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COHOMOLOGY GROUPS OF LOCALLY q-COMPLETE MORPHISMS WITH r-COMPLETE BASE

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To the memory of my Professor Martin Jurchescu

1. Introduction.

Vanishing theorems are important in complex analysis. One general way to obtain them was given by Andreotti and Grauert ([1]), where they showed that every q-complete complex space is cohomologically q-complete. (For the definitions, see below.)

Our main concern in this paper is to prove a variant of this theorem for families of q-complete spaces. We consider the following situation:

(♣) *Let $\pi : X \to S$ be a holomorphic map of complex spaces such that its fibres $\pi^{-1}(s), s \in S$, are q-complete. What can be said about the vanishing of the cohomology groups $H^i(X, \mathscr{F}), \mathscr{F} \in \mathrm{Coh}(X)$, for i in a suitable range?*

Simple examples show that there are holomorphic maps $\pi : X \to S$ of complex manifolds such that S and all the fibes of π are Stein, and, however, $H^{n-1}(X, \mathscr{O})$ does not vanish, where n is the complex dimension of X; e,g. $X = \mathbf{C}^n \setminus \{0\}, S = \mathbf{C}^{n-1}$, and π the projection onto the first $n-1$ coordinates. Therefore, to answer our question, we have to make additional assumptions on the dependence of the family of q-complete spaces on the base points. In this way we are lead to *locally q-complete morphins*, and, the vanishing theorem holds; viz. theorem 1 in §3, which says that X is cohomologically $(q+r)$-complete provided that π and S are locally q-complete and r-complete respectively. See also corollary 2 in §3.

As consequences (viz., theorem 2 and corollaries 3, 4, and 6), one gets vanishing theorems for the cohomology of locally q-complete open sets in r-complete spaces.

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Particular cases of our results were treated by various authors; Ballico ([2]), Bolondi ([3]), and Jennane ([6], [7], [8]).

Throughout this paper all complex spaces are assumed to be reduced and with countable topology.

2. Preliminaries.

Let D be an open subset of $\mathbb{C}^n$ and $f \in C^\infty(D, \mathbb{R})$. Let $z_1, \ldots, z_n$ be the complex coordinates of $\mathbb{C}^n$. For every point $w \in D$, the quadratic form

$$\mathbb{C}^n \ni \xi \mapsto L(f, w)(\xi) := \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}(w)\xi_i \bar{\xi}_j \in \mathbb{R},$$

is called the *Levi form* of f at w. We say that f is q-convex on D if its Levi form $L(f, w)$ has at least $n - q + 1$ positive eigenvalues for every $w \in D$.

Let X be a complex space. A function $\varphi : X \to \mathbb{R}, X$ a complex space, is said to be *q-convex* if for each point $x \in X$ there is an holomorphic imbedding $\iota : U \to \hat{U}, U \ni x, \hat{U} \subset \mathbb{C}^n$ open, and a q-convex function $\hat{\varphi} \in C^2(\hat{U}, \mathbb{R})$ which extends $\varphi|_U$.

We say that X is *q-complete* (resp. *q-convex*) if there exists a smooth exhaustion function $\varphi : X \to \mathbb{R}$ which is q-convex on $X \backslash K$ for some suitable compact subset K of X). The normalization is such that Stein spaces correspond to 1-complete spaces.

Finitely many q-convex functions $\varphi_1, \ldots, \varphi_k$, on X have *the same positivity directions* if for each point $x_0 \in X$ there are: an holomorphic imbedding $\iota : U \to \hat{U}, U \ni x_o, \hat{U} \subset \mathbb{C}^n$ open; q-convex extensions $\hat{\varphi}_j$ of $\varphi_j, j = 1, \ldots, k$; and a complex vector subspace E of $\mathbb{C}^n$ of dimension at least $n - q + 1$ such that all the Levi forms $L(\hat{\varphi}_1, z_o), \ldots, L(\hat{\varphi}_k, z_o), z_o = \iota(x_0)$ are positive definite when restricted to E.

The motivation for this notion is the following: Let φ_1 and φ_2 be q-convex with the same positivity directions. Then $\varphi_1 + \varphi_2$ is again q-convex and $\max(\varphi_1, \varphi_2)$ can be approximated in the C^o-topology by q-convex functions ([16]).

Definition. Let $\pi : X \to S$ be a holomorphic map between complex spaces. (We consider S as a space of parameters.)

(•) We say that π is: *q-convex* (resp. *q-complete*) if there exists a smooth function $\varphi : X \to \mathbb{R}$ and a real number $c_\sharp$ (resp. $c_\sharp = -\infty$) such that:

(i) φ is q convex on the open set $\{x \in X; \varphi(x) > c_\sharp\}$;

(ii) For every real number λ, the restriction of π from $\{x \in X; \varphi(x) \leq \lambda\}$ to S is proper.

We call φ the *exhaustion function of X* and $c_\sharp$ the *convexity bound.*

($\bullet$) We say that π is: *locally q-convex* (resp. *locally q-complete*) if every point $s \in S$ has an open neighborhood U such that the restriction of π from $\pi^{-1}(U)$ to U is q-convex (resp. q-complete).

REMARKS. 1) If $\pi : X \to S$ is locally q-convex (resp. locally q-complete), then its fibres $X_s := \pi^{-1}(s), s \in S$, are q-convex (resp. q-complete) complex spaces. In particular, if S is a point, the definitions of q-complete and q-convex spaces are regained.

2) One verifies readily that $\pi : X \to S$ is locally q-complete if, and only if, every point $s \in S$ has a neighborhood U such that $\pi^{-1}(U)$ is q-complete. (An analogous situation does not occur for q-convex mappings as the example 1 from below shows).

