube in R^n which is homeomorphic to the Cantor set, has Lebesgue measure α and lies in one of the classes $\mathscr{K}^n_{\eta,\kappa}$.*

Proof. Assume that $n = 1$ (the general case may be dealt with by a consideration of product sets). Choose numbers $a_n \in]0, \frac{1}{3}]$, $n \geqq 1$, such that $\prod_{n=1}^{+\infty} (1 - a_n) = \alpha$.

Construct $C = \bigcap_{n=1}^{+\infty} C_n$ in a similar way as the Cantor set: Let $C_0 = [0, 1]$ and let C_n be a union of 2^n disjoint closed subintervals of $[0, 1]$, all of the same length. If $[a, b]$ is one of the 2^{n-1} components of C_{n-1}, then $[a, b]$ contains two components of C_n, viz. $[a, c]$ and $[d, b]$ with a "hole" $]c, d[$ of length $d - c = a_n \cdot (b - a)$. Clearly, C is homeomorphic to the usual Cantor set and $|C| = \alpha$. Note also that for any n and any component A of C_n one has $|A \cap C| \geqq \alpha |A|$.

We shall prove that if (η, κ) satisfies the condition

$$(9) \qquad 2\eta + 6\alpha^{-1}\kappa \leqq 1,$$

then $C \in \mathscr{K}^1_{\eta,\kappa}$. Assume (9) and let $I \subseteq \mathsf{R}$ be any interval. If the interior of I contains no point of C, then (4) holds (with $B = I$ and $K = C$).

If the interior of I contains a point of C, we let n be the smallest number for which I contains one of the 2^n components of C_n. Let A be such a component.

Assume that neither 0 nor 1 belong to A. Then there is a component A^- of C_n lying closest to A among all components in C_n to the left of A, and there is a component A^+ of C_n lying closest to A to the right of A. We may assume that A^- and A are "neighbours" (i.e. their convex hull is a component in C_{n-1}). Let A^{++} be the neighbour of A^+. Then I is contained in the convex hull of $A^- \cup A^{++}$. Let x denote the common value of $|A^-|$, $|A|$, $|A^+|$ and $|A^{++}|$, and let y denote the common length of the hole between A^- and A, and between A^+ and A^{++}. Note that $y \leqq x$, because $\alpha_n \leqq \frac{1}{3}$. Furthermore, let J be the hole between A and A^+, i.e.

$$J = \operatorname{conv}\{A \cup A^+\} \setminus (A \cup A^+),$$

and put $z = |I \cap J|$.

Choose k such that

$$\frac{6\eta}{1 - \eta} \leqq k \leqq \frac{\alpha}{\kappa} - 6.$$

If $z \leqq kx$, then

$$\frac{|I \cap C|}{|I|} \geqq \frac{|A|}{4x + 2y + kx} \geqq \frac{\alpha x}{6x + kx} \geqq \kappa,$$

so (3) holds. If $z > kx$, then

$$\frac{|\ker(I \setminus C)|}{|I|} \geqq \frac{|I \cap J|}{|I|} \geqq \frac{z}{4x + 2y + z} \geqq \frac{k}{6 + k} \geqq \eta,$$

so (4) holds in this case.

The analysis of the cases in which A contains either 0 or 1 (or both) is left to the reader.

4. Proof of Theorem 1.

We shall derive Theorem 1 from the more general Theorem 2 below. We first prove

LEMMA 1. *Let $\mathscr{D}$ be a family of subsets of R^n such that*

$$\sum\{|D| \mid D \in \mathscr{D}\} < +\infty.$$

Let ψ be a nonnegative function on $]0, +\infty[$ satisfying

$$\psi(r) \leqq r^n \quad \textit{and} \quad \int_0^1 \frac{\psi(r)}{r^n}\frac{dr}{r} = +\infty.$$

For any given $t > 0$ and $\alpha \in]0, 1]$, each of the following properties of a set $E \subset \mathsf{R}^n$ ensures that $|E| = 0$:

(i) *For every $x \in E$ and every $r < t$ the ball $B[x, r]$ contains a set $D \in \mathscr{D}$ with $|D| \geqq \psi(r)$.*

