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THE DETERMINANT IN THE LAURENT RING AND THE L^2-REIDEMEISTER-FRANZ TORSION

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Abstract.

We prove that the L^2-Reidemeister-Franz torsion of Carey and Mathai is a non-zero algebraic number for infinite cyclic coverings.

1. Introduction.

The L^2-Reidemeister-Franz torsion has been introduced by Carey and Mathai [1]. In the definition of their torsion invariant they required the weak invertibility of the combinatorial Laplacian. It essentially means that the Fuglede-Kadison determinant of the Laplacian is non-zero. See also [7].

In this paper we shall study the Fuglede-Kadison determinant of the von Neumann algebra $N\mathbf{Z}$ [4]. Our main result is that the determinant is non-zero on the non-zero elements of the $R[z, z^{-1}] \subset N\mathbf{Z}$ subring. We shall also obtain an algebraic formula for the determinant. As a consequence of this formula, we shall prove that for infinite cyclic coverings the combinatorial Laplacian is always weakly invertible if it has trivial kernel. We shall also see that the L^2-Reidemeister-Franz torsion is an algebraic number. For the sake of completeness, we recall some basic facts about the combinatorial L^2-theory.

Let K be a finite simplicial complex of dimension n and let $\tilde{K}$ be a normal simplicial covering of K, with the group Γ of deck transformations. $|K|_p$ resp. $|\tilde{K}|_p$ stands for the set of p-simplices in K resp. in $\tilde{K}$. Let $C^p_{(2)}$ denote the Hilbert space of square-integrable real-valued oriented p-cochains on $\tilde{K}$. The scalar product on $C^p_{(2)}$ is given by

$$\langle f, g \rangle = \sum_{\sigma \in |\tilde{K}|_p} f(\sigma) g(\sigma).$$

Note that for an oriented p-cochain: $f(-\sigma) = -f(\sigma)$. The combinatorial Laplace operator $\Delta_p = d_p^* d_p + d_{p-1} d_{p-1}^*$ is a bounded, non-negative, self-adjoint operator on the Hilbert space $C^p_{(2)}$ [3].

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Let $D_p(\sigma,\tau)$ be the operator kernel of Δ_p, i.e.

(1) $$\Delta_p(f)(\sigma) = \sum_{\tau\in|\tilde{K}|_p} D_p(\sigma,\tau)f(\tau)$$

for any $\sigma\in|\tilde{K}|$ and $f\in C^p_{(2)}$.

The Γ-invariant bounded operators on $C^p_{(2)}$ form a von Neumann algebra, which has a finite trace Tt_Γ such that $\mathrm{Tr}_\Gamma(\mathrm{Id}) = 1$ [1]. Then,

$$\mathrm{Tr}_\Gamma(\Delta_p) = \sum_{\sigma\in|\tilde{K}|_p} D_p(\tilde{\sigma},\tilde{\sigma})$$

where $\pi(\tilde{\sigma}) = \sigma$ for the $\pi\colon \tilde{K}\to K$ covering map. For any $\tilde{\sigma}\in|\tilde{K}|_p$, $\chi_{\tilde{\sigma}}$ shall denote the characteristic function of $\tilde{\sigma}$.

In this paper we study only acyclic complexes i.e. for which $\ker\Delta_p = 0$, for any p. For an acyclic complex, the Laplacian is called weakly invertible [1] [5] if

$$\int_0^\infty \log\lambda\, d\phi_\lambda > -\infty,$$

where $\Delta_p = \int_p^\infty \lambda\, dE_\lambda$ is the spectral representation of Δ_p and $\phi_\lambda = \mathrm{Tr}_\Gamma E_\lambda$. In this case the Fuglede-Kadison determinant of the Laplacian is

$$|\mathrm{Det}|_\Gamma(\Delta_p) = \exp\left(\int_0^\infty \log\lambda\, d\phi_\lambda\right).$$

The L^2-Reidemeister-Franz torsion is defined as

$$T_{(2)}(\tilde{K}) = \prod_{j=0}^{n} (|\mathrm{Det}|_\Gamma(\Delta_j))^{\frac{j(-1)^{j+1}}{2}}.$$

REMARK. Lück and Rothenberg [6] defined the Reidemeister-von Neumann torsion as an element of $K_1^w(N\Gamma)^{Z/2}$. They observed that if the Laplacians are weakly invertible, the L^2-Reidemeister-Franz torsion is just the Fuglede-Kadison determinant of the Reidemeister-von Neumann torsion.