Moreover, in this case, if $V \subset U$ is an arbitrary Stein open set, then $\pi^{-1}(V)$ is again q-complete.

3) Also, it is easy to check that if π is q-complete and S is r-complete (resp. r-convex), then X is $(q+r-1)$-complete (resp. $(q+r-1)$-convex). (This does not hold for locally q-complete mappings as the example 2 in § 3 shows.)

4) Locally 1-complete morphisms are also called *locally Stein* morphisms ([6], [7]).

EXAMPLE 1. Let $A \subset \mathbf{C}^n$ be a closed submanifold of pure dimension $d \leq n-2$, and $\pi : X \to \mathbf{C}^n$ the blowing-up of $\mathbf{C}^n$ at A. Then π is 1-convex.

Indeed, if $A = \{f_1 = \ldots = f_m = 0\}$ for some holomorphic functions $f_1, \ldots, f_m$ on $\mathbf{C}^n$, we set $h : \mathbf{C}^n \to \mathbf{R}$ by $h(z) = (|f_1(z)|^2 + \ldots + |f_m(z)|^2) \exp(\| z \|^2)$. Then, $\varphi = h(\pi)$ together with $c_\sharp = 0$ as convexity bound show the 1-convexity of π. However, if $d > 0$, then for each open subset $U \subset \mathbf{C}$ with $U \cap A \neq \emptyset, \pi^{-1}(U)$ is not 1-convex. (In fact, not even $(n-d-1)$-convex.)

By extending the usual notion of Runge domains in Stein spaces, we say that an open subset D of a complex space X is *q-Runge* if for every compact set $K \subset D$ there is a q-convex exhaustion function $\varphi : X \to \mathbf{R}$ (which may depend on K) such that

$$K \subset \{x \in X; \varphi(x) < 0\} \Subset D.$$

(Note that X is q-complete if and only if the empty set is q-Runge in X.) With this definition, we reinterprete a result due to Andreotti and Grauert ([1]).

PROPOSITION 1. *Let D be a q-Runge domain in a q-complete complex space*

X. Then, for every coherent analytic sheaf $\mathscr{F}$ on X, $H^i(D,\mathscr{F})$ vanishes for $i \geq q$, and the restriction map $H^{q-1}(X,\mathscr{F}) \to H^{q-1}(D,\mathscr{F})$ has dense range for the natural topology.

A complex space X is said to be *cohomologically q-complete* (resp. *cohomologically q-convex*) if the cohomology groups $H^i(X,\mathscr{F}), \mathscr{F} \in \mathrm{Coh}(X)$ vanish (resp. are finite dimensional complex vector spaces) for every $i \geq q$.

An open subset D of a complex space X is said to be *locally q-complete* if for each point $x \in \partial D$ there exists an open neighborhood U of X such that $U \cap D$ is q-complete. Equivalently, the inclusion map $j : D \to X$ is locally q-complete.

In the sequel, topological vector spaces are such that its zero element has a countable base of open neighborhoods. For such a topological vector space E we denote by E_{sep} the separated topological space associated with E, namely; the quotient of E modulo the closure of its zero-element. The following result is evident.

LEMMA 1. *Let $u : E \to F$ be a continuous map of topological vector spaces. The following statements are equivalent one another:*

a) *u has the lifting property of sequences,* i.e., *for every sequence $\{f_n\} \subset u(E)$ with $f_n \to 0$ there is another sequence $\{e_n\} \subset u(E), e_n \to 0$ and* $u(e_n) = f_n, n \geq 1$.

b) *u is quasi-open,* i.e., *the induced map $u' : E \to u(E)$ is open where $u(E) \subset F$ is endowed with the trace topology which comes from F.*

By diagram chasing, the following is a consequence of the preceding lemma.

COROLLARY 1. *Assume we have a commutative diagram of topological vector spaces with exact rows*

$$\begin{array}{ccccccc} E & \to & F & \to & G & \to & 0 \\ a\downarrow & & b\downarrow & & c\downarrow & & \\ E' & \to & F' & \xrightarrow{v} & G' & & \end{array}$$

where a, b, c, v are continuous linear maps. Suppose a and c have dense range. Then b has dense range provided that v is quasi-open.

Let X be a complex space. A Stein open covering $\mathscr{U} = (U_i)_{i\in I}$ of X is said to be *a special covering of X* if $\mathscr{U}$ is a countable base of open subsets of X. If $D \subset X$ is open, we let $\mathscr{U}_{|D} := \{U \in \mathscr{U}; U \subseteq D\}$. Obviously, $\mathscr{U}_{|D}$ is a special covering of D.

Now, we let $\mathscr{F} \in \mathrm{Coh}(X)$. Since the spaces of Čech cochains $C^{p+1}(\mathscr{U},\mathscr{F})$, $p = 0, 1, \ldots,$ are Fréchet spaces, and the coboundary maps

$\delta = \delta^p : C^p(\mathcal{U},\mathcal{F}) \to C^{p+1}(\mathcal{U},\mathcal{F})$ are continuous, the spaces of cocycles $Z^p(\mathcal{U},\mathcal{F}) := \operatorname{Ker} \delta^p$ are also Fréchet spaces.

Thus we have a simple:

LEMMA 2. *For every non-negative integer p, the following statements are equivalent:*

a) *The restriction map* $H^p(X,\mathcal{F}) \to H^p(D,\mathcal{F})$ *has dense range.*

b) *The restriction map* $Z^p(\mathcal{U},\mathcal{F}) \to Z^p(\mathcal{U}_{|D},\mathcal{F})$ *has dense range for any special covering* $\mathcal{U}$ *of* X.

c) *Statement* b) *holds for one special covering* $\mathcal{V}$ *of* X.