(ii) *For every $x \in E$ and every $r < t$ the ball $B[x, r]$ either contains a set $D \in \mathscr{D}$ from $\mathscr{D}$ with $|D| \geqq \psi(r)$, or it contains a ball B with $|B| \geqq \psi(r)$ and $|B \cap E| \leqq (1 - \alpha)|B|$.*

PROOF. (i) The condition implies that

$$E \subset \bigcup\{B[D, r] \mid D \in \mathscr{D}, \operatorname{diam}(D) \leqq 2r, |D| \geqq \psi(r)\},$$

whenever $r < r_0$. We introduce $N(s) = \operatorname{card}\{D \in \mathscr{D} \mid |D| > s\}$. Then

$$|E| \leqq (2r)^n |B[0, 1]| \cdot N(\psi(r)).$$

The lemma follows, if we can prove that $\liminf_{r \to 0} r^n N(\psi(r)) = 0$.

Let us assume that this is not the case, i.e. we assume that $N(\psi(r)) \geqq c \cdot r^{-n}$ for $r \in]0, r_0[$. Let $M(t) = \sup\{s \mid t \leqq N(s)\}$. Then $\psi(r) \leqq M(c \cdot r^{-n})$ for $r \in]0, r_0[$, from which we deduce that

$$\int_0^{r_0} \frac{\psi(r)}{r^n}\frac{dr}{r} \leqq \int \frac{M(cr^{-n})}{r^{n+1}}\, dr = nc \int M(s)\, ds$$

$$= nc \int N(s)\, ds = c \sum\{|D| \mid D \in \mathscr{D}\} < +\infty.$$

contradicting the assumption on ψ.

(ii) Let us assume that $|E| > 0$. By the density theorem one can find a bounded set $E_0 \subset E$ and $s \in]0, t[$, such that $|E_0| > 0$ and

$$|E \cap B[x, r]| > (1 - 4^{-n}r)|B[x, r]| \quad \text{for} \quad x \in E_0 \quad \text{and} \quad r \in]0, s[.$$

Let $\mathscr{B}$ be the family of all closed balls $B[x,r] \subseteq B[E_0,s]$ for which $r \in]0,s/4[$ and $|E \cap B[x,r]| \leqq (1-\alpha)|B[x,r]|$. According to Theorem 2.8.4 in [2] there exists a disjointed subfamily $\mathscr{C}$ of $\mathscr{B}$ such that for every ball $B \in \mathscr{B}$ there is a $C \in \mathscr{C}$ satisfying

$$C \cap B \neq \emptyset \quad \text{and} \quad |B| \leqq 2|C|.$$

We claim that for every $x \in E_0$ and every $r \in]0,s/4[$ we can find a $D \in \mathscr{D} \cup \mathscr{C}$, such that

$$D \subset B[x,r] \quad \text{and} \quad |D| \geqq \tfrac{1}{2}\psi\left(\frac{r}{2}\right).$$

In fact, our assumptions imply that either there is a $D \in \mathscr{D}$, such that $D \subset B[x,r/2]$ and $|D| \geqq \psi(r/2)$, or there is a $B \in \mathscr{B}$ such that $B \subset B[x,r/2]$ and $|B| \geqq \psi(r/2)$. In the former case there is nothing to prove. In the latter case we choose $D \in \mathscr{D}$, such that

$$\emptyset \neq D \cap B \subset B \cap B[x,r/2] \quad \text{and} \quad |B| \leqq 2|D|.$$

Let $\tilde{B}$ be the smallest ball of centre x containing D. Since the radius of $\tilde{B}$ is less than s, we have

$$|E \cap \tilde{B}| > (1 - 4^{-n}\alpha)|\tilde{B}|. \tag{10}$$

If $D \backslash B[x,r] \neq \emptyset$, then $\operatorname{diam}(D) \geqq \frac{1}{4}\operatorname{diam}(\tilde{B})$, so

$$|E \cap \tilde{B}| \leqq |\tilde{B} \backslash D| + |E \cap D| \leqq (1 - 4^{-n}\alpha)|\tilde{B}|,$$

contradicting (10). Hence, $D \subset B[x,r]$, which proves our claim.

