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CONVERGENCE OF TIGHT ASYMPTOTIC MARTINGALES IN A BANACH SPACE

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Abstract.

In this paper we show that a L^1-bounded martingale $(X_n, F_n, n \geqq 1)$ taking values in a Banach space E converges almost surely iff a family (P_{X_n}) of distribution of (X_n) is tight. A L^1-bounded tight asymptotic martigale (amart) need not converge a.s., although it always converges in probability and, if E^* is separable, it also converges weakly (in the sense of weak convergence in a Banach space) with probability 1.

1. Introduction.

A classical problem in the theory of martingales is to give conditions which assure their almost sure (a.s.) convergence. In [5] and [7] one can find theorems solving this problem in terms of vector measures and the Radon-Nikodym derivatives. In [8] another approach has been presented: a L^1-bounded asymptotic martingale (X_n) taking values in a Banach space E converges a.s. in norm iff it is strongly tight, i.e. for every $\varepsilon > 0$ there exists a compact set K_ε such that $P\left(\bigcap_{n=1}^{\infty} [X_n \in K_\varepsilon]\right) > 1 - \varepsilon$. It is natural to pose a question: can we replace strong tightness by tightness of a family of distributions of (X_n)? In this paper we prove that it is, in general, false: a tight L^1-bounded asymptotic martingale in a Banach space converges in probability to a Bochner integrable r.v. X, moreover, if the dual space E^* is separable, it also converges weakly for almost every $\omega \in \Omega$, but convergence in norm need not hold even in a separable Hilbert space and if E^* is not separable, weak convergence with probability 1 need not hold. However, it can be proved that every tight L^1-bounded martingale taking values in a Banach space converges a.s.

2. Notation and definitions.

Let N denote a set of natural numbers, i.e. $N = \{1, 2, 3, \ldots\}$. Let (Ω, A, P) be a probability space. We can always assume that it is complete, i.e. for every $B \in A$

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such that $P(B) = 0$ and for every $C \subset B$ we have $C \in A$ [3]. Let $(F_n, n \geqq 1)$ be an increasing sequence of sub-σ-fields of A (i.e. $F_n \subset F_{n+1} \subset A$ for every $n \in N$). Let $F_\infty = \sigma\left(\bigcup_{n=1}^{\infty} F_n\right)$. A mapping $\tau : \Omega \to N \cup \{\infty\}$ will be called a stopping time with respect to (F_n) iff for every $n \in N$ the event $[\tau = n]$ belongs to F_n. A stopping time τ will be called bounded iff there exists $M \in N$ such that $P[\tau \leqq M] = 1$. A set of all bounded stopping times will be denoted by T. Let E be a Banach space with a norm $\| \ \|$. Let E^* be its dual and let $\| \ \|_*$ be a norm in E^*. The set of all Bochner integrable r.v.s with values in E (more precisely, the set of all their equivalence classes) will be denoted by L^1_E or simply by L^1, where it does not lead to confusion. Let F be a sub-σ-field of A. Definitions and basic properties of the Bochner integral EY and the conditional expectation $E^F Y$ of a r.v. $Y \in L^1_E$ can be found e.g. in [9].

Throughout this paper, let $(X_n, F_n, n \geqq 1)$ be an adapted sequence of Bochner integrable random variables with values in a Banach space E.

DEFINITION 1. A sequence $(X_n, F_n, n \geqq 1)$, will be called a *martingale* if, for every $n \in N$, $E^{F_n} X_{n+1} = X_n$ a.s.

DEFINITION 2 ([6]). A sequence $(X_n, F_n, n \geqq 1)$ is called an *asymptotic martingale* (amart) iff for every $\varepsilon > 0$ there exist $\tau_0 \in T$ such that for every $\tau, v \in T, \tau, v \geqq \tau_0$ a.s. we have

(1) $\|\int X_\tau dP - \int X_v dP\| < \varepsilon.$

Obviously, every martingale is an asymptotic martingale.

