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DERIVED DE RHAM COMPLEX AND CYCLIC HOMOLOGY*

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Let $A \to B$ be a homomorphism of commutative rings. D. Quillen [22, 8.1] constructed a convergent spectral sequence

$$E^2_{p,q} = H_p(\wedge^q_B \mathsf{L}_{B\backslash A}) \Rightarrow \operatorname{Tor}^{B \otimes^L_A B}_{p+q}(B, B)$$

where $\mathsf{L}_{B\backslash A}$ is the cotangent complex of André-Quillen of $A \to B$ [22], $\wedge^q_B$ is the qth exterior power functor (applied dimension-wise to the simplicial B-module $\mathsf{L}_{B\backslash A}$), and $B \otimes^L_A B$ is the derived tensor product [21, p.II.6.8.] (in particular, if B is flat as A module, we can remove the L in the spectral sequence and the abutment is Hochschild homology of $A \to B$). If $\mathbf{Q} \subset B$, this spectral sequence is degenerate and there are isomorphisms

$$\bigoplus_{p+q=n} E^2_{p,q} = \operatorname{Tor}^{B \otimes^L_A B}_n(B, B)$$

This spectral sequence can be useful in two ways: to deduce results on Hochschild homology from results on the cotangent complex, and to obtain information from the cotangent complex by computing Hochschild homology.

There is a close relationship between Hochschild homology and cyclic homology [15]. It would be useful to obtain a similar spectral sequence converging to cyclic homology, and whose second term were related to $H_p(\wedge^q_B \mathsf{L}_{B\backslash A})$ in such a way that the abutment of this relationship were the existing one between cyclic and Hochschild homologies. It should be also degenerate if $\mathbf{Q} \subset B$.

This spectral sequence is easy to construct and the E^2 term is the homology of some quotients, which we shall denote $\mathsf{L}\Omega^{(m)}_{B\backslash A}$, of the derived de Rham complex [11, chapitre VIII].

The aim of this paper is to study cyclic homology of commutative algebras

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from this point of view, which I consider, for some particular purposes, more natural and easy. In fact, all the contents of this paper can be considered as easy consequences of results by D. Quillen [22] and L. Illusie [11, chapitre VIII] and in order to evidence this fact, I have chosen only some aspects of cyclic homology. So it must not be considered a rigorous study of this homology theory. Also, part of this relationship was hinted in [6] although that paper is in the characteristic zero context.

Another advantage of this point of view, is that the necessary formalism to globalize, was well developed in [11], so we can define $\mathsf{L}\Omega^{(m)}_{X\backslash Y}$ for a morphism of ringed toposes.

In the first sections of this paper, we introduce and study $\mathsf{L}^{(m)}_{B\backslash A}$ and $\mathsf{L}\Omega^{(m)}_{B\backslash A}$. These complexes are finer invariants than their homology modules, so as in [22] or [11] we will concentrate in them rather than their homology. In section 5 we study the relationship between their homologies, and Hochschild and cyclic homologies.

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1. Definitions and first properties.

All rings in this paper will be commutative with unit (sometimes they will be graded and shall be understood to be anticommutative).

(1.1) Let $\psi\colon A \to B$ be a homomorphism of rings. Consider A and B as constant simplicial rings, and let $A \to X \to B$ be a factorization of ψ, where $A \to X$ is a cofibration and $X \to B$ a trivial fibration [21, II § 4. Prop. 3, p.4.5]. This factorization is unique up to simplicial homotopy [21, II § 2. Prop. 4, p.2.4] and up to homotopy depends functorially on ψ. We will say that $A \to X \to B$ is a cofibrant factorization of ψ. We have a simplicial DG A-algebra [11, VIII.2.1.1] depending only on ψ up to simplicial homotopy

$$\Omega^\bullet_{X\backslash A} = \begin{array}{ccccccc} X_0 & \leftrightarrows & X_1 & \leftleftarrows & X_2 & \cdots \\ \downarrow d_{\mathrm{DR}} & & \downarrow d_{\mathrm{DR}} & & \downarrow d_{\mathrm{DR}} & \cdots \\ \Omega^1_{X_0\backslash A} & \leftrightarrows & \Omega^1_{X_1\backslash A} & \leftleftarrows & \Omega^1_{X_2\backslash A} & \\ \downarrow d_{\mathrm{DR}} & & \downarrow d_{\mathrm{DR}} & & \downarrow d_{\mathrm{DR}} & \cdots \\ \Omega^2_{X_0\backslash A} & \leftrightarrows & \Omega^2_{X_1\backslash A} & \leftleftarrows & \Omega^2_{X_2\backslash A} & \\ \downarrow d_{\mathrm{DR}} & & \downarrow d_{\mathrm{DR}} & & \downarrow d_{\mathrm{DR}} & \\ \vdots & & \vdots & & \vdots & \end{array}$$

where the ith column is the Rham algebra of $A \to X_i$, and the jth row is the simplicial X-module $\Omega^j_{X\backslash A}$. We will denote the associated fourth quadrant double complex [5, 2.1] by $(\mathsf{L}\Omega_{B\backslash A})_{*,*}$, where $(\mathsf{L}\Omega_{B\backslash A})_{p,q} = \Omega^{-q}_{X_p\backslash A}$.