Finally, (ii) is deduced from (i) with ψ, $\mathscr{D}$ and t replaced by $\psi(r/2)/2$, $\mathscr{D} \cup \mathscr{C}$ and $s/4$, respectively.

THEOREM 2. *Assume that $\kappa \in]0,1]$ and that φ and η are nonnegative functions on $]0,+\infty[$ satisfying*

$$\varphi(r) \leqq r^n,\ \eta(r) \leqq r \quad \textit{and} \quad \int_0^1 \varphi(\eta(r)/2) r^{-n-1}\, dr = +\infty.$$

Let $\mathscr{K}$ be a family of compact subsets of R^n, such that whenever $K \in \mathscr{K}$ every ball $B[x,r]$ in R^n satisfies at least one of the following two properties:

(i) *$B[x,r]$ contains a ball B of measure at least $\varphi(\frac{1}{2}\eta(r))$ such that $|K \cap B| \geqq \kappa|B|$.*

(ii) *$B[x,r] \backslash K$ contains a ball of radius $\eta(r)$.*

Then $\mathscr{K}$ contains a countable disjointed subfamily $\mathscr{C}$ covering almost all of the set of those $x \in \mathsf{R}^n$ for which

$$\exists r_0 \forall r < r_0 \exists K \in \mathscr{K} : K \subseteq B[x,r] \textit{ and } |K| \geqq \varphi(r). \tag{11}$$

PROOF. We may assume that the union of $\mathscr{K}$ is bounded. Using the standard procedure from Theorem 2.8.4 in [2] we can find a maximal subfamily $\mathscr{C}$ of $\mathscr{K}$, such that

(a) whenever C_1 and C_2 are different sets from $\mathscr{C}$, then

$$(12) \qquad B[C_1, \operatorname{diam}(C_1)/4] \cap C_2 = \emptyset \quad \text{or} \quad B[C_2, \operatorname{diam}(C_2)/4] \cap C_1 = \emptyset,$$

(b) for every $K \in \mathscr{K}$ either $B[K, \frac{1}{4}\operatorname{diam}(K)] \cap C = \emptyset$ for all $C \in \mathscr{C}$, or $B[K, \frac{1}{4}\operatorname{diam}(K)] \cap C \neq \emptyset$ and $|K| \leqq 2|C|$ for some $C \in \mathscr{C}$.

The standard argument in [2] then shows that for every $K \in \mathscr{K}$ there is a $C \in \mathscr{C}$, such that $B[K, \frac{1}{4}\operatorname{diam}(K)] \cap C \neq \emptyset$ and $|K| \leqq 2|C|$ for some $C \in \mathscr{C}$. Note that since $\mathscr{C}$ is a disjointed family of measurable sets and $\cup \mathscr{C}$ is bounded we have $\sum \{|C| \mid C \in \mathscr{C}\} < +\infty$.

We shall prove that $\mathscr{C}$ has the required property of the theorem. Let $r_0 > 0$ and let E denote the set of all $x \in \mathrm{R}^n \setminus \cup \mathscr{C}$ such that for every $r < r_0$ the ball $B[x, r]$ contains a set $K \in \mathscr{K}$ with $|K| \geqq \varphi(r)$. The theorem follows if we can prove that $|E| = 0$.

In the proof of $|E| = 0$ we shall use the statement (ii) of Lemma 1 with ψ, $\mathscr{D}$ and α replaced by $4^{-n}\varphi(\frac{1}{2}\eta(2r/27))$, $\mathscr{C}$, r_0 and κ, respectively. Thus we start by showing that the assumptions of (ii) in Lemma 1 are satisfied.

Let $x \in E$ and $r \in]0, r_0[$. Then we can find $K \in \mathscr{K}$ such that $K \subset B[x, 2r/27]$ and $|K| \geqq \varphi(2r/27)$. Choose $C_0 \in \mathscr{C}$ such that $|C_0| \geqq \frac{1}{2}|K|$ and

$$\emptyset \neq B[K, \operatorname{diam}(K)/4] \cap C_0 \subset B[x, r/9] \cap C_0.$$

If $C_0 \subset B[x, r]$, then $|C_0| \geqq \psi(r)$, and the required property is verified.