2. The representation of the Laplacian.

Let g be a greater of Z and let $\sigma_1,\sigma_2,\dots,\sigma_k$ be p-simplices, such that any p-simplex in $\tilde{K}$ can be uniquely represented as $g^{a_i}\cdot\sigma_i$, where $a_1\in\mathsf{Z}$, $1\leq i\leq k$ and $g^{a_i}\cdot\sigma_i$ denotes the left action of Z on $\tilde{K}$. Note that the Laurent ring $\mathsf{R}[z,z^{-1}]$ is Z-isomorphic to the group algebra RZ with the group action $g^m\cdot[f(z)] = z^m f(z)$. Let us denote by $H[z]$ the space of Laurent series $\sum_{i\in\mathsf{Z}} a_i z^i$ such that $\sum_{i\in\mathsf{Z}} |a_i|^2 < \infty$. Then $H[z]^k$ is Z-isomorphic to $C^p_{(2)}$ via the following Z-isomorphism ψ,

$$\psi(f_1(z), f_2(z), \ldots, f_k(z))(g^j \cdot \sigma_i) = a_{ij},$$

where $f_i(z) = \sum_{j \in Z} a_{ij} z^j$.

The trace τ on $\mathsf{R}[z, z^{-1}]$ is defined as $\tau(\sum_{k=-\infty}^{\infty} c_k z^k) = c_0$. It gives us an inner product on $\mathsf{R}[z, z^{-1}]$ by $\langle f, g \rangle = \tau(f(z) \cdot g(z^{-1}))$.

We can extend the inner product to $H[z]$, which becomes the Hilbert closure of the pre-Hilbert space $\mathsf{R}[z, z^{-1}]$. Note that for $f(z) = \sum_{i \in Z} a_i z^i$, $\|f\|^2 = \sum_{i \in Z} |a_i|^2$ and ψ is an isometry from $H[z]^k$ to $C^p_{(2)}$. The inner product defines a $*$-structure on $\mathrm{Mat}_k(\mathsf{R}[z, z^{-1}])$ by $L^*(z) = L^T(z^{-1})$, where L^T stands for the usual matrix transposition.

DEFINITION 2.1. *For any $A \in \mathrm{Mat}_k(H[z])$ we have a densely defined* Z*-invariant operator $\Psi(A)$ on $C^p_{(2)}$.*

$$\Psi(A)(\chi_{g^m \cdot \sigma_j}) = \sum_{i,l} a_{ij}^{(l-m)} \chi_{g^l \cdot \sigma_i}$$

where $(A)_{ij} = \sum_{n \in Z} a_{ij}^{(n)} z^n$ is the i, j-entry of A.

The bounded Z-invariant operators on $C^p_{(2)}$ form a von Neumann algebra. This von Neumann algebra is isomorphic to $\mathrm{Mat}_k(NZ)$ where NZ is the von Neumann algebra of Z. The composition of the matrix trace and the type-II trace on NZ gives us the trace Tr_Z on $\mathrm{Mat}_k(NZ)$. Let us list some results on the representation Ψ [2], [6].

PROPOSITION 2.2. 1. *$\Psi(A)$ is bounded, if and only, each A_{ij} converges almost everywhere to a bounded function on the unit circle.*

2. *If $\Psi(A)$ and $\Psi(B)$ are bounded operators;*

a. $\Psi^*(A) = \Psi(A^*)$.

b. $\Psi(AB) = \Psi(A)\Psi(B)$, $\Psi(A + B) = \Psi(A) + \Psi(B)$.

c. *For any $v \in H[z]^k$, $\psi(A(v)) = \Psi(A)(\psi(v))$.*

d. $\tau(\mathrm{tr}(A)) = \mathrm{Tr}_Z(\Psi(A))$, *where* tr *is the matrix trace.*

3. *If $A = A^*$ non-negative definite on the unit circle, then*

a. *$\Psi(A)$ is a self-adjoint non-negative operator.*

b. Ker $\Psi(A) = 0$ *if and only if* $\det A \neq 0$, *where* det *is the matrix determinant.*

Note that we can define $A_p \in \mathrm{Mat}_k(R[z, z^{-1}])$ such that $\Psi(A_p) = \Delta_p$, the following way.

$$(A_p)_{ij} = \sum_{k \in Z} a_{ij}^{(k)} z^k,$$

where, $a_{ij}^{(k)} = D_p(\sigma_i, g^{-k} \sigma_j)$. By Proposition 2.2, if $\mathrm{Ker}(\Delta_p) = 0$, then $\det A_p \neq 0$.

3. The Fuglede-Kadison determinant on the Laurent ring.

First, let us recall the lemma on "the continuity of the roots".