The next lemma is probably well known, but since we did not found any reference in the literature we give it here.

LEMMA 3. *Let* X *be a complex space and* X_1, X_2 *be open subsets. Let also* $\mathcal{F} \in \operatorname{Coh}(X)$ *and* p *a non-negative integer such that* $H^p(X_1 \cap X_2,\mathcal{F})$ *is Hausdorff. Then the natural map which comes from the Mayer-Vietoris sequence*

$$H^p(X_1 \cup X_2,\mathcal{F}) \to H^p(X_1,\mathcal{F}) \oplus H^p(X_2,\mathcal{F})$$

is quasi-open. In particular, this holds if $H^p(X_1 \cap X_2,\mathcal{F})$ *vanishes.*

PROOF. Since the case $p = 0$ is clear, we may assume $p \geq 1$; and without any loss of generality, let $X = X_1 \cup X_2$. We break the proof into three steps.

Step 1. There is an special covering $\mathcal{U} = (U_i)_{i\in I}$ of X such that for the next three sets of indices $I_1 := \{i \in I; U_i \subset X_1\}, I_2 := \{i \in I; U_i \subset X_2\}$ and $I_{12} := \{i \in I; U_i \subset X_1 \cap X_2\}$, it holds: (♠) If $i \in I_1 \backslash I_{12}$ and $j \in I_2 \backslash I_{12}$, then $\overline{U_i} \cap \overline{U_j} = \emptyset$.

Indeed, first select $\mathcal{U}_{12}$ an arbitrary special covering of $X_1 \cap X_2$. Then there are disjoint open sets $D_1 \subset X_1$ and $D_2 \subset X_2$ such that $X_1 \backslash X_2 \subset D_1$ and $X_2 \backslash X_1 \subset D_2$. Further choose $\mathcal{U}_1$ and $\mathcal{U}_2$ special coverings of D_1 and D_2 respectively. Finally, set $\mathcal{U} =$ the collection of all open sets from $\mathcal{U}_1, \mathcal{U}_2$ and $\mathcal{U}_{12}$. Note that, if $U_{i_1} \cap \ldots \cap U_{i_s} \neq \emptyset$, then $U_{i_1} \cup \ldots \cup U_{i_s}$ is either contained in X_1 or in X_2.

Step 2. With the notations from above , there is a commutative diagram

$$\begin{array}{ccccc} C^{p-1}(\mathcal{U}_{12},\mathcal{F}) \oplus Z^p(\mathcal{U},\mathcal{F}) & \stackrel{u}{\to} & Z^p(\mathcal{U}_1,\mathcal{F}) \oplus Z^p(\mathcal{U}_2,\mathcal{F}) & \stackrel{v}{\to} & Z^p(\mathcal{U}_{12},\mathcal{F}) \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow \\ H^p(X_1 \cup X_2,\mathcal{F}) & \stackrel{u'}{\to} & H^p(X_1,\mathcal{F}) \oplus H^p(X_2,\mathcal{F}) & \stackrel{v'}{\to} & H^p(X_1 \cap X_2,\mathcal{F}), \end{array}$$

where β, γ, u', v' are the natural maps, α is obtained by extending with zero the natural map $Z^p(\mathcal{U},\mathcal{F}) \to H^p(X_1 \cup X_2,\mathcal{F})$, and v, u are to be constructed as follows:

For

$$(\xi_1, \xi_2) \in Z^p(\mathscr{U}_1, \mathscr{F}) \oplus Z^p(\mathscr{U}_2, \mathscr{F}),$$

set

$$v(\xi_1, \xi_2) := \xi_{2|X_1 \cap X_2} - \xi_{1|X_1 \cap X_2}.$$

For

$$(\eta, \xi) \in C^{p-1}(\mathscr{U}_{12}, \mathscr{F}) \oplus Z^p(\mathscr{U}, \mathscr{F}),$$

set

$$u(\eta, \xi) := (\xi_{|X_1}, \xi_{|X_2} + \delta\tilde{\eta}),$$

where $\tilde{\eta} \in C^{p-1}(\mathscr{U}_{12}, \mathscr{F})$ is the trivial extension of η to $C^{p-1}(\mathscr{U}_2, \mathscr{F})$.

We claim that: Im $u = v^{-1}$ (Ker γ).

Only "$\supseteq$" needs a proof. Let $\xi_1 \in Z^p(\mathscr{U}_1, \mathscr{F}), \xi_2 \in Z^p(\mathscr{U}_2, \mathscr{F})$ and $\eta \in C^{p-1}(\mathscr{U}_{12}, \mathscr{F})$ be such that $\xi_{2|X_1 \cap X_2} - \xi_{1|X_1 \cap X_2} = \delta\eta$. Now let $\tilde{\eta} \in C^{p-1}(\mathscr{U}_2, \mathscr{F})$ be the trivial extension of η. Then $(\xi_2 - \delta\tilde{\eta})_{|X_1 \cap X_2} = \xi_{1|X_1 \cap X_2}$. We define $\xi \in Z^p(\mathscr{U}, \mathscr{F})$ by

$$\begin{cases} \xi_{|X_1} := \xi_1 \\ \xi_{|X_2} := \xi_2 - \delta\tilde{\eta}. \end{cases}$$

This is well-defined because of (♠), and we have $u(\eta, \xi) = (\xi_1, \xi_2)$.

Step 3. Here we conclude the proof of the lemma. By hypothesis Ker γ is separated; and by the above claim u has closed image since v is continuous. On the one hand, by the Banach open theorem u is quasi open. On the other hand, from the commutativity of the left square in the diagram from step 2 u' results quasi-open.