If $C_0 \setminus B[x, r] \neq \emptyset$, we either get a ball $B_0 \subset B[x, 2r/27]$ of measure at least $\varphi(\eta(2r/27)/2)$, such that $|C_0 \cap B_0| \geqq \kappa |B_0|$, or a point y such that $B[y, \eta(2r/27)] \subset B[x, 2r/27] \setminus C_0$.

In the former case the required property is obvious, so consider the latter case. Since $|B[y, \frac{1}{4}\eta(2r/27)]| \geqq \psi(r)$, the required property also holds if $B[y, \frac{1}{4}\eta(2r/27)] \cap E = \emptyset$. We may therefore assume that we can find $z \in B[y, \frac{1}{4}\eta(2r/27)] \cap E$. Then there exists a $K_1 \in \mathscr{K}$, such that $K_1 \subset B[z, \frac{1}{2}\eta(2r/27)]$ and $|K_1| \geqq \varphi(\frac{1}{2}\eta(2r/27))$. Choose $C_1 \in \mathscr{C}$, such that $B[K_1, \frac{1}{4}\operatorname{diam}(\mathrm{K}_1)] \cap \mathrm{C}_1 \neq \emptyset$ and $|C_1| \geqq \frac{1}{2}|K_1|$. Since $B[K_1, \frac{1}{4}\operatorname{diam}(K_1)] \subset B[y, \eta(2r/27)]$, we get $B(y, \eta(2r/27)] \cap C_1 \neq \emptyset$. This implies that $C_1 \neq C_0$ and $C_1 \cap B[x, r/9] \neq \emptyset$. We note that $C \cap B[x, r/9] \neq \emptyset$ and $C \setminus B[x, r] \neq \emptyset$ imply that $B[C, \frac{1}{4}\operatorname{diam}(C)] \supset B[x, r/9]$. Since $C_0 \cap B[x, r/9] \neq \emptyset$, we infer that $C_1 \cap B[C_0, \frac{1}{4}\operatorname{diam}(C_0)] \neq \emptyset$. By (12) this implies that $C_0 \cap B[C_1, \frac{1}{4}\operatorname{diam}(C_1)] = \emptyset$, and the argument above shows that $C_1 \subset B[x, r]$. Hence, we have shewn that the assumptions of (ii) in Lemma 1 are fulfilled.

By applying (ii) in Lemma 1 we finally get $|E| = 0$.

PROOF OF THEOREM 1, SUFFICIENCY. Let $\kappa, \eta > 0$ be constants satisfying (1), and let $\varphi \in \Phi^n$ satisfy (2). Then every point $x \in \mathsf{R}^n$ satisfies (11) with $\mathscr{K} = \mathscr{K}^n_{\eta,\kappa}$.

We define $\eta(r) = \eta \cdot r$. Then obviously, $\int_0 \varphi(\frac{1}{2}\eta(r))r^{-n-1}\,dr = +\infty$. Furthermore, for each $K \in \mathscr{K}^n_{\eta,\kappa}$ and every ball $B[x,r]$, either $|B[x,r] \cap K| \geqq \kappa\,|B[x,r]|$, or there exists a ball $B \subseteq B[x,r]$ such that $B \cap K = \emptyset$ and $\operatorname{diam}(B) \geqq \eta \cdot \operatorname{diam}(B[x,r]) = 2\eta r$. Thus, $\mathscr{K}^n_{\eta,\kappa}$ satisfies (i) and (ii) of Theorem 2, and Theorem 1 follows from Theorem 2.