Let $\mathsf{L}^{(m)}_{B\backslash A}$ be the complex of projective B-modules associated to $\Omega^m_{X\backslash A} \otimes_X B$. It represents a well defined object of $D(B)$ (up to isomorphism). When $m = 1$, it is the cotangent complex of $A \to B$ and was studied in [1], [22].

The simplicial DG A-algebra $\Omega^\bullet_{X\backslash A}$ has a Hodge filtration defined by $F^m\Omega^\bullet_{X\backslash A} = (0 \to 0 \to \ldots \to 0 \to \Omega^m_{X\backslash A} \xrightarrow{d_{\mathrm{DR}}} \Omega^{m+1}_{X\backslash A} \xrightarrow{d_{\mathrm{DR}}} \ldots)$. Let $\mathsf{L}\Omega^{(m)}_{B\backslash A}$ be the mth suspension of the simple complex associated to $\Omega^\bullet_{X\backslash A}/F^{m+1}\Omega^\bullet_{X\backslash A}$ (which is concentrated in nonnegative degrees). It represents also a well defined object of $D(A)$.

Since $X \to B$ is a quasi-isomorphism of simplicial X-modules and $\Omega^m_{X\backslash A}$ is X-projective, we have a quasi-isomorphism of simplicial A-modules $\Omega^m_{X\backslash A} \to \Omega^m_{X\backslash A} \otimes_X B = \mathsf{L}^{(m)}_{B\backslash A}$ [11, I.3.3.2.1]. So the exact sequence

$$0 \to F^m\Omega^\bullet_{X\backslash A}/F^{m+1}\Omega^\bullet_{X\backslash A} \to \Omega^\bullet_{X\backslash A}/F^{m+1}\Omega^\bullet_{X\backslash A} \to \Omega^\bullet_{X\backslash A}/F^m\Omega^\bullet_{X\backslash A} \to 0$$

gives a natural distinguished triangle in $D(A)$.

$$\mathsf{L}\Omega^{(m-1)}_{B\backslash A} \to \mathsf{L}^{(m)}_{B\backslash A} \to \mathsf{L}\Omega^{(m)}_{B\backslash A} \to \mathsf{L}\Omega^{(m-1)}_{B\backslash A}[1] \tag{1.2}$$

(1.3) Since $\Omega^m_{-\backslash A} \otimes_- M$ preserves colimits, in particular exact sequences $\bullet \rightrightarrows \bullet \to \bullet$, we have

$$H_0(\mathsf{L}^{(m)}_{B\backslash A} \otimes_B M) = \Omega^m_{B\backslash A} \otimes_B M$$

and so, by (1.2) $H_0(\mathsf{L}\Omega^{(m)}_{B\backslash A}) = \Omega^m_{B\backslash A}/d_{\mathrm{DR}}\Omega^{m-1}_{B\backslash A}$.

In particular the last terms of the homology exact sequence associated to (1.2) are

$$H_1(\mathsf{L}\Omega^{(m)}_{B\backslash A}) \to \Omega^{m-1}_{B\backslash A}/d_{\mathrm{DR}}\Omega^{m-2}_{B\backslash A} \xrightarrow{d_{\mathrm{DR}}} \Omega^m_{B\backslash A} \to \Omega^m_{B\backslash A}/d_{\mathrm{DR}}\Omega^{m-1}_{B\backslash A} \to 0$$

If $B = A/I$, we can take $X_0 = A$, and similar reasoning together with Quillen's results [11, I.4.3.2.1] [21, II § 6. Prop. 1, p.6.3] give

$$H_m(\mathsf{L}^{(m)}_{B\backslash A}) = \Gamma^m_B(I/I^2), \quad H_i(\mathsf{L}^{(m)}_{B\backslash A}) = 0 \quad \text{if } i < m$$

where Γ^m_B is the mth devided power functor.

(1.4) For $m = 1$ we obtain $H_n(\mathsf{L}\Omega^{(1)}_{B\backslash A}) = H_n(A, B, B)$ if $n > 1$ (André-Quillen homology groups) and an exact sequence

$$0 \to H_1(A, B, B) \to H_1(\mathsf{L}\Omega^{(1)}_{B\backslash A}) \to B \xrightarrow{d_{\mathrm{DR}}} \Omega^1_{B\backslash A} \to \Omega^1_{B\backslash A}/dB \to 0$$

In particular, an exact sequence $0 \to H_1(A, B, B) \to H_1(\mathsf{L}\Omega^{(1)}_{B\backslash A}) \to H^0_{\mathrm{DR}}(B) \to 0$, where $H_{\mathrm{DR}}(B)$ is the cohomology of the de Rham complex of $A \to B$.