For the bumping techniques we shall need the next

LEMMA 4. *Let Y be a complex space and Y_1, Y_2 open subsets such that $Y = Y_1 \cup Y_2$. Let $\mathscr{F} \in \mathrm{Coh}(Y)$ and q a positive integer. Then $H^q(Y, \mathscr{F})_{\mathrm{sep}}$ vanishes if the subsequent two statements hold.*

(a) *$H^q(Y_1, \mathscr{F})_{\mathrm{sep}}$ vanishes.*

(b) *$Y_1 \cap Y_2$ is q-Runge in Y_2.*

PROOF. We let $\mathscr{U}$ be a special covering of Y and set: $\mathscr{U}_1 := \mathscr{U}|_{Y_1}, \mathscr{U}_2 := \mathscr{U}|_{Y_2}$, and $\mathscr{U}_{12} := \mathscr{U}|_{Y_1 \cap Y_2}$. Statement (a) means that the natural coboundary map $\delta : C^{q-1}(\mathscr{U}|_{Y_1}, \mathscr{F}) \to Z^q(\mathscr{U}|_{Y_1}, \mathscr{F})$ has dense range. We have to show that $\delta : C^{q-1}(\mathscr{U}, \mathscr{F}) \to Z^q(\mathscr{U}, \mathscr{F})$ has dense range. For this, we consider the map

$$\rho : C^{q-1}(\mathscr{U}_1, \mathscr{F}) \oplus C^{q-1}(\mathscr{U}_2, \mathscr{F}) \to Z^q(\mathscr{U}_{12}, \mathscr{F})$$

defined by $\rho(\xi_1, \xi_2) := \delta(\xi_1|_{Y_{12}} - \xi_2|_{Y_{12}})$ where $Y_{12} := Y_1 \cap Y_2$.

We claim that ρ is surjective. To show this, note that by proposition 1 one has $Z^q(\mathscr{U}_{12}, \mathscr{F}) = \delta C^{q-1}(\mathscr{U}_{12}, \mathscr{F})$. Now let $\alpha \in C^{q-1}(\mathscr{U}_{12}, \mathscr{F})$ and consider $\xi_1 := \tilde{\alpha}, \xi_2 := 0$, where $\tilde{\alpha}$ is the trivial extension of α to $C^{q-1}(\mathscr{U}_1, \mathscr{F})$. Then $\rho(\xi_1, \xi_2) = \delta\alpha$, whence the surjectivity of ρ; hence ρ is open. Consequently, it has the lifting property of sequences. In order to finish the proof of the lemma, we fix $\xi \in Z^q(\mathscr{U}, \mathscr{F})$ arbitrary. Then choose a sequence $\{\theta_1^{(n)}\}_n \subset C^{q-1}(\mathscr{U}_1, \mathscr{F})$, $\delta\theta_1^{(n)} \to \xi|_{Y_1}$. Let $\theta_2 \in C^{q-1}(\mathscr{U}_2, \mathscr{F}), \delta\theta_2 = \xi|_{Y_2}$. (Note that Y_2 is q-complete.) Therefore in $Z^q(\mathscr{U}_{12}, \mathscr{F})$ one has $\delta(\theta_1^{(n)}|_{Y_{12}}) - \delta(\theta_2|_{Y_{12}}) \to 0$

Now choose sequences $\{\alpha_1^{(n)}\}_n \subset C^{q-1}(\mathscr{U}_1, \mathscr{F})$ and $\{\alpha_2^{(n)}\}_n \subset C^{q-1}(\mathscr{U}_2, \mathscr{F})$ which converge to zero and such that

$$\delta(\alpha_1^{(n)}) - \delta(\alpha_2^{(n)}) = \delta(\theta_1^{(n)}) - \delta(\theta_2).$$

Thus $u^{(n)} := \alpha_2^{(n)} - \theta_2 - \alpha_1^{(n)} + \theta_1^{(n)} \in Z^q(\mathscr{U}_{12}, \mathscr{F})$. Now $Z^{q-1}(\mathscr{U}_2, \mathscr{F}) \to Z^{q-1}(\mathscr{U}_{12}, \mathscr{F})$ has dense range from (b), proposition 1, and lemma 2. Thus there exists a sequence

$$\{h^{(n)}\}_n \subset Z^{q-1}(\mathscr{U}_2, \mathscr{F}), \text{ with } h^{(n)}|_{Y_{12}} - u^{(n)} \to 0$$

on Y_{12}. Let $\tilde{h}_1^{(n)}, \tilde{u}^{(n)}$ be the trivial extensions to $C^{p-1}(\mathscr{U}_1, \mathscr{F})$. Define a sequence $\{\eta^{(n)}\}_n$ in $C^{q-1}(\mathscr{U}, \mathscr{F})$ by:

$$\eta^{(n)} := \begin{cases} \theta_1^{(n)} - \alpha_1^{(n)} + \tilde{h}_1^{(n)} - \tilde{u}^{(n)} & \text{on } \mathscr{U}_1 \\ \theta_2 - \alpha_2^{(n)} + h_2^{(n)}, & \text{on } \mathscr{U}_2 \end{cases}$$

Then $\delta\eta^{(n)} \to \xi$, whence the lemma.