In Section 2 we introduced the classes $\mathscr{K}^n_{\eta,\kappa}$, where η: $]0, +\infty[\,\rightarrow\,]0,1]$ is a function and $\kappa \in]0,1]$ is a constant. Let η^* be given by (6), i.e. $\eta^*(d) = d \cdot \eta(d)$. Then $\mathscr{K}^n_{\eta,\kappa}$ denotes the class of compact sets $K \subset \mathsf{R}^n$, such that for every closed ball B,

$$|B \cap K| \geqq \kappa\,|B| \quad \text{or} \quad \operatorname{diam}(\ker(B \backslash K)) \geqq \eta^*(\operatorname{diam}(B)).$$

THEOREM 3. *Assume that $\mathscr{K}^n_{\eta,\kappa}$ contains all closed balls where η is a nondecreasing function and κ a constant. Let $\varphi \in \Phi^n$. If*

$$\int_0^1 \frac{\varphi(r\eta(r))}{r^n}\frac{dr}{r} = \int_0^1 \frac{\varphi(\eta^*(r))}{r^n}\frac{dr}{r} = +\infty,$$

then the Vitali system $\mathscr{V}^n[\varphi, \mathscr{K}^n_{\eta,\kappa}]$ has the packing property.

PROOF. Just repeat the proof of Theorem 1 with the constant η replaced by the function $\eta(r)$.

As to the kind of density theorems and differentiation theorems one can derive as immediate corollaries from Theorem 1 and Theorem 3, we refer to [4], [7].

5. A Nullset Criterion.

We shall, by applying Theorem 1, obtain a sufficient criterion for a set $C \subseteq \mathsf{R}^n$, $n \geqq 2$, to be a nullset.

Let ϱ be a nondecreasing positive function on $]0, +\infty[$. Let ℓ be an oriented halfline in R^n with $\mathcal{O}$ as endpoint and let $R > 0$. Consider the set $T = T(\varrho, \ell, R) \subset \mathsf{R}^n$ consisting of all points $P \in \mathsf{R}^n$ for which the orthogonal projection P' of P onto the line determined by ℓ satisfies the properties

$$P' \in \ell, \quad \|\mathcal{O}P'\|_2 \leqq R, \quad \|P'P\|_2 \leqq \varrho(\|\mathcal{O}P'\|_2),$$

where $\|\cdot\|_2$ denotes the Euclidean norm. Such a set is called a *ϱ-trumpet*. The point $\mathcal{O}$ is the *vertex* of the ϱ-trumpet. We refer to ϱ as the *shape function*. The class of all ϱ-trumpets in R^n is denoted by $\mathscr{T}^n_\varrho$.

LEMMA 2. *There exist positive constants η and κ such that $\mathscr{T}^n_\varrho \subseteq \mathscr{K}^n_{\eta,\kappa}$ for any shape function ϱ.*

PROOF. Let $T = T(\varrho, \ell, R) \in \mathscr{T}_\varrho^n$ and consider any ball B. There exists a cube $Q \subseteq B$ such that ℓ is parallel to one of the edges in Q and such that $|Q| \geqq a|B|$, where a is a positive constant depending only on n and the metric. Divide Q into 3^n subcubes of the same size. An easy geometrical consideration shows that if all the 3^n subcubes of Q intersect T, then the middle subcube is entirely contained in T. Hence, there exists at least one of these subcubes Q', such that either $Q' \subseteq T$ or $T \cap Q' = \emptyset$. Now, Q' contains a ball $\tilde{B}$ such that $|\tilde{B}| \geqq b|Q'|$ for some positive constant b depending only on n and the metric, so $|\tilde{B}| \geqq ab \cdot 3^{-n}|B|$, proving the lemma for $\kappa = \eta^n = ab \cdot 3^{-n}$.

THEOREM 4. *Let ϱ be a shape function such that*

$$\int_0^1 \left(\frac{\varrho(r)}{r}\right)^{n-1} \frac{dr}{r} = +\infty. \tag{13}$$

If a set $C \subseteq \mathsf{R}^n$ has the property that to every $x \in C$ there exists a ϱ-trumpet with vertex x and no other points in common with C, then C is a nullset.