It is well known that every (strongly) measurable r.v. with values in E is essentially separable valued (see [5], theorem 2.1.2). Thus, considering a sequence (indexed by elements of N) of such r.v.s, we can always, without loss of generality, assume that they take values in a separable subspace of E. We shall use this fact, assuming in proof (without loss of generality in the statements of the results) that E is itself separable.

DEFINITION 3. We shall say that a sequence $(X_n, n \geqq 1)$ of E-valued r.v.s is L^1_E (or simply L^1)-*bounded* iff $\sup_n E\|X_n\| < \infty$ and that is *strongly tight* iff for every $\varepsilon > 0$ there exists a compact subset K_ε of E such that

(2) $P\left(\bigcap_{n=1}^{\infty} [X_n \in K_\varepsilon]\right) > 1 - \varepsilon.$

Let us recall that an indexed family $(\mu_t, t \in T)$ of probability measures defined on the σ-field $B(E)$ of the Borel subsets of E is called tight iff for every $\varepsilon > 0$ there exists a compact set $K \subset E$ such that for every $t \in T$ we have $\mu_t(K) > 1 - \varepsilon$. Obviously if a sequence $(X_n, n \in N)$ is strongly tight, the family of their distributions $(\mu_{X_n}: n \in N)$ is tight, but the reverse implication does not hold, e.g. take a sequence of i.i.d. real r.v.s having a standard normal distribution.

3. Main results.

In [8] the following theorem was proved.

THEOREM 1. *An L^1-bounded asymptotic martingale in a Banach space converges almost surely if and only if it is strongly tight.*

Now we shall investigate convergence of tight martigales in a Banach space. We begin with a following lemma.

LEMMA 1. *Let $(X_n, F_n, n \geqq 1)$ be a L^1-bounded asymptotic martingale taking values in a Banach space. If the family (P_{X_n}) of distributions of (X_n) is tight, (X_n) converges in law.*

PROOF. It is well known that a tight family of distributions in a complete and separable metric space is conditionally compact in the Prokhorov metric and that convergence in this metric is equivalent to convergence in law (c.f. e.g. [10]). Thus the closure of the family (P_{X_n}) is compact in the Prokhorov metric. Let us suppose that (X_n) does not convergence in law. In this case there exist subsequences (X_{m_k}), (X_{m_l}) and two distinct probabilitites measures P_1, P_2 such that $P_{X_{m_k}} \xrightarrow{D} P_1$, $P_{X_{m_l}} \xrightarrow{D} P_2$.

Let us consider a countable family $\Gamma \subset E^*$ which separates points of E, i.e. such that for every $x \in E$ $x = 0$ iff $x^*x = 0$ for all $x^* \in E^*$ (for example, a family $\Gamma_0 = \{x^*_{j,k} \in E^*: \|x^*_{j,k}\|_* = 1, x^*_{j,k}(e_j - e_k) = \|e_j - e_k\|, j, k \in N, j \neq k\}$, where (e_n) is a countable dense subset of E). By theorem 2.1 [10] the Borel σ-field $B(X)$ coincides with a cyllindrical σ-field $C(X, \Gamma)$ generated by cyllindrical sets $C = \{x \in E: (x^*_{k_1}(x), \ldots x^*_{k_n}(x)) \in C'\}$, where $n, k_1, \ldots k_n \in N$, $x^*_{k_1}, \ldots x^*_{k_n} \in \Gamma$ and $C' \in B(R^n)$. By the Dynkin theorem [3] there exist a cyllindrical set C such that $P_1(C) \neq P_2(C)$. Applying the Dynkin theorem once again to a π-system of finite intersections of open balls in R^n and a λ-system R^n we find that there exists a cyllinder $G = \{x \in E: (x^*_{k_1}(x), \ldots x^*_{k_n}(x)) \in G'\}$, where G' is a finite intersection of open balls in R^n and $P_1(G) \neq P_2(G)$.