(1.5) Filtering the double complex E^0 whose simple complex is $\mathsf{L}\Omega^{(m)}_{B\backslash A}$ by rows, we obtain a convergent first quadrant spectral sequence

$$E^1_{p,q} = \begin{Bmatrix} H_p(\mathsf{L}^{(m-q)}_{B\backslash A}) & \text{if } 0 \le q \le m,\ p \ge 0 \\ 0 & \text{in other case} \end{Bmatrix} \Rightarrow H_{p+q}(\mathsf{L}\Omega^{(m)}_{B\backslash A})$$

(1.6) Filtering now by columns, we obtain a convergent spectral sequence with

$$E^1_{p,q} = \begin{cases} H^{m-q}_{\mathrm{DR}}(X_p) & \text{if } p \ge 0,\ 0 < q \le m \\ \Omega^m_{X_p\backslash A}/d_{\mathrm{DR}}\Omega^{m-1}_{X_p\backslash A} & \text{if } p \ge 0,\ q = 0 \\ 0 & \text{in other case} \end{cases}$$

In particular, if A contains the rational numbers, by Poincaré's lemma, the first term vanishes for $q \neq 0,\ m$ and the second term is

$$E^2_{p,q} = \begin{cases} A & \text{if } p = 0,\ q = m \\ H_p(\Omega^m_{X\backslash A}/d_{\mathrm{DR}}\Omega^{m-1}_{X\backslash A}) & \text{if } p \ge 0,\ q = 0 \\ 0 & \text{if } q \neq 0,\ m \end{cases}$$

So we have isomorphisms

$$H_n(\mathsf{L}\Omega^{(m)}_{B\backslash A}) = H_n(\Omega^m_{X\backslash A}/d_{\mathrm{DR}}\Omega^{m-1}_{X\backslash A}) \quad \text{if } n \neq m,\ m+1$$

and an exact sequence

$$0 \to H_{m+1}(\mathsf{L}\Omega^{(m)}_{B\backslash A}) \to H_{m+1}(\Omega^m_{X\backslash A}/d_{\mathrm{DR}}\Omega^{m-1}_{B\backslash A}) \xrightarrow{\phi} A$$
$$\to H_m(\mathsf{L}\Omega^{(m)}_{B\backslash A}) \to H_m(\Omega^{(m)}_{X\backslash A}/d_{\mathrm{DR}}\Omega^{m-1}_{X\backslash A}) \to 0$$

If $A \to B$ is injective then $\mathrm{Im}(X_1 \to X_0) \cap \mathrm{Ker}(X_0 \xrightarrow{d_{\mathrm{DR}}} \Omega^1_{X_0\backslash A}) = 0$, i.e., $\mathrm{Im}(E^0_{1,m} \to E^0_{0,m}) \cap \mathrm{Ker}(E^0_{0,m} \to E^0_{0,m-1}) = 0$ and since $E^0_{0,m+1} = 0$ an easy computation shows that the differential ϕ: $E^{m+1}_{m+1,0} \to E^{m+1}_{0,m}$ is zero. So in this case we have isomorphisms

$$H_n(\mathsf{L}\Omega^{(m)}_{B\backslash A}) = H_n(\Omega^m_{X\backslash A}/d_{\mathrm{DR}}\Omega^{m-1}_{X\backslash A}) \quad \text{if } n \neq m$$

and an exact sequence

$$0 \to A \to H_m(\mathsf{L}\Omega^{(m)}_{B\backslash A}) \to H_m(\Omega^{(m)}_{X\backslash A}/d_{\mathrm{DR}}\Omega^{m-1}_{X\backslash A}) \to 0$$

(1.7) Since any functor extended from the category of modules to the category of simplicial modules [5, 1.11] (we are thinking in the mth exterior power functor) preserves homotopy equivalences [5, 1.15, 3.31] using the dual of [10, I.4.7] we can extend the results of base change [22, 5.3] to $\mathsf{L}^{(m)}$:

Let B and C be A-algebras such that $\mathrm{Tor}^A_q(B, C) = 0$ for $q > 0$. Then there are isomorphisms in $D(B \otimes_A C)$:

$$(C \otimes_A B) \otimes_B \mathsf{L}^{(m)}_{B\backslash A} \simeq \mathsf{L}^{(m)}_{B\otimes_A C\backslash C}$$

(1.8) Let A, B, C be as in (1.7). We have isomorphisms in $D(C)$

$$C \otimes_A \mathsf{L}\Omega^{(m)}_{B\backslash A} \simeq \mathsf{L}\Omega^{(m)}_{C\otimes_A B\backslash C}$$

We have a morphism of distinguished triangles

$$\begin{array}{ccccccc} C\otimes_A \mathsf{L}\Omega^{(m-1)}_{B\backslash A} & \to & C\otimes_A \mathsf{L}^{(m)}_{B\backslash A} & \to & C\otimes_A \mathsf{L}\Omega^{(m)}_{B\backslash A} & \to & C\otimes_A \mathsf{L}\Omega^{(m-1)}_{B\backslash A}[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathsf{L}\Omega^{(m-1)}_{C\otimes_A B\backslash C} & \to & \mathsf{L}^{(m)}_{C\otimes_A B\backslash C} & \to & \mathsf{L}\Omega^{(m)}_{C\otimes_A B\backslash C} & \to & \mathsf{L}\Omega^{(m-1)}_{C\otimes_A B\backslash C}[1] \end{array}$$

The result follows then from (1.7) by induction on m.