PROOF. We just outline the essential argument. Assume for simplicity that the metric is induced by the maximum norm. Let T_x be a ϱ-trumpet with vertex $x \in C$ and no other point in common with C, and let $\mathscr{S}$ be the family of all such ϱ-trumpets. Assume, as we may, that $\varrho(r) \leqq r$ for all r. Let δ be sufficiently small and assume for simplicity that the axis of T is parallel to an edge in $B[x, \delta]$. Then $T \cap B[x, \delta]$ is a ϱ-trumpet of measure

$$\varphi(\delta) = c \int_0^\delta \varrho(s)^{n-1}\, ds, \tag{14}$$

where c only depends on n. Then $(C, \mathscr{S}) \in \mathscr{V}^n[\varphi, \eta, \kappa]$ with η and κ as in Lemma 2 and φ given by (14). By Theorem 1, it only remains to verify (2). This is an easy exercise because of (13).

REMARKS. If the trumpets are ordinary cones corresponding to a shape function of the form $\varrho(\delta) = c \cdot \delta$, Theorem 4 gives a nullset criterion which also follows immediately from Lebesgue's density theorem.

If C is any curve in R^2 without double points, and $|C| > 0$, and ϱ is any shape function such that for every $x \in C$ one can find a ϱ-trumpet T_x of vertex x and no other point in common with C, then we get by contraposition of Theorem 4,

$$\int_0^1 \frac{\varrho(r)}{r} \frac{dr}{r} < +\infty,$$

which shows that the family of all ϱ-trumpets with only their vertices in common with C must have a very "thin" shape near their vertices.

If we had applied the weaker Theorem 2 of [4] instead of Theorem 1 we would have obtained a weaker criterion with (13) replaced by

$$\int_0^1 \left(\frac{\varrho(r)}{r}\right)^n \frac{dr}{r} = +\infty.$$

We believe that condition (13) is close to the best possible condition for a nullset criterion of the type considered.

6. A Cantor-set Construction in R.

For simplicity we restrict our attention to the one-dimensional case $n = 1$. Let

$$2^{(\mathsf{N})} = \bigcup_{n=0}^{+\infty} 2^n$$

denote the set of finite multi-indices of 0's and 1's including the empty multi-index. To a given sequence $\mathbf{a} = (a_n)_{n \geqq 0}$ with $0 \leqq a_n < 1$, we construct two systems (*Hausdorff schemes*) of intervals $C(\varepsilon)$ and $J(\varepsilon)$, $\varepsilon \in 2^{(\mathsf{N})}$, in $[0, 1]$ according to the following rules:

1. $C(\emptyset) = [0, 1]$.
2. $J(\varepsilon_1, \ldots, \varepsilon_n)$ is the open middle subinterval of $C(\varepsilon_1, \ldots, \varepsilon_n)$ of length

$$|J(\varepsilon_1, \ldots, \varepsilon_n)| = a_n \cdot |C(\varepsilon_1, \ldots, \varepsilon_n)|,$$

3. $C(\varepsilon_1, \ldots, \varepsilon_n, 0)$ is the left component, and $C(\varepsilon_1, \ldots, \varepsilon_n, 1)$ is the right component of the set $C(\varepsilon_1, \ldots, \varepsilon_n) \backslash J(\varepsilon_1, \ldots, \varepsilon_n)$.

We put

$$C_n = \bigcup_{\varepsilon \in 2^n} C(\varepsilon), \quad n \geqq 0,$$

and

$$C = C_{\mathbf{a}} = \bigcap_{n=0}^{+\infty} C_n.$$

In case of $a_n = 0$ it is understood that $J(\varepsilon) = \emptyset$, and $C(\varepsilon 0)$ is the left half and $C(\varepsilon 1)$ is the right half of $C(\varepsilon)$.

LEMMA 3. *Let K be a compact subset of* R, *denote by J the convex hull of K and put $\ell = |J|$. Assume that there are only finitely many components of $J \backslash K$ and denote by Δ the minimal length of these components.*

Let $\kappa \leqq 1/3$ and let $\eta\colon]0, +\infty[\to]0, 1/3]$ be a function such that η^* defined by (6) is never decreasing. If $\eta^*(3\ell)) \leqq \Delta$, then $K \in \mathscr{K}^1_{\eta,\kappa}$.