A random vector $((x^*_{k_1}(X_m), \ldots x^*_{k_n}(X_m)),\ m \geqq 1)$ converges a.s. (so it also converges in law to some distribution P' on R^n), because for every $x^* \in E^*$ a sequence $x^*(X_m)$ is a L^1-bounded real asymptotic martingale and thus it converges a.s. [1]. Let $x_0 \in G'$. Let us denote $\lambda \cdot G' = \{x_0 + \lambda(x - x_0): x \in G'\}$. It is easy to see that if $0 \leqq \lambda_1 \leqq \lambda_2$, then $\lambda_1 \cdot G' \subset \lambda_2 \cdot G'$ and that if we take an increasing sequence $\lambda_n \to 1$ as $n \to \infty$, then $\bigcup_{n=1}^{\infty} \lambda_n \cdot G' = G'$. Let us denote $\lambda \cdot G = (x^*_{k_1}, \ldots x^*_{k_n})^{-1}(\lambda \cdot G')$. Using the axion of continuity we can choose such $\lambda < 1$ that $P_1[\lambda \cdot G] \neq P_2[\lambda \cdot G]$, and $P_1[(x^*_{k_1},\dots x^*_{k_n})^{-1}(\delta(\lambda\cdot G'))] = P_2[(x^*_{k_1},\dots x^*_{k_n})^{-1}(\delta(\lambda\cdot G'))] = P'[\delta(\lambda\cdot G)] = 0$, where δC denotes the boundary of C. Thus we have (see [10], theorem 3.5)

$$P[X_{m_k}\in\lambda\cdot G] = P[(x^*_{k_1},\dots x^*_{k_n})(X_{m_k})\in\lambda\cdot G'] \to P_1[\lambda\cdot G]$$

and

$$P[X_{m_l}\in\lambda\cdot G] = P[(x^*_{k_1},\dots x^*_{k_n})(X_{m_l})\in\lambda\cdot G'] \to P_2[\lambda\cdot G],$$

But $P[(x^*_{k_1},\dots x^*_{k_n})(X_m)\in\lambda\cdot G'] \to P'[\lambda\cdot G']$, so $P_1[\lambda\cdot G] = P_2[\lambda\cdot G]$. This contradiction completes the proof.

LEMMA 2. *Let $(X_n, F_n, n \geqq 1)$ be a tight L^1-bounded asymptotic martingale. There exist an F_∞-measurable, Bochner integrable* r.v. *X such that (X_n) converges to X in law and scalarly.*

PROOF. By hypothesis, for every $m\in N$ there exists a compact set $K_{\frac{1}{m}} \subset E$ such that for every natural n $P[X_n\in K_{\frac{1}{m}}] \geqq 1 - \dfrac{1}{m}$.

We can assume that for $m_1 \leqq m_2$ $K_{\frac{1}{m_1}} \subset K_{\frac{1}{m_2}}$. Let $A_m = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}[X_k\in K_{\frac{1}{m}}]$ denote an event that infinitely many of points $X_n(\omega)$ belong to $K_{\frac{1}{m}}$. Obviously $P(A_m) \geqq 1 - \dfrac{1}{m}$.