We conclude this paragraph with the next:

Lemma 5 *Let $\pi : X \to S$ be a holomorphic map of complex spaces and $D \Subset S$ an open subset such that $\pi^{-1}(D)$ is q-complete. Let also $\varphi_1, \varphi_2 \in C^\infty(S, \mathbf{R})$ be two r-convex functions with the same positivity directions. Set $U_i := \{s \in D; \varphi_i(s) < 0\}, i = 1,2,$ and $p = q + r - 1$. Then $\pi^{-1}(U_1 \cap U_2)$ is p-complete and p-Runge in $\pi^{-1}(U_2)$.*

Proof. Let $\psi : \pi^{-1}(D) \to (0, \infty)$ be q-convex and exhaustive. For every real number $C > 0$ define a family of continuous exhaustion functions $\psi_C : \pi^{-1}(U_2) \to \mathbf{R}$ by

$$\psi_C := \psi - 1/(\varphi_2 \circ \pi) + C \cdot \max(\varphi_1 \circ \pi, \varphi_2 \circ \pi).$$

If $K \subset \pi^{-1}(U_1 \cap U_2)$ is a compact set, then, with a large enough $C > 0$, we get

$$K \subset \{\psi_c < 0\} \Subset \pi^{-1}(U_2).$$

A suitable smooth p-convex approximation of ψ_C in the C^o-topology ([16]) enables us to conclude that $\pi^{-1}(U_1 \cap U_2)$ is p-Runge in $\pi^{-1}(U_2)$.

The p-completeness of $\pi^{-1}(U_1 \cap U_2)$ results if one approximates the function

$$\psi - 1/\max(\varphi_1 \circ \pi, \varphi_2 \circ \pi) \in C^o(\pi^{-1}(U_1 \cap U_2), \mathrm{R})$$

in the C^o-topology by smooth p-convex functions ([16]).

3. The results.

Here is our relative vanishing theorem for families of q-complete complex spaces.

THEOREM 1. *Let $\pi : X \to S$ be a locally q-complete morphism of complex spaces. If S is r-complete, then X is cohomologically $(q+r)$-complete. Moreover, $H^{q+r-1}(X, \mathcal{F})_{\text{sep}}$ vanishes for every coherent analytic sheaf $\mathcal{F}$ on X.*

PROOF. We consider ([16]) a r-convex exhaustion function $h : S \to \mathrm{R}$ such that for every real number λ if $S(\lambda) := \{s \in S; h(s) < \lambda\}$, then the set

$$\{s \in S; h(s) = \lambda\} \backslash \partial S(\lambda)$$

contains at most one point. Correspondingly, define the sets $X(\lambda) := \pi^{-1}(S(\lambda))$.

Put $p = q + r - 1$ and let $\mathcal{F} \in \mathrm{Coh}(X)$. We claim that for every pair of real numbers $\lambda < \mu$ we have:

(a) The restriction $H^p(X(\mu), \mathcal{F}) \to H^p(X(\lambda), \mathcal{F})$ has dense range;
(b) $H^i(X(\lambda), \mathcal{F})$ vanishes for all $i \geq q + r$;
(c) $H^p(X(\mu), \mathcal{F})_{\text{sep}}$ vanishes.

First we show that (a) holds. For this, we define $T \subseteq \mathrm{R}$ to be the set of all real numbers μ such that the restriction map $H^p(X(\mu), \mathcal{F}) \to H^p(X(\lambda), \mathcal{F})$ has dense image for every real number λ with $\lambda < \mu$.

Obviously, T is not empty. In fact if $\mu_* := \min\{h(s); s \in S\}$, then one sees easily that $(-\infty, \mu_*] \subset T$. Also, by lemma 2 and a standard argument of Fréchet spaces, T is closed. To prove T is open, we use the bumping technique of Andreotti and Grauert. To begin with, fix some $\mu \in T$. We shall find $\epsilon_o > 0$ such that $\mu_o + \epsilon_o \in T$. Recall that $\{h = \mu_o\} \backslash \partial S(\mu_o)$ is empty or equals $\{s_o\}$ for some $s_o \in S$. We treat only the second case since the first one is similar (so we omit its proof).

Let $U \subset S$ be a Stein open neighborhood of s_o such that $\pi^{-1}(U)$ is q-complete and $\overline{U} \cap \overline{S(\mu_o)} = \emptyset$. Choose finitely many Stein open sets $\{U_j\}, j = 1, \dots, k$, disjoint from U, which cover $\partial S(\mu_o)$ and such that $\pi^{-1}(U_j)$ are q-complete. Let $V_j \Subset U_j$ be also open Stein sets such that $\{V_j\}_j$ still covers $\partial S(\mu_o)$. Then select $\{\rho_j\} \in C^\infty(U_j, \mathrm{R}), \rho_j \geq 0$, and $\rho_j \equiv 1$ on $V_j, j = 1, \dots, k$. Define smooth functions $h_j : X \to \mathrm{R}$ by

$$h_j := h - \sum_{\nu=1}^{j} c_\nu \rho_\nu, j = 1, \dots, k,$$

where $c_\nu > 0$ are small enough constants such that $h_o := h, h_1, \dots, h_k$, are r-convex with the same positivity directions. Set

$$S_j := \{s \in S; h_j(s) < \mu_o\}, j = 1, \dots, k \text{ and } S_o := S(\mu_o).$$

Obviously, $S_j \backslash S_{j-1} \Subset U_j$. Also since h is proper, there exists $\epsilon_o > 0$ with $S(\mu_o + \epsilon_o) \subseteq S_k \cup U$. We define for an arbitrary real number μ and integer $j = 0, \dots, k$, the set

$$(1) \qquad X_j(\mu) := \pi^{-1}(S_j \cap S(\mu)).$$

Since $S(\mu) = (S(\mu) \cap S_k) \cup (S(\mu) \cap U)$ we get: $X(\mu) = X_k(\mu) \cup V(\mu)$, where $V(\mu) := \pi^{-1}(S(\mu) \cap U)$ is p-complete by lemma 4. Moreover, we remark that

$$(2) \qquad X_k(\mu) \cap V(\mu) \text{ is } p\text{-Runge in } V(\mu).$$

Therefore $H^p(X(\mu), \mathscr{F}) = H^p(X_k(\mu), \mathscr{F}) \oplus H^p(V(\mu), \mathscr{F}) = H^p(X_k(\mu), \mathscr{F})$. Now fix μ and λ with $\mu_o < \mu \leq \mu_o + \epsilon_o$ and $\lambda < \mu$. To get (a) we show inductively on j that