LEMMA 3.1. *Let $U \subset \mathbf{C}$ be an open set and let $\{f_n\}_{n=1,2,\dots}$ be holomorphic functions on U. Suppose that $\{f_n\}_{n=1,2,\dots}$ converges uniformly to a function $f \neq 0$. Let $W \subset \bar{W} \subset U$ be an other open set and $p \in W$. Suppose that p is the only zero of f in U and p has multiplicity k. Then for sufficiently large n, f_n has exactly k zeroes (with multiplicity) in W.*

Let S denote the set of all non-zero $f \in R[z, z^{-1}]$, which has real and non-negative value on the unit circle. Then, for any $f \in S$, $f(z) = f(z^{-1})$. Therefore, if $p \neq \pm 1$ is a zero of f. then p^{-1} is also a zero of f, with the same multiplicity.

LEMMA 3.2. *Let $f \in S$. Suppose $p = \pm 1$ is a zero of f with multiplicity k. Then k is an even number.*

PROOF. Let $f_n = f + \frac{1}{n}$, $n \in \mathbf{N}$. Choose a U open set, such that p is the only zero of f in U. Let $W \subset \bar{W} \subset U$ be a symmetric neighbourhood of p i.e. if $z \in W$, then $z^{-1} \in W$, too. Since f_n is positive on the unit circle for all n, by our previous remark, f_n has even amount of roots in W. $\lim_{n \to \infty} f_n = f$, therefore the lemma follows from the "continuity of roots".

PROPOSITION 3.3. *For any $f \in S$, we can construct $F \in \mathbf{C}[z]$ such that*

$$f(z) = F(z)F(z^{-1})$$

and F has no zeroes inside the unit circle.

PROOF. Let $f(z) = \lambda \frac{1}{z^l} \prod_{i=1}^{k} (z - a_i)$; $|a_i| > 0$. Then,

$$\lambda \frac{1}{z^l} \sum_{i=1}^{k} (z - a_i) = \lambda z^l \sum_{i=1}^{k} (z^{-1} - a_i)$$

$$= \lambda z^l \sum_{i=1}^{k} (z - a_i^{-1})(-a_i z^{-1}) = \lambda \frac{z^l}{z^k} \left((-1)^k \prod_{i=1}^{k} a_i \right) \prod_{i=1}^{k} (z - a_i^{-1}).$$

Hence, we can conclude that $k = 2l$. Let us define $F(z)$ by

$$F(z) = \sqrt{\lambda} \prod_{j=1}^{m} (z - a_j) \sqrt{\prod_{j=1}^{m} (-a_j)^{-1}}$$

where the product is taken the following way. Any root of f outside the unit circle is counted with its full multiplicity. If ± 1 is a root, it is counted with half of its multiplicity. For roots w on the unit circle such that $w \neq \pm 1$, we choose one from the (w, w^{-1}) pair with full multiplicity. Obviously, $l = m = \frac{1}{2}k$.

$$F(z)F(z^{-1}) = \lambda \prod_{j=1}^{m} (z - a_j) \prod_{j=1}^{m} (z^{-1} - a_j) \prod_{j=1}^{m} (-a_j)^{-1}$$

$$= \lambda \prod_{j=1}^{m} (z - a_j) \prod_{j=1}^{m} (z - a_j^{-1}) z^{-m} = \lambda \frac{1}{z^l} \prod_{i=1}^{k} (z - a_i) = f(z).$$

The construction above does not give us a unique polynomial $F(z)$. However, $|F(0)|$ does not depend on the choice of the roots on the unit circle and the choice of the square root.

DEFINITION 3.4. *For* $f \in S$,

$$\mathrm{Det}_L(f) = |F(0)|^2 \in \mathbb{R}^+.$$

Note, $\mathrm{Det}_L(f) = |\lambda| \cdot \prod |a_i|$, where the product is taken for the roots outside the unit circle with their full multiplicity. Therefore, by the continuity of the roots,

$$\mathrm{Det}_L(f) = \lim_{n \to \infty} \mathrm{Det}_L\left(f + \frac{1}{n}\right). \tag{2}$$

Note that Det_L can be extended to all non-zero element of the Laurent ring by

$$\mathrm{Det}_L(f) = \sqrt{\mathrm{Det}_L(ff^*)}$$

LEMMA 3.5. *If* $f \in S$ *is strictly positive on the unit circle, then*

$$\tau(\log f) = \log \mathrm{Det}_L(f)$$

PROOF. By Proposition 3.3, $f(z) = F(z)F(z^{-1})$, where F has no zeroes in a neighbourhood of the unit disc. Therefore, we can define $\log F$ in a neighbourhood of the unit disc. Let us consider the Taylor-series of $\log F$.