PROOF. Let I be a compact interval, and decompose I into three subintervals,

$I = I_1 \cup I_2 \cup I_3$, of equal length (with I_2 as the central interval). If one of the subintervals is disjoint with K, then (5) holds with B replaced by I. In the contrary case, there must exist $k_1 \in K \cap I_1$ and $k_3 \in K \cap I_3$. Thus $|I| \leqq 3\ell$. If k_1 and k_3 belong to the same component of K, then $I_2 \subseteq K$, and hence (3) holds with B replaced by I. If k_1 and k_3 belong to different components of K, then I must contain a component of $J \setminus K$ which has length at least Δ. As

$$\Delta \geqq \eta^*(3\ell) \geqq \eta^*(|I|),$$

we see that (7) holds in this case.

LEMMA 4. *Assume that η^* and φ are nondecreasing, that $\eta^*(r) \leqq r/3$ for all r and that $\kappa \leqq 1/3$. Assume that for all $(\varepsilon_n) \in 2^{(\mathrm{N})}$ and all $n \geqq 1$ the following inequality holds*

$$\max_{m \geqq n} |C(\varepsilon_1, \ldots, \varepsilon_m) \setminus C_{f(m)}| \geqq \varphi(|C(\varepsilon_1, \ldots, \varepsilon_{n-1})|), \tag{15}$$

where $f(n)$ denotes the largest integer such that

$$\eta^*(3 \cdot |C(\varepsilon_1, \ldots, \varepsilon_n)|) \leqq |C[\varepsilon_1, \ldots, \varepsilon_{f(n)}]|. \tag{16}$$

Then C is $\mathscr{V}^1[\varphi, \eta, \kappa]$-meagre.

PROOF. The common value of $|C(\varepsilon_1, \ldots, \varepsilon_n)|$ for all $(\varepsilon_1, \ldots, \varepsilon_n) \in 2^n$ is denoted by δ_n. Given $x \in C$ and $\delta > 0$, determine $(\varepsilon_n) \in 2^{(\mathrm{N})}$ and $n \geqq 1$ such that $x \in \bigcap_{n=1}^{+\infty} C(\varepsilon_1, \ldots, \varepsilon_n)$ and $\delta_n \leqq \delta < \delta_{n-1}$. Put

$$K_m = C(\varepsilon_1, \ldots, \varepsilon_m) \setminus C_{f(m)}, \quad m \geqq n.$$

Then $K_m \subseteq B[x, \delta] \setminus C$ and $K_m \in \mathscr{K}^1_{\eta, \kappa}$ for all $m \geqq n$. This follows from (16) and Lemma 3. Finally, (15) shows that $|K_m| \geqq \varphi(\delta_{n-1}) \geqq \varphi(\delta)$ for some $m \geqq n$.

THEOREM 5. *Let $\kappa \leqq 1/3$ and let η^* be nondecreasing. If $\mathscr{V}^1[\varphi, \eta, \kappa]$ has the packing property for every $\varphi \in \Phi^1$ which satisfies the condition*

$$\int_0^1 \frac{\varphi(r)}{r} \frac{dr}{r} = \int_0^1 \varphi(r) r^{-2}\, dr = +\infty, \tag{17}$$

or, if just every $\mathscr{V}^1[\varphi, \eta, \kappa)$-meagre set with φ of this type is a nullset, then there exists a constant $\eta_0 > 0$ such that $\eta^*(\delta) \geqq \eta_0 \cdot \delta$ for all $0 < \delta \leqq 1$.

PROOF. Choose $\varepsilon > 0$, such that

$$\tfrac{1}{2} \sum_{\nu=0}^{+\infty} \alpha_\nu \leqq 1 - \prod_{\nu=0}^{+\infty} (1 - \alpha_\nu) \leqq 2 \sum_{\nu=0}^{+\infty} \alpha_\nu \tag{18}$$

for every sequence $(\alpha_\nu)_{\nu \geqq 0}$ of nonnegative numbers with $\sum_{\nu=0}^{+\infty} \alpha_\nu \leqq \varepsilon$. Then choose $\beta_\nu \geqq 0$, $\nu \geqq 0$, such that $\sum_{\nu=0}^{+\infty} \beta_\nu \leqq \varepsilon$ and $\sum_{\nu=0}^{+\infty} \nu\beta_\nu = +\infty$. Put