$$(\heartsuit) \qquad H^p(X_j(\mu), \mathscr{F}) \to H^p(X_j(\lambda), \mathscr{F})$$

has dense range. For $j = 0$ this is clear since $\mu_o \in T$. Now let $j \geq 1$. We have

$$(3) \qquad X_j(\mu) = X_{j-1}(\mu) \cup V_j(\mu)$$

where $V_j(\mu) := \pi^{-1}(U_j \cup S(\mu))$. Note also that

$$(4) \qquad X_{j-1}(\mu) \cap V_j(\mu) \text{ is } p\text{-complete and } p\text{-Runge in } V_j(\mu).$$

This is a consequence of lemma 5 for $D := U_j, \varphi_1 := h_{j-1} - \mu_o$, and $\varphi_2 := h - \mu$. Now, from Mayer-Vietoris sequence, one gets the subsequent commutative diagram with exact rows (Note that $V_j(\mu)$ and $V_j(\lambda)$ are p-complete)

$$\begin{array}{ccccccc} H^{p-1}(X_j(\mu)\cap V_j(\mu),\mathscr{F}) & \to & H^p(X_j(\mu),\mathscr{F}) & \to & H^p(X_{j-1}(\mu),\mathscr{F}) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H^{p-1}(X_j(\lambda)\cap V_j(\lambda),\mathscr{F}) & \to & H^p(X_j(\lambda),\mathscr{F}) & \to & H^p(X_{j-1}(\lambda),\mathscr{F}). & & \end{array}$$

Lemma 3 applied for $X_1 := X_{j-1}(\lambda), X_2 := V_j(\lambda)$; the p-completeness of X_2 and $X_1 \cap X_2$; and corollary 1 yield (♡).

Statement (b) is similar to (a), (and, in fact easier) so we omit its proof.

In order to prove statement (c), one chooses a a special covering $\mathscr{U}$ of X and showes that the coboundary map

$$\delta : C^{q+r-2}(\mathscr{U}_{|_{X(\mu)}}, \mathscr{F}) \to Z^{q+r-1}(\mathscr{U}_{|_{X(\mu)}}, \mathscr{F})$$

has dense range. This folows by lemma 4 and the proof of (a) given above by using (1) to (4).

Now the cohomological statement of the theorem follows in a standard way, because for all $v = 0, 1, \dots,$ we have that $H^i(X(v), \mathscr{F})$ vanishes for all $i \geq q + r$, and the restriction maps $H^{q+r-1}(X(v+1), \mathscr{F}) \to H^{q+r-1}(v), \mathscr{F})$ have dense image.

In order to obtain "the moreover", note that for every special covering $\mathscr{U}$ of X, the restriction maps $C^{p-1}(\mathscr{U}_{|_{X(v+1)}}, \mathscr{F}) \to C^{p-1}(\mathscr{U}_{|_{X(v)}}, \mathscr{F}), v = 0, 1, \dots,$ are surjective. Also by lemma 2 and (a) from above, the restrictions $Z^p(\mathscr{U}_{|_{X(v+1)}}, \mathscr{F}) \to Z^p(\mathscr{U}_{|_{X(v)}}, \mathscr{F})$ have dense range. A standard argument of Fréchet spaces yields that the coboundary map $\delta : C^{p-1}(\mathscr{U}, \mathscr{F}) \to Z^p(\mathscr{U}, \mathscr{F})$ has dense range.

Here we give an improvement of theorem 1 in a particular case.

COROLLARY 2 *Let $\pi : X \to S$ be q-convex and locally q-complete. If S is r-complete, then X is cohomologically $(q + r - 1)$-complete.*

PROOF. Let $\mathscr{F} \in \mathrm{Coh}(X)$. One has to check that the cohomology group $H^{q+r-1}(X, \mathscr{F})$ is separated. If $r = 1$, this follows from ([11]); for $r \geq 2$ one applies ([10]).

REMARKS. 1) ([10], p. 995]) Let $\pi : X \to S$ be a q-convex morphism. Then the canonical topologies on $H^i(X, \mathscr{F}), \mathscr{F} \in \mathrm{Coh}(X)$, are separated for all indices $i \geq q + 1$. (No further assumption on S !) It is unknown if this is true for $i = q$.

2) If $\pi : X \to S$ is locally q-convex and $\mathscr{F} \in \mathrm{Coh}(X)$, then the higher direct image sheaf $R^i\pi_*(\mathscr{F})$ is coherent for all $i \geq q$. However, this and the Leray spectral sequence do not imply our theorem, since $R^i\pi_*(\mathscr{F})$ for $i < q$ may not be coherent.

The result from theorem 1 is sharp, in fact we show:

EXAMPLE 2. *For every positive integers q, r, there exists a holomorphic fi-*

bration π from X to S with typical fiber F such that S and F are r-complete and q-complete respectively, and nevertheless, $H^{q+r-1}(X, \mathcal{O}_X)$ does not vanish. (In fact, it has infinite dimension over $\mathbf{C}$.)

Before getting involved with the example, we recall some facts:

($\bullet$) Let $\mathcal{F}$ and $\mathcal{G}$ be coherent analytic sheaves on the complex spaces Y and Z. We denote by $\mathcal{F} \square \mathcal{G}$ the (coherent analytic) sheaf $p_Y^*\mathcal{F} \hat{\otimes} p_Z^*\mathcal{G}$ on $Y \times Z$, where p_Y and p_Z stands for the canonical projections on Y and Z respectively. E.g. $\mathcal{O}_Y \square \mathcal{O}_Z = \mathcal{O}_{Y \times Z}$.