(1.10) Let $A \to B \to C$ be ring homomorphisms. Let P be a cofibrant factorization of $A \to B$, and Q a cofibrant factorization of the composition $P \to B \to C$. As in [22, proof of 5.1] we have exact sequence of simplicial C-modules

$$0 \to (\Omega_{P\backslash A} \otimes_P B) \otimes_B C \to \Omega_{Q\backslash A} \otimes_Q C \to \Omega_{Q\otimes_P B\backslash B} \otimes_{Q\otimes_P B} C \to 0$$

Applying the functor $\wedge^m_C(-)$ we obtain an exact sequence

$$0 \to dP \wedge \Omega^{m-1}_{Q\backslash A} \otimes_Q C \to \Omega^m_{Q\backslash A} \otimes_Q C \to \Omega^m_{Q\otimes_P B\backslash B} \otimes_{Q\otimes_P B} C \to 0$$

where the left term is the simplicial C-submodule of $\Omega^m_{Q\backslash A} \otimes_Q C$ generated by the elements $dx_1 \wedge \ldots \wedge dx_m \otimes c$ with some $x_i \in P$, and the other terms are $\mathsf{L}^{(m)}_{C\backslash A}$ and $\mathsf{L}^{(m)}_{C\backslash B}$.

In particular, we can filter $\Omega^m_{Q\backslash A} \otimes_Q C, 0 \subset F^0 \subset F^1 \subset \ldots \subset F^m = \Omega^m_{Q\backslash A} \otimes_Q C$ with $F^t = \{dx_1 \wedge \ldots \wedge dx_m \otimes c \in \Omega^m_{Q\backslash A} \otimes_Q C$ s.t. at most t elements x_i are not in $P\}$, obtaining a convergent spectral sequence:

$$E^1_{p,q} = H_{p+q}(F^p/F^{p-1}) \Rightarrow H_{p+q}(\mathsf{L}^{(m)}_{C\backslash A})$$

with $E^1_{0,q} = H_q(F^0) = H_q(\mathsf{L}^{(m)}_{B\backslash A} \otimes_B C), E^1_{m,q} = H_{m+q}(\mathsf{L}^{(m)}_{C\backslash B})$. This spectral sequence was obtained in [12]

(1.11) As a consequence of (1.7), (1.8), we obtain: let φ: $A \to B$ be a ring homomorphism, $S \subset A$ a multiplicative subset. We have isomorphisms in $D(S^{-1}A \otimes_A B)$ and $D(S^{-1}A)$ respectively

$$S^{-1}A \otimes_A \mathsf{L}^{(m)}_{B\backslash A} \simeq \mathsf{L}^{(m)}_{S^{-1}A \otimes_A B\backslash S^{-1}A}$$

$$S^{-1}A \otimes_A \mathsf{L}\Omega^{(m)}_{B\backslash A} \simeq \mathsf{L}\Omega^{(m)}_{S^{-1}A \otimes_A B\backslash S^{-1}A}$$

Also, taking $B = C = S^{-1}A$ in (1.7), (1.8) we have $\mathsf{L}^{(m)}_{S^{-1}A\backslash A} \simeq 0$, $\mathsf{L}\Omega^{(m)}_{S^{-1}A\backslash A} \simeq 0$.

If $T \subset B$ is another multiplicative subset such that $\varphi(S) \subset T$, using (1.10) for $m = 1$ (i.e. [22, 5.1]) with the sequences $A \to B \to T^{-1}B$ and $A \to S^{-1}A \to T^{-1}B$ we obtain an isomorphism in $D(T^{-1}B)$

$$\mathsf{L}^{(m)}_{B\backslash A} \otimes_B T^{-1}B \simeq \mathsf{L}^{(m)}_{T^{-1}B\backslash S^{-1}A}.$$

(1.12) Since $\mathsf{L}^{(m)}_{B\backslash A}$ (resp. $\mathsf{L}\Omega^{(m)}_{B\backslash A}$) are complexes of projective B-modules (resp. A-modules), for a B-module (resp. A-module) M we have universal coefficient spectral sequences

$$E^2_{p,q} = \operatorname{Tor}^B_p\Big(H_q(\mathsf{L}^{(m)}_{B\backslash A}), M\Big) \Rightarrow H_{p+q}(\mathsf{L}^{(m)}_{B\backslash A} \otimes_B M)$$

$$E^2_{p,q} = \operatorname{Tor}^A_p\Big(H_q(\mathsf{L}\Omega^{(m)}_{B\backslash A}), M\Big) \Rightarrow H_{p+q}(\mathsf{L}\Omega^{(m)}_{B\backslash A} \otimes_A M)$$

and similarly with $\operatorname{Hom}(-, M)$, $\operatorname{Ext}(-, M)$ instead of $-\otimes_M$, $\operatorname{Tor}(-, M)$.