$$p = \prod_{\nu=0}^{+\infty} (1 - \beta_\nu).$$

Then $p > 0$. For $n \geqq 0$ let $g(n)$ be the largest integer, for which

$$\eta^*(3 \cdot 2^{-n}) \leqq \tfrac{1}{2} \cdot p \cdot 2^{-g(n)}. \tag{19}$$

Assume that the conclusion of the theorem does not hold. Then we can find an increasing sequence $(n_k)_{k \geqq 0}$ of positive integers such that

$$g(n_k) - n_k > k, \quad k \geqq 0, \tag{20}$$

and such that the sequence $(g(n_k))_{k \geqq 0}$ is also increasing.

Define $\mathbf{a} = (a_\nu)_{\nu \geqq 0}$ by

$$a_{g(n_k)} = \beta_k,\ k \geqq 0, \quad \text{and} \quad a_\nu = 0 \text{ for } \nu \notin \{g(n_0), g(n_1), \ldots\},$$

and carry out the Cantor construction based on this sequence.

For $n \geqq 1$ put

$$\varphi_n = \tfrac{1}{2} \cdot p \cdot 2^{-(n+1)} \cdot \sum_{n+1 \leqq \nu \leqq g(n+1)} a_\nu,$$

and define $\varphi \in \Phi^1$ by

$$\varphi(\delta) = \sup_{m \geqq n} \varphi_m \quad \text{for} \quad 2^{-n} \leqq \delta < 2^{-(n-1)}, \quad n \geqq 1.$$

We claim that $C = C_{\mathbf{a}}$ is $\mathscr{V}^1[\varphi, \eta, \kappa]$-meagre, that φ satisfies (17) and that $|C| > 0$. The latter is obvious since $|C| = p$. To verify (17), note that

$$\begin{aligned} \sum_{n=0}^{+\infty} \frac{\varphi(2^{-n})}{2^{-n}} &\geqq \sum_{n=0}^{+\infty} \frac{\varphi_n}{2^{-n}} = \frac{p}{4} \sum_{n=0}^{+\infty} \sum_{n+1 \leqq \nu \leqq g(n+1)} a_\nu \\ &\geqq \frac{p}{4} \sum_{k=0}^{+\infty} \sum \{a_{g(n_k)} \mid n+1 \leqq g(n_k) \leqq g(n+1)\} \\ &\geqq \frac{p}{4} \sum_{k=0}^{+\infty} k\beta_k = -\infty, \end{aligned}$$

which is equivalent to (17).

Finally, we prove that C is $\mathscr{V}^1[\varphi, \eta, \kappa]$-meagre. First note that for $(\varepsilon_n) \in 2^{(\mathrm{N})}$ and $n \geqq 1$,

$$\eta^*(3 \cdot |C(\varepsilon_1, \ldots, \varepsilon_n)|) \leqq \eta^*(3 \cdot 2^{-n}) \leqq \tfrac{1}{2} \cdot p \cdot 2^{-g(n)} \leqq |C(\varepsilon_1, \ldots, \varepsilon_{g(n)+1})|,$$

so $f(n) \geqq g(n) + 1$, where f is defined as in Lemma 4. Then (15) may be verified as follows:

$$\max_{m \geqq n} |C(\varepsilon_1, \ldots, \varepsilon_m) \backslash C_{f(m)}| \geqq \max_{m \geqq n} |C(\varepsilon_1, \ldots, \varepsilon_m) \backslash C_{g(m)+1}|$$

$$= \max_{m \geqq n} \left(1 - \prod_{v=m}^{g(m)} (1 - a_v)\right) \cdot |C(\varepsilon_1, \ldots, \varepsilon_m)| \geqq \max_{m \geqq n} \frac{p}{2} \sum_{v=m}^{g(m)} a_v \cdot 2^{-m}$$

$$= \max_{m \geqq n} \varphi_{m-1} = \varphi(2^{-(n-1)}) \geqq \varphi(|C(\varepsilon_1, \ldots, \varepsilon_{n-1})|).$$

It then follows from Lemma 4 that C is meagre.

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