$$\log F(z) = \log(F(0)) + c_1 z + c_2 z^2 + \dots$$

Then, $\log(F(0)) + c_1 z^{-1} + c_2 z^{-2} + \dots$ is the Laurent series of $\log F(z^{-1})$ that converges in a neighbourhood of the unit circle, f is positive on the unit circle, thus we can also define $\log f$ in a neighbourhood of the unit circle and we can suppose that it has real value on the unit circle. Obviously,

$$\log f(z) = \log F(z) + \log F(z^{-1}) + 2\pi i s,$$

where $s \in \mathbb{Z}$. Hence,

$$\tau(\log(f)) = 2\log(F(0)) + 2\pi i s.$$

Let us note that $\tau(\log f)$ is real, therefore

$$\tau(\log(f)) = \operatorname{Re}\log(F^2(0)) = \log \mathrm{Det}_L(f)$$

Now, we can state our main theorem.

THEOREM 1. *Let* $B \in \mathrm{Mat}_k(R[z, z^{-1}])$, *such that* $\det B \neq 0$. *Then,*

$$|\mathrm{Det}|_\Gamma \Psi(B) = \mathrm{Det}_\Gamma(\det B).$$

PROOF. We may suppose that B is non-negative definite on the unit circle. Let $B_n = B + \frac{1}{n}$. By the definition of the Fuglede-Kadison determinant,

$$|\mathrm{Det}|_\Gamma \Psi(B) = \exp \int_{[0,\infty]} \log \lambda \, d\phi_\lambda$$

where $\int_{(0,\infty]} \lambda \, dE_\lambda$ is the spectral decomposition of $\Psi(B)$ and $\phi_\lambda = \mathrm{Tr}_Z E_\lambda$. Therefore,

$$|\mathrm{Det}|_\Gamma \Psi(B) = \lim_{n\to\infty} \exp \int_{[0,\infty]} \log\left(\lambda + \frac{1}{n}\right) d\phi_\lambda = \lim_{n-\infty} \exp(\mathrm{Tr}_Z(\log \Psi(B_n))).$$

By Proposition 2.2, $\log \Psi(B_n) = \Psi(\log B_n)$ and $\mathrm{Tr}_Z(\Psi(\log(B_n)) = \tau(\mathrm{tr}(\log B_n))$. On the other hand, $\mathrm{tr}(\log B_n) = \log(\det B_n)$. Hence, by Lemma 3.5,

$$|\mathrm{Det}|_\Gamma \Psi(B) = \lim_{n\to\infty} \exp \tau(\log(\det B_n)) = \lim_{n\to\infty} \mathrm{Det}_\Gamma(\det B_n).$$

by (2), $\lim_{n\to\infty} \mathrm{Det}_L(\det B_n) = \mathrm{Det}_L(\det B)$. Therefore,

$$|\mathrm{Det}|_\Gamma \Psi(B) = \mathrm{Det}_L(\det B)$$

When we apply our theorem to the Laplacians Δ_p, we obtain the following corollaries.

COROLLARY 3.6. *If* $\mathrm{Ker}\, \Delta_p = 0$, *then* $|\mathrm{Det}|_\Gamma \Delta_p = \mathrm{Det}_L(\det A_p) \neq 0$. *Thus* Δ_p *is weakly acyclic.*

COROLLARY 3.7. *The* L^2*-Reidemeister-Franz torsion of an acyclic* Z*-covering is always a non-zero algebraic number.*

REFERENCES

1. A. Carey and V. Mathai, *L^2-Torsion invariants*, J. Funct. Anal. 110 (1992), 377–409.
2. J. Dixmier, *von Neumann Algebras*, North-Holland, Amsterdam (1981).
3. J. Dodziuk, *de Rham-Hodge theory for L^2-cohomology of infinite coverings*, Topology 16 (1977), 157–165.

4. R. Fuglede and R. V. Kadison, *Determinant theory in finite factors*, Ann. of Math. 55 (1952), 520–530.
5. W. Lück, *Approximating, L^2-invariants by their finite dimensional analogues*, preprint 1993.
6. W. Lück and M. Rothenberg, *Reidemeister torsion and the K-Theory on von Neumann Algebras*, K-Theory 5 (1991), 215–264.
7. S. P. Novikov and M. A. Shubin, *Morse inequalities and von Neumann* II_1 *Factors*, Dokl. Akad. Nauk. Sssr 289 (1986), 289–292.

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