Let $\omega\in A_m$. By definition of A_m and compactness of $K_{\frac{1}{m}}$ there exist a subsequence (n_k) and a point $x_1 = X(\omega,(n_k),K_{\frac{1}{m}})$ such that $X_{n_k}\in K_{\frac{1}{m}}$ and $X_{n_k}(\omega)\to x_1$ as $k\to\infty$. Let Γ_0 be the same as in lemma 1 and let C be a set of such ω that for every $x^*\in\Gamma_0$ a sequence $x^*X_n(\omega)$ converges. As it was mentioned in the proof of lemma 1, for every $x^*\in E^*$ x^*X_n is a L^1-bounded real amart and thus converges a.s., so $P(C) = 1$. Let $B_m = A_m\cap C$. Obviously $P(B_m) = P(A_m)$. Now we shall show that for every $\omega\in B_m$ and every subsequence (η_l) such that $X_{n_l}(\omega)\in K_{\frac{1}{m}}$ and $X_{n_l}(\omega)\to X(\omega,(\eta_l),K_{\frac{1}{m}}) = x_2$ as $l\to\infty$ $x_1 = x_2$, so the limit depends only on ω and $K_{\frac{1}{m}}$. If $\|x_1 - x_2\| = \varepsilon > 0$, there exist e_i, e_j belonging to the countable set dense in E which has been used in the definition of Γ_0 such that $\|x_1 - e_i\| < \dfrac{\varepsilon}{5}$ and $\|x_2 - e_j\| < \dfrac{\varepsilon}{5}$. Thus, for k, l sufficiently large, $|x^*_{i,j}X_{n_k}(\omega) - x^*_{i,j}e_i| < \dfrac{\varepsilon}{5}$, $|x^*_{i,j}X_{n_l}(\omega) - x^*_{i,j}e_j| < \dfrac{\varepsilon}{5}$ and, by definition of $x^*_{i,j}$, $|x^*_{i,j}(e_i - e_j)| = \|e_i - e_j\| > \dfrac{3\varepsilon}{5}$. By the triangle inequality we have $|x^*_{i,j}X_{n_k}(\omega) - x^*_{i,j}X_{n_l}(\omega)| > \dfrac{\varepsilon}{5}$ for k, l sufficiently large, so $\omega\notin C$, contradiction. Thus $x_1 = x_2$.

It is obvious that if $m_1 < m_2$, then $B_{m_1}\subset B_{m_2}$ and if points $X_{n_k}(\omega)$ belong to

$K_{\frac{1}{m_1}}$ and converge, they also belong to $K_{\frac{1}{m_2}}$ and converge in this set to the same limit. Thus for $\omega \in B_{m_1}$ $X(\omega, K_{\frac{1}{m_1}}) = X(\omega, K_{\frac{1}{m_2}}) = X(\omega)$. Thus we have defined a mapping $X: B \to E$, where $B = \bigcup_{m=1}^{\infty} B_m$. Obviously $P(B) = 1$. Put $X(\omega) = 0$ for $\omega \notin B$. By definition, X is almost separably (even separably!) valued. By the definition of B, for every $\omega \in B$ there exists a subsequence (n_k) such that $X_{n_k}(\omega) \to X(\omega)$, so for every $x^* \in E^*$ and $\omega \in B$ such that an L^1-bounded real amart $x^* X_n(\omega)$ converges (a set of such ω has probability 1) we have $x^* X_n(\omega) \to x^* X(\omega)$. By this fact and completeness of the probability space, $x^* X$ is F_∞-measurable for every $x^* \in E^*$, so, by the Pettis Measurability Criterion (see e.g. [5]), X is F_∞-(strongly) measurable.

It remains to show that $X_n \xrightarrow{D} X$ and that X is integrable. Let us remark that for every $x^*_{k_1}, \ldots x^*_{k_n} \in \Gamma_0$ a random vector $(x^*_{k_1}, \ldots x^*_{k_n})(X_n)$ converges almost surely to $(x^*_{k_1}, \ldots x^*_{k_n})(X)$, so, by reasoning similar to that given in lemma 1, a limit distribution of (X_n) (which exists by lemma 1) and the distribution of X coincide. Now let $a > 0$. We have

$$\infty > \sup_n E\,\|X_n\| \geqq \int_\Omega \min(\|X_n\|, a)\,dP = \int_E \min(\|x\|, a)\,dP_{X_n} \to$$

$$\int_E \min(\|x\|, a)\,dP_X = \int_\Omega \min(\|X\|, a)\,dP,$$

letting $a \to \infty$ we obtain $E\|X\| \leqq \sup_n E\,\|X_n\|$. The proof is complete.

We are now ready to prove our main result.