($\bullet$) The following Künneth formula due to Cassa ([4]) holds. Assume that the cohomology groups $H^j(Z, \mathcal{G}), j = 0, 1, \ldots,$ are Hausdorff. Then for every non-negative integer k there exists a *topological* isomorphism

$$H^k(Y \times Z, \mathcal{F} \square \mathcal{G}) \cong \bigoplus_{i+j=k} \Big((H^i(Y, \mathcal{F})_{\text{sep}} \ \hat{\otimes} H^j(Z, \mathcal{G})) \oplus R_{ij} \Big)$$

where R_{ij} are complex vector space of infinite dimension (with the trivial topology) if $H^i(Y, \mathcal{F})$ is not-Hausdorff and $H^j(Z, \mathcal{G})$ does not vanish; otherwise $R_{ij} = \{0\}$.

Now, the example goes as follows. Skoda ([14]) produced a locally trivial holomorphic fibration $f : M \to D$ with fibre $\mathbf{C}^2$ and base $D \subset \mathbf{C}$ open, such that M is not Stein. Notice that $H^1(M, \mathcal{O}_M)$ is not separated ([7]).

Set $X = M \times (\mathbf{C}^r \backslash \{0\}) \times (\mathbf{C}^q \backslash \{0\}), S = D \times (\mathbf{C}^r \backslash \{0\})$ and π from X to S canonically induced by f and the natural projection on $\mathbf{C}^r \backslash \{0\}$. It is evident that π is a fibre bundle with q-complete fibre and r-complete base space. Now, the above Künneth formula says that $H^{q+r-1}(X, \mathcal{O}_X)$ is infinitely dimensional; a fortiori $H^{q+r-1}(X, \mathcal{O}_X) \neq 0$.

Here we give some immediate consequences to theorem 1.

COROLLARY 3. *Every locally q-complete open subset of a r-complete complex space is cohomologically $(q + r)$-complete.*

COROLLARY 4. *Let $E \to S$ be a holomorphic fibre bundle with fiber F. Suppose that F and S are q-complete and r-complete respectively. Then E is cohomologically $(q + r)$-complete.*

COROLLARY 5. *Let X be a r-complete complex space and $D \subset X$ an open set such that the inclusion map $\iota : D \to X$ is q-convex. Then D is $(q + r - 1)$-complete.*

PROOF, Let $\psi : X \to \mathbf{R}$ be r-convex and exhaustive, and $\varphi : D \to \mathbf{R}$ the function which gives the q-convexity of ι. Then the set $\{x \in D; \varphi(x) \leq c_\sharp\}$ is *closed* in X; therefore, by standard arguments there exists a smooth rapidly increasing and convex function $\chi : \mathrm{R} \to \mathrm{R}$ such that the function $\tilde{\varphi} : D \to \mathrm{R}$ defined by $\tilde{\varphi} = \chi(\psi) + \varphi$ is $(q + r - 1)$-convex and exhaustive.

REMARK. This corollary does not hold for arbitrary q-convex mappings. (See the example 1 in § 2.)

The same method used for the proof of theorem 1, together with the subsequent two lemmas

LEMMA 6. *Let Y be a p-complete complex space of dimension n and U a p-Runge domain. Then $H_i(Y, U; G) = 0$ for $i \geq n + p$ and every abelian group G.*

LEMMA 7. *Let Y be a p-complete complex space of dimension n which is locally a set theoretic complete intersection and U a p-Runge domain. Then for every abelian group G, $H_c^i(Y; G) = 0$ for $i \leq n - p$ and the natural map $H_c^{n-p+1}(U; G) \to H_c^{n-p+1}(Y; G)$ is injective.*

from [15] and [16] respectively give us, mutatis mutandis, the following result concerning the vanishing of other cohomology groups on X, namely;

THEOREM 2. *Let $\pi : X \to S$ be locally q-complete. Let $n = \dim(X)$ and assume that S is r-complete. Then, for every abelian group G we have:*

(1) *$H_i(X, G)$ vanishes for $i \geq n + q + r - 1$.*

(2) *$H_c^i(X, G)$ vanishes for $i \leq n - (q + r) + 1$ if X is of pure dimension and locally a set theoretic complete intersection.*

(We recall that a complex space Y of pure dimension n is said to be *locally a set theoretic complete intersection* if each point $y \in Y$ admits a local chart $\iota : V \to \hat{V} \subset \mathrm{C}^N$ with $\hat{V}$ open such that $\iota(V) \subset \hat{V}$ is an analytic subset given by precisely $N - n$ equations.)

As an interesting application of our method, we have:

COROLLARY 6. *Let D be a locally q-complete open subset of a r-complete complex space X of dimension n. Set $p = q + r - 1$. Then $H_{n+p-1}(D; \mathrm{Z})$ is torsion free and $H_i(D; \mathrm{Z})$ vanishes for $i \geq n + p$. Moreover, if ∂D is real-analytic, then $H_{n+p-1}(D, \mathrm{Z})$ is free.*

REMARK. The first part of corollary 6 was proved by Bolondi ([3]) in the case $q = r = 1$.

4. Some remarks on locally 1-convex maps.

Motivated by what we proved by now, one should ask if there are also similar global finiteness theorems for X, when $\pi : X \to S$ is locally q-convex and S enjoys some convexity properties, like r-convexity.