2. Smooth algebras.

(2.1) Let A be a ring, B a flat A-algebra such that $B \otimes_A B$ is a noetherian ring (e.g. if A is noetherian and B an A-algebra essentially of finite type). The following are equivalent:

i) B is a smooth A-algebra

ii) $\mathsf{L}^{(m)}_{B\backslash A} \simeq \Omega^m_{B\backslash A}$ in $D(B)$ for all $m \geq 0$

iii) There exists some $p \geq 1$ such that $H_p(\mathsf{L}^{(m)}_{B\backslash A}) = 0$ for all $m \in [p+1, p+$ ext.rk.$(\Omega_{B\backslash A})]$, where ext.r.k.$(\Omega_{B\backslash A}) = \max\{n \text{ s.t. } \Omega^n_{B\backslash A} \neq 0\}$.

iv) $\mathsf{L}\Omega^{(m)}_{B\backslash A} \simeq (\Omega^m_{B\backslash A} \overset{d_{\mathrm{DR}}}{\longleftarrow} \Omega^{m-1}_{B\backslash A} \longleftarrow \ldots \longleftarrow \Omega_{B\backslash A} \overset{d_{\mathrm{DR}}}{\longleftarrow} B)$ in $D(A)$.

v) There exists some $p \geq 1$ such that $H_p(\mathsf{L}\Omega^{(m)}_{B\backslash A}) \to H_{p-1}(\mathsf{L}\Omega^{(m-1)}_{B\backslash A})$ is a monomorphism and $H_{p+1}(\mathsf{L}\Omega^{(m)}_{B\backslash A}) \to H_p(\mathsf{L}\Omega^{(m-1)}_{B\backslash A})$ is an epimorphism for all $m \in [p+1,\ p+$ ext.rk.$(\Omega_{B\backslash A})]$. These conditions are satisfied if (and this is what happens a posteriori) $H_p(\mathsf{L}\Omega^{(m)}_{B\backslash A}) = 0$ for all $m \in [p+1,\ p+$ ext.rk.$(\Omega_{B\backslash A})]$ and $H_{p+1}(\mathsf{L}\Omega^{(p+1)}_{B\backslash A}) \to H_p(\mathsf{L}\Omega^{(p)}_{B\backslash A})$ is surjective.

For the proof, i) $\Rightarrow$ ii) is in [22, 5.4.iii)] since $\Omega_{B\backslash A}$ is a projective B-module. ii) $\Rightarrow$ iii) and iv) $\Rightarrow$ v) are trivial. v) $\Rightarrow$ iii) is consequence of (1.2), ii) $\Rightarrow$ iv) can be obtained by induction on m in (1.2). Finally, iii) $\Rightarrow$ i) is in [17, corollary 6] (see [17, theorem 5] for a more general result).

Implications i) $\Rightarrow$ ii), i) $\Rightarrow$ iii) are also valid in some similar situations, e.g. for a regular homomorphism, as a consequence of [1, Suppl. 30] or [20, theorem 2.5] (=[24, theorem 1.3]) and using [11, I.4.2.2].

(2.2) Now we will study some étale descent results as in [27]. If $A \to B \to C$ are ring homomorphisms with $B \to C$ étale (of finite type), by [22, 5.1, 5.4 ii)] we have a natural homotopy equivalence $\mathsf{L}^{(m)}_{B\backslash A} \otimes_B C \simeq \mathsf{L}^{(m)}_{C\backslash A}$ for all $m \geq 0$. Also, since $B \to C$ is flat we have $H_p(\mathsf{L}^{(m)}_{B\backslash A}) \otimes_B C = H_p(\mathsf{L}^{(m)}_{C\backslash A})$. This result was obtained (at least in characteristic zero) in [12].

(2.3) Now let $A \to B \to C$ be ring homomorphisms with $B \to C$ faithfully flat and étale. By faithful flatness, the augmented Amitsur cosimplicial B-algebra

$$(2.4) \quad 0 \to B \to C \underset{s_0^0}{\overset{d_1^0 d_1^1}{\rightleftarrows}} C \otimes_B C \underset{s_1^0 s_1^1}{\overset{d_2^0 d_2^1 d_2^2}{\rightleftarrows}} C \otimes_B C \otimes_B C \ldots$$

$$d_n^i(c_0 \otimes \ldots \otimes c_{n-1}) = c_0 \otimes \ldots \otimes c_{i-1} \otimes 1 \otimes c_i \otimes \ldots \otimes c_{n-1}$$
$$s_n^i(c_0 \otimes \ldots \otimes c_{n+1}) = c_0 \otimes \ldots \otimes c_i c_{i+1} \otimes \ldots \otimes c_{n+1}$$

is exact [8, p.18].