THEOREM 2. *Let $(X_n, F_n, n \geqq 1)$ be a L^1-bounded tight asymptotic martingale taking values in a Banach space E. There exists an integrable* r.v. *X such that (X_n) converges to X in probability. Moreover, if the dual space E^* is separable, then for almost all $\omega \in \Omega$ a sequence $(X_n(\omega))$ converges weakly to $X(\omega)$.*

PROOF. Let X be a E-valued random variable fulfilling the conditions given in lemma 2. It is well known that $E^{F_n}X \to E^{F_\infty}X = X$ almost surely and in L^1 [9]. Let $Y_n = X_n - E^{F_n}X$. (Y_n) is a L^1-bounded asymptotic martingale with respect to (F_n). It is easy to see that if K_1, K_2 are compact subsets of E, then $K_1 - K_2 = \{x_1 - x_2 : x_1 \in K_1, x_2 \in K_2\}$ is compact, so (Y_n) is tight. Thus, by lemma 2, (Y_n) converges in law, the method of construction of a limit given in its proof assures that $Y_n \xrightarrow{D} X - X = 0$. Thus $Y_n \xrightarrow{P} 0$, because convergence in law to a constant is equivalent to convergence in probability to the same limit. Thus $X_n = Y_n + E^{F_n}X \xrightarrow{P} X$.

The second statement of the theorem follows from the fact that, by lemma 2, for every $x^* \in X^*$ $x^*X_n \xrightarrow{a.s.} x^*X$ and that, by $X_n \xrightarrow{P} X$, $P[\sup \|X_n\| < \infty] = 1$. The remainder of the proof is the same as the end of the proof of the theorem 5.2a) from [6].

EXAMPLE 1. *Almost sure convergence need not hold even in a separable Hilbert space.* Let $(\Omega, A, P) = ([0,1], B([0,1]), \mu)$, where μ is the Lebesgue measure. Let $E = l^2$ and let $(e_n^i, n \in N, i = 1, \dots 2^n)$ be a standard orthonormal basis in l^2 in some order. Let, for every $n \in N$, A_n^i, $i = 1, \dots 2^n$, be such sets that $A_n^i \cap A_n^j = \emptyset$ for $i \neq j$, $P(A_n^i) = \frac{1}{2^n}$, $i = 1, \dots 2^n$, and $\bigcup_{i=1}^{2^n} A_n^i = \Omega$. Let $Y_n^i = e_n^i I_{A_n^i}$, $n \in N$, $i = 1, \dots 2^n$. Put $X_{2^n - 2 + i} = Y_n^i$ and let $F_n = B([0,1])$ for every natural n and $i = 1, \dots 2^n$. We shall show that $(X_n, F_n, n \geqq 1)$ is a L^1-bounded asymptotic martingale.

It is obvious that $\|X_n\| \leqq 1$ a.s., so $\sup_n E\|X_n\| < \infty$. Let $\tau \in T$ and let

$$I_\tau = \{(n,i) : n \in N, i \in \{1, \dots 2^n\}, P[\tau = 2^n - 2 + i] > 0\}.$$

It is easy to see that if $\tau \geqq 2^{k+1} - 2$ a.s., then for every $(n,i) \in I_\tau$ we have $k \leqq n$. Let $B_n^i = A_n^i \cap [\tau = 2^n - 2 + i]$, $n \in N$, $i \in \{1, \dots 2^n\}$. Let us remark that $B_n^i \cap B_m^j = \emptyset$ for $m \neq n$ or $i \neq j$ and that obviously $P(B_n^i) \leqq P(A_n^i)$. Therefore $X_\tau = \sum_{(n,i) \in I_\tau} Y_n^i I_{[\tau = 2^n - 2 + i]} = \sum_{(n,i) \in I_\tau} e_n^i I_{B_n^i}$, so $EX_\tau = \sum_{(n,i) \in I_\tau} e_n^i P(B_n^i)$ and

$$\|EX_\tau\|^2 = \sum_{(n,i) \in I_\tau} P(B_n^i)^2 \leqq \frac{1}{2^k} \sum_{(n,i) \in I_\tau} P(B_n^i) \leqq \frac{1}{2^k}.$$

Thus (X_n) is an amart.

It is clear that for $n \geqq n_0$ $P[Y_n^i = 0] = 1 - \frac{1}{2^n} \geqq 1 - \frac{1}{2^{n_0}}$, so if $K_{n_0} = \{0, e_m^i, m < n_0, i = 1, \dots 2^m\}$, (it is finite, hence compact), K_{n_0} contains all the values taken by $Y_m^i, m < n_0, i = 1, \dots 2^m$ and thus $P[Y_n^i \in K_{n_0}] \geqq 1 - \frac{1}{2^{n_0}}$ for all $n \in N$. We have proved that (X_n) is tight.