In general, this is not true, e.g. let $\pi : X \to \mathbf{C}^n$ be as in example 1 from § 2 with A an infinite discrete set. Then π is 1-convex, $\mathbf{C}^n$ is Stein, and X fails to be even cohomologically $(n-1)$-convex. The situation does not improve even if we assume S compact. A simple example is the canonical map $\pi : X = \mathbf{C}^n \backslash \{0\} \to S = \mathbf{P}^{n-1}$, which, of course, is locally 1-convex, and, again X fails to be cohomologically $(n-1)$-convex. (However, without any further assumption on S, if $\pi : X \to S$ is q-convex and $\pi(X)$ is relatively compact in S, then X is q-convex.)

There is one particular case of the situation (♣) considered in the introduction which may be of some interest, namely; Let $\pi : X \to S$ be locally 1-convex. Then for every $s \in S$ the fiber $X_s := \pi^{-1}(s)$ is a 1-convex space which contains an exceptional compact analytic set E_s. Put $q_s := \dim(E_s)$ if X_s is not Stein; otherwise we take $q_s = 0$. We assume $q := q(\pi) := \sup_{s \in S} q_s < \infty$.

Recall the relative Stein factorization from ([9]). There exist: a complex space Y together with a proper surjective holomorphic map $\rho : X \to Y$ such that $\rho_*(\mathcal{O}_X) = \mathcal{O}_Y$, and Stein morphism $\sigma : Y \to S$ such that $\sigma \circ \rho = \pi$. Moreover, if $E = \{x \in X; \dim(\rho^{-1}(\rho(x))) > 0\}$ denotes the degeneracy set of ρ, then the restriction of π from E into S is proper. In particular, $\pi(E)$ is an analytic subset of S, and the restriction σ from $\rho(E)$ onto $\pi(E)$ is a finite map. One checks easily that $q = \sup\{\dim \pi^{-1}(\pi(x)); x \in E\}$.

Proposition 2. *If $q > 0$, then X is cohomologically $(q+r)$-complete.*

Proof. Since S is r-complete, $\rho(E)$ is r-complete and then E is $(q+r)$-complete ([17]); hence $\rho(E)$ and E have a fundamental systems of r-complete and $(q+r)$-complete neighborhoods respectively. By theorem 1, Y is cohomologically $(r+1)$-complete; hence from a well-known exact sequence one has $H^i_\Phi(Y\backslash\rho(E), \mathcal{G}) = 0$ if $i \geq r+1$ and $\mathcal{G} \in \mathrm{Coh}(Y)$, where "$\Phi$" means the family of supports made up from all subsets of $Y\backslash\rho(E)$ which are closed in Y. Similarly, one gets surjections $H^j_\Psi(X\backslash E, \mathcal{F}) \to H^j(X, \mathcal{F})$ for $j \geq q+r$ and $\mathcal{F} \in \mathrm{Coh}(X)$, where "$\Psi$" means the family of supports made up from all suubsets of $X\backslash E$ which are closed in X. Since π is proper, by Grauert's coherence theorem, $\mathcal{G} := \rho_*(\mathcal{F})$ is coherent. On the other hand, as π is closed and $X\backslash E \cong Y\backslash\rho(E)$, we may identify Φ and Ψ. Therefore $H^k_*(X\backslash E, \mathcal{F}) \cong H^k_*(Y\backslash\rho(E), \rho_*(\mathcal{F}))$ for every k, whence the proposition.

Remark. For $q = 0$ we deduce only the cohomological $(r+1)$-completeness of X.

Now, we extend a result from [17] to families of 1-convex spaces.

PROPOSITION 3. *If* $\pi : X \to S$ *is 1-convex and* S *is r-complete, then* X *is* $(q+r)$*-complete.*

PROOF. (Sketch) We consider $\varphi : X \to \mathbb{R}$ and $c_\sharp$ according to the definition. By replacing φ with $\chi(\varphi)$ for a smooth convex function $\chi : \mathbb{R} \to \mathbb{R}$ such that $(-\infty, c_\sharp] = \{\chi = 0\}$ and χ is strictly increasing on $[c_\sharp, \infty)$, we may assume that $c_\sharp = 0$ and φ is plurisubharmonic on the whole space X. For every non-negative real number c we set $X(c) := \{x \in X; \varphi(x) \leq c\}$.

Let $\psi : S \to \mathbb{R}$ be r-convex and exhaustive. Then for every closed subset $T \subset Y$ such that the restriction of σ from T to S is proper, there is a smooth function f on Y which is r-convex on T. In fact, if $\{V_i\}_{i\in I}$ is a locally finite open covering of S such that $Y_i := \sigma^{-1}(V_i), i \in I$, are Stein, and $f_i : Y_i \to \mathbb{R}$ are 1-convex and exhaustive, we put

$$f := \phi \circ \sigma + \sum \epsilon_i (\lambda_i \circ \sigma) f_i$$

where $\{\lambda_i\}_{i\in I}$ is a partition of unity subordinate to the covering $\{V_i\}_{i\in I}$ and $\epsilon_i > 0, i \in I$, are sufficiently small constants (which depend on T).

Now, since $E \subset X(c_\sharp)$ and for $T := \rho(X(c_\sharp))$ with some fixed $c_* > c_\sharp$, there exists a smooth function Φ on X which is $(q+r)$-convex on $X(c_*)$, and here Φ and $\psi \circ \pi + \varphi$ have the same positivity directions ([17]).

Since $\psi \circ \pi + \varphi$ is exhaustive for X and r-convex on $X \backslash X(c_\sharp)$, there exist a smooth rapidly increasing convex function $\mu : \mathbb{R} \to \mathbb{R}$ and $\theta \in C^\infty(X, \mathbb{R})$ which equals 1 on $X(c_\sharp)$ and $\sup(\theta) \subset X(c_*)$ such that the function $\mu(\psi \circ \pi + \varphi) + \theta\Phi : X \to \mathbb{R}$ is $(q+r)$-convex and exhaustive.

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