By (2.2), applying $H_p(\mathsf{L}^{(m)}_{-\backslash A})$ to this cosimplicial B-algebra we obtain

$$(2.5) \quad 0 \to H_p(\mathsf{L}^{(m)}_{B\backslash A}) \to H_p(\mathsf{L}^{(m)}_{B\backslash A}) \otimes_B C \rightleftarrows H_p(\mathsf{L}^{(m)}_{B\backslash A}) \otimes_B C \otimes_B C \ldots$$

Since $B \to C$ is faithfully flat, (2.5) is exact if and only if (2.5) $\otimes_B C$ is. But (2.5) $\otimes_B C$ is isomorphic to $H_p(\mathsf{L}^{(m)}_{B\backslash A}) \otimes_B$ (2.4) $\otimes_B C$, and so it is exact, since (2.4) $\otimes_B C$ is homotopically trivial. Therefore $H_p(\mathsf{L}^{(m)}_{-\backslash A})$ satisfies faithfully flat étale descent.

(2.6) Now let $A \to B \to C$ be ring homomorphisms with $B \to C$ a Galois extension with group $G \subset \mathrm{Aut}_B(C)$ [13]. Then G acts on $H_p(\mathsf{L}^{(m)}_{B\backslash A} \otimes_B C)$ and on $H_p(\mathsf{L}^{(m)}_{C\backslash A})$ and we have isomorphisms

$$H_p(\mathsf{L}^{(m)}_{B\backslash A}) \simeq H_p(\mathsf{L}^{(m)}_{B\backslash A} \otimes_B C^G) \simeq H_p(\mathsf{L}^{(m)}_{B\backslash A} \otimes_B C)^G \simeq H_p(\mathsf{L}^{(m)}_{C\backslash A})^G$$

by (2.2).

(2.7) Asume now that $\frac{1}{|G|} \in A$, and so $(-)^G$ is exact. By induction on m on the homology exact sequence associated to the triangle (1.2) we obtain an isomorphism.

$$H_*(\mathsf{L}\Omega^{(m)}_{B\backslash A}) \simeq H_*(\mathsf{L}\Omega^{(m)}_{C\backslash A})^G.$$

3. Complete intersections.

In this section we shall give one result from [11, chapitre VIII]. First two definitions:

(3.1) Let B be a ring, I an ideal of B. We say that I is quasi-regular [22, 6.10] if I/I^2 is a flat B/I-module and the canonical homomorphism of graded B/I-algebras

$$\wedge^*_{B/I}(I/I^2) \to \mathrm{Tor}^B_*(B/I,\ B/I)$$

is an isomorphism. Equivalently, if I/I^2 is a flat B/I-module and $\mathsf{L}^{(1)}_{(B/I)\backslash B} \simeq I/I^2[1]$ in $D(B/I)$.

If B is noetherian this is the same as saying that I is locally generated by a regular sequence.

(3.2) Let C be a ring and $u\colon E \to F$ a C-module homomorphism. There exists an unique C-derivation d of degree $(-1,1)$ on the bigraded C-algebra $\Gamma^* E \otimes \wedge^* F$ such that $d(\gamma_k(x) \otimes 1) = \gamma_{k-1}(x) \otimes ux$, $d(1 \otimes x) = 0$. We have $d^2 = 0$. We denote the DGC-algebra $(\Gamma^* E \otimes \wedge^* F, d)$ by $\mathrm{Kos}_\bullet(u)$ [11,I.4.3.1.2]. Sometimes we regard $\mathrm{Kos}_\bullet(u)$ as a complex, the homological degree being the divided powers degree.

(3.3) [11,VIII, 2.1.2.2] Let $A \to B \to C$ be ring homomorphisms such that $B \to C$ is surjective with kernel a quasi-regular ideal I, and that the canonical morphism $\mathsf{L}^{(1)}_{B\backslash A} \to \Omega_{B\backslash A}$ is a quasi-isomorphism and $\Omega_{B\backslash A}$ is a flat B-module (e.g. if $A \to B$ is a regular homomorphism of noetherian rings and I is locally generated by a regular sequence). Then there exists a graded isomorphism in $D(C)$

$$\mathsf{L}^{(m)}_{C\backslash A} \simeq \mathrm{Kos}_\bullet(I/I^2 \to \Omega^1_{B\backslash A} \otimes_B C)$$

In order to make clear the grading, note that it induces isomorphisms

$$H_q(\mathsf{L}^{(m)}_{C\backslash A}) = H_q\Big(\Gamma_C(I/I^2) \otimes_B \Omega^*_{B\backslash A}\Big)_m$$

where this last module is a subquotient of $\Gamma^q_C(I/I^2) \otimes_B \Omega^{m-q}_{B\backslash A}$.