Let us consider an arbitrary $\omega \in \Omega$. For every n there exist $i_n, j_n \in \{1, \dots 2^n\}$ such that $\omega \in A_n^{i_n}$ and $\omega \notin A_n^{j_n}$. Obviously $\|Y_n^{i_n}(\omega)\| = 1$, $\|Y_n^{j_n}(\omega)\| = 0$. Thus a sequence $(X_n(\omega))$ does not convergence for any ω. By theorem 1 it is not strongly tight, moreover, it is easy to see that for every compact set $K \subset E$ $\bigcap_{n=1}^{\infty} [X_n \in K] = \emptyset$. Indeed, for every $\omega \in \Omega$ $Y_n^{i_n}(\omega) = e_n^{i_n}$ and a set $\{e_n^{i_n}, n \in N\}$ does not have a compact closure, because for $m \neq n$ $\|e_m^{i_m} - e_n^{i_n}\| = \sqrt{2}$.

Thus a tight L^1-bounded asymptotic martingale need not converge a.s. (or, equivalently, be strongly tight) even under some additional assumptions (separability of E^*, the Radon-Nikodym property of both E and E^*, etc.

EXAMPLE 2. *If E^* is not separable, weak convergence with probability* 1 *need not hold.* Let (Ω, A, P) and (X_n) be like in example 1, but now $E = l^1$. By the Schwarz inequality $\|x\|_1 \leqq (\|x\|_2)^{\frac{1}{2}}$, where $\| \ \|_1$ and $\| \ \|_2$ denote the l^1 and l^2-norm respectively. Thus X_n is again a tight L^1-bounded asymptotic martingale such that for every $\omega \in \Omega$ the sequence $(X_n(\omega))$ does not converge in norm. It is well known that in l^1 weak convergence is equivalent to convergence in norm [2], so it does not converge weakly, either. Let us remark that $(l^1)^* = l^\infty$ is not separable.

Now we shall show that every L^1-bounded tight martingale converges a.s.

THEOREM 3. *An L^1-bounded martingale in a Banach space converges almost surely if and only if it is tight.*

PROOF. It is obvious that every sequence of r.v.s which converges a.s. is tight. Conversely, if (X_n) is a tight L^1-bounded martingale in a Banach space, then, by lemma 2, there exist a r.v. X such that (X_n) converges to X scalarly. It is known (see [7], Proposition 5.3.21) that for L^1-bounded martingales scalar convergence and almost sure convergence are equivalent, so (X_n) converges to X almost surely.

COROLLARY 1. *A Banach space E has the Radon-Nikodym property iff every L^1-bounded martingale with values in E is tight.*

It is known that every real amart (X_n) has the so called "Riesz decomposition", i.e. it can be (uniquely) written as $X_n = Y_n + Z_n$, where (Y_n) is a martingale and (Z_n) is an amart which converges to 0 a.e. and in L^1 [6]. Corollary 2 and its proof show that it is, in general, not true even in a separable Hilbert space, although it has the Radon-Nikodym property. Thus the structure of asymptotic martingales in Banach spaces is more complicated than in the real case.

COROLLARY 2. *There are asymptotic martingales taking values in a separable Hilbert space which do not have the Riesz decomposition.*

PROOF. Consider the amart (X_n) constructed in example 1. Suppose that it has the Riesz decomposition $X_n = Y_n + Z_n$. Thus, by L^1-boundedness and tightness of (X_n) and (Z_n), (Y_n) is L^1-bounded and tight (compare the proof of theorem 2). Thus, by theorem 3, (Y_n) converges a.s., so (X_n) converges a.s. The obtained contradiction ends the proof.

This result can be compared with (5.2.27) and (5.2.29) from [7], where a slightly different definition of the Riesz decomposition was given.

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