In particular, $H_q(\mathsf{L}^{(m)}_{C\backslash A}) = 0$ if $q > m$. A partial converse of this has been given in [26].

(3.4) EXAMPLE. Let k be a field, $B = k[x_1, \ldots, x_n]$, $f \in B$ a polynomial, $f \notin k$, and $C = B/(f)$. Then $(f)/(f)^2$ is a free C-module of rank one, and so $(\Gamma^*_C((f)/(f)^2) \otimes_B \Omega^*_{B\backslash A}, d)$ is the usual Koszul complex $K\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}; B\right)$ [14,6]. In particular, $H_q(\mathsf{L}^{(m)}_{C\backslash A}) = H_{q+i}(\mathsf{L}^{(m+i)}_{C\backslash A})$ for $q > 0$, $i \geq 0$.

Assume moreover that the characteristic of k is zero, and C is reduced. Then it is easy to prove (as in [18, 4.10]) that

$$H_q(\mathsf{L}^{(m)}_{C\backslash A}) = T(\Omega^{m-q}_{C\backslash A}) \quad \text{if } q > 0$$

where $T(\Omega^{m-q}_{C\backslash A})$ is the torsion submodule of $\Omega^{m-q}_{C\backslash A}$.

4. Local noetherian rings.

(4.1) Let (A, m, k) be a noetherian local ring. The following are equivalent:

i) A is regular

ii) $\mathsf{L}^{(m)}_{k\backslash A} \simeq \Gamma^m_k(\mathrm{m}/\mathrm{m}^2)[m]$ in $D(k)$

iii) $H_p(S^q_k \mathsf{L}^{(1)}_{k\backslash A}) = 0$ for some $p \geq 2$ and some $q \in \left[\frac{p}{2}, \frac{p+e}{2}\right]$ where $e = \dim_k \mathrm{m}/\mathrm{m}^2$ is the embedding dimension of A.

i) ⇔ ii) is clear (see e.g. [22, 6.14]). For i) ⇔ iii) see [17, Remark 8].

5. Comparison with Hochschild and cyclic homology.

(5.1) Let $R \to S$ be a ring homomorphism, and let $B^R(S)$ be the double complex $B^R(S)_{p,q} = S \otimes_R \ldots \otimes_R S(q - p + 1 \text{ times})$ [15, p.56]:

$$\begin{array}{ccccc}
\downarrow b & & \downarrow b & & \downarrow b \\
S \otimes_R S \otimes_R S & \xleftarrow{B} & S \otimes_R S & \xleftarrow{B} & S \\
\downarrow b & & \downarrow b & & \\
S \otimes_R S & \xleftarrow{B} & S & & \\
\downarrow b & & & & \\
S & & & &
\end{array}$$

The cyclic homology of $R \to S$ is $\mathrm{HC}_*^R(S) := H_*(B^R(S))$.

The homology of the first column is Hochschild homology $\mathrm{HH}_*^R(S)$. It S is R-flat, $\mathrm{HH}_*^R(S) = \mathrm{Tor}_*^{S \otimes_R S}(S, S)$.

Let $R \to X \to S$ be a cofibrant factorization, and consider the triple complex associated to $B^R(X)$, where $B^R(X)$ is the simplicial double complex obtained applying $B^R(-)$ to X dimension-wise. Denote $\overset{L}{\mathrm{HC}}{}_*^R(S) = H_*(B^R(X))$ since it does not depend on the choice of X, up to isomorphism, and $\overset{L}{\mathrm{HH}}{}_*^R(S)$ the homology of the double complex associated to $B^R(X)_{0,*}$. If S is R-flat, by Eilenberg-Zilber, $\overset{L}{\mathrm{HC}}{}_*^R(S) = \mathrm{HC}_*^R(S)$ and $\overset{L}{\mathrm{HH}}{}_*^R(S) = \mathrm{HH}_*^R(S)$.

(5.2) Filtering $B^R(X)_{0,*}$ by lines so that in the associated spectral sequence E^1 is the homology for the differential b, we obtain a convergent spectral sequence of R-modules [22, 8.1]

$$E_{p,q}^2 = H_p(\mathsf{L}_{S \backslash R}^{(q)}) \Rightarrow \overset{L}{\mathrm{HH}}{}_{p+q}^R(S)$$

Note that replacing in $B^R(X)_{0,*}$ the first factor X_i of each tensor product by S, it is easy to construct the same spectral sequence with S-module structure.

(5.3) Filtering similarly the triple complex $B^R(X)$, and having in mind that the map B induces the de Rham differential in the columns of E^1 [15, 2.3.3], we have a convergent spectral sequence

$$E_{p,q}^2 = H_p(\mathsf{L}\Omega_{S \backslash R}^{(q)}) \Rightarrow \overset{L}{\mathrm{HC}}{}_{p+q}^R(S)$$

Note that these spectral sequences take the homology exact sequence associated to (1.2) and the spectral sequence (1.5), in the Connes' exact sequence [15, 2.2.1] and the Hochschild to cyclic spectral sequence [15, 2.1.7].

(5.4) If R contains the rational numbers, we have a quasi-isomorphism (since it is a morphism inducing quasi-isomorphisms on the columns) of triple complexes

$$\beta\colon B^R(X) \to D^R(X)$$

where $D^R(X)$ is the triple complex associated to the simplicial double complex

$$\begin{array}{ccccc}
\downarrow 0 & & \downarrow 0 & & \downarrow 0 \\
\Omega^2_{X\backslash R} & \stackrel{d_{\rm DR}}{\longleftarrow} & \Omega^1_{X\backslash R} & \stackrel{d_{\rm DR}}{\longleftarrow} & X \\
\downarrow 0 & & \downarrow 0 & & \\
\Omega^1_{X\backslash R} & \stackrel{d_{\rm DR}}{\longleftarrow} & X & & \\
\downarrow 0 & & & & \\
X & & & &
\end{array}$$

given by $\beta\colon X_i \otimes \overset{(n+1)\ \text{times}}{\ldots\ldots} \otimes X_i \to \Omega^n_{X_i\backslash R}$, $\beta(x_0 \otimes \ldots \otimes x_n) = \frac{1}{n!} x_0 dx_1 \wedge \ldots \wedge dx_n$.

Therefore the spectral sequences (5.2) and (5.3) are degenerate and we have isomorphisms:

$$\bigoplus_{p+q=n} H_p(\mathsf{L}^{(q)}_{S\backslash R}) = \overset{L}{\mathrm{HH}}{}^R_n(S)$$

$$\bigoplus_{p+q=n} H_p(\mathsf{L}\Omega^{(q)}_{S\backslash R}) = \overset{L}{\mathrm{HC}}{}^R_n(S)$$

Moreover, using (1.6) and assuming for simpicity that $R \to S$ is injective, we can write

$$\overset{L}{\mathrm{HC}}{}^R_n(S) = \bigoplus_{p+q=n} H_p\left(\Omega^q_{X\backslash R}/d_{\rm DR}\Omega^{q-1}_{X\backslash R}\right) \quad \text{if } n \text{ is odd}$$

and an exact sequence

$$0 \to R \to \overset{L}{\mathrm{HC}}{}^R_n(S) \to \bigoplus_{p+q=n} H_p\left(\Omega^q_{X\backslash R}/d_{\rm DR}\Omega^{q-1}_{X\backslash R}\right) \to 0 \quad \text{if } n \text{ is even.}$$

These decompositions are well known [22], [6], [7], [2], [16], [19], [4]. They were obtained first (I think) by D. Quillen [22, 8.6] for Hochschild homology and by B.L. Feigin and B.L. Tsygan [6, lemma 10] for cyclic homology. In

[6] this decomposition are in terms of differential graded algebras, but coincides with this simplicial approach, using essentially a commutative analogue to [23, I.4] (I am grateful to A. Roig and P. Pascual for pointing out this fact to me). Later, other decompositions were obtained by using λ-operations on the Hochschild and cyclic complexes. They also coincide with the decompositions by means of DG algebras as it was shown by M. Vigué-Poirrier [25]. There is also a recent paper by M.O. Ronco (see [15, 4.5.13]) where it is shown directly that the λ-operations decomposition coincides with the simplicial one.

Note also that a decomposition exists (in any characteristic) for Hochschild homology of complete intersections [9].

(5.5) REMARK. We can define a new complex $\mathsf{L}\Gamma\Omega^{(m)}_{B\backslash A}$ similarly to $\mathsf{L}\Omega^{(m)}_{B\backslash A}$ but whose columns are the folowing: let $A \to X \to B$ be a cofibrant factorization where $X_i = S_A(V_i)$, V_i being a free A-module. Then the ith column of $\mathsf{L}\Gamma\Omega^{(m)}_{B\backslash A}$ is

$$\Gamma_A(V_i) \to \Gamma_A(V_i) \otimes_{S_A(V_i)} \Omega_{S_A(V_i)\backslash A} \to \Gamma_A(V_i) \otimes_{S_A(V_i)} \Omega^2_{S_A(V_i)\backslash A} \to \cdots$$

the differential being induced by

$$\gamma_n(v) \otimes dx_1 \dots dx_t \to \gamma_{n-1}(v) \otimes dv dx_1 \dots dx_t$$

In characteristic zero we have a canonical isomorphism $\mathsf{L}\Gamma\Omega^{(m)}_{B\backslash A} \simeq \mathsf{L}\Omega^{(m)}_{B\backslash A}$ and in general the columns of $\mathsf{L}\Gamma\Omega^{(m)}_{B\backslash A}$ have non-trivial homology only in dimension zero and m.

However, we do not consider this case since, as we say in the introduction, our purpose is to study cyclic homology, and in this new setting we would lose the spectral sequence (5.3).

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