MATH. SCAND. 79 (1996), 209–223

ORTHOGONAL POLYNOMIALS, L^2-SPACES AND ENTIRE FUNCTIONS.

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Abstract.

We show that for determinate measures μ having moments of every order and finite index of determinacy, (i.e., a polynomial p exists for which the measure $|p|^2\mu$ is indeterminate) the space $L^2(\mu)$ consists of entire functions of minimal exponential type in the Cartwright class.

1. Introduction.

Let $\mathscr{M}^*$ denote the set of positive Borel measures on the real line having moments of every order and infinite support. We are interested in finding conditions on $\mu \in \mathscr{M}^*$ such that $L^2(\mu)$ consists of entire functions in the following sense: (i) There exists a continuous linear injection $E : L^2(\mu) \to \mathscr{H}(\mathbf{C})$, where $\mathscr{H}(\mathbf{C})$ denotes the set of entire functions with the topology of compact convergence. (ii) For all $f \in L^2(\mu)$ we have $E(f) = f$ μ-a.e.. We say that E is a *realization* of $L^2(\mu)$ as entire functions. In the discussion of this problem we need for $\mu \in \mathscr{M}^*$ the corresponding sequence of orthonormal polynomials (p_n). It is uniquely determined by the orthonormality condition and the requirement that p_n is a polynomial of degree n with positive leading coefficient. The sequence of orthonormal polynomials depends only on the moments of μ, so if μ is *indeterminate*, i.e. there are other measures having the same moments as μ, all these measures lead to the same sequence (p_n).

When the measure μ is indeterminate, the Fourier expansion of $f \in L^2(\mu)$

$$\sum_{n=0}^{\infty}\left(\int f(t)p_n(t)d\mu(t)\right)p_n(z) \tag{1.1}$$

converges in $L^2(\mu)$ and uniformly on compact subsets of $\mathbf{C}$ to an entire function $F(f)(z)$, which is the orthogonal projection of f onto the closure in $L^2(\mu)$ of the set $\mathbf{C}[t]$ of polynomials. We recall that $z \mapsto (p_n(z))_n$ is an entire

The work of the second author was supported by DGICYT ref. PB 93–0926.
Received April 6, 1995.

function with values in the Hilbert space ℓ^2, so in particular $(p_n^{(m)}(z))_n \in \ell^2$ for all $z \in \mathbf{C}$, $m \in \mathbf{N}$, cf. [BD1]. By a theorem of M. Riesz ([R], [A]) $F(f)$ is of minimal exponential type. If the indeterminate measure μ is *Nevanlinna extremal* (*N-extremal* in short), which means that $\mathbf{C}[t]$ is dense in $L^2(\mu)$, then μ is discrete and $F(f)(x) = f(x)$ for $x \in \operatorname{supp}(\mu)$. This means that $F(f)$ is an extension of f to an entire function of minimal exponential type.

Furthermore $f \mapsto F(f)$ is a continuous injection of $L^2(\mu)$ into $\mathscr{H}(\mathbf{C})$. In fact, for any compact set $K \subseteq \mathbf{C}$ we find by (1.1) and Parsevals formula

$$\sup_{z\in K} |F(f)(z)| \leq \|f\|_2 \sup_{z\in K} \rho(z),$$

where

$$\rho(z) = \left(\sum_{k=0}^{\infty} |p_k(z)|^2\right)^{\frac{1}{2}}$$

is continuous. Riesz ([R]) also showed that

$$\int_{-\infty}^{\infty} \frac{\log \rho(t)}{1+t^2}\, dt < \infty,$$

and it follows that

$$\int_{-\infty}^{\infty} \frac{\log^+ |F(f)(t)|}{1+t^2}\, dt < \infty.$$

For a survey of the interplay between entire functions and indeterminate moment problems see [B].

In the following we denote by $\mathscr{C}_0$ the class of entire functions f of minimal exponential type satisfying

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(t)| dt}{1+t^2} < \infty.$$

It is the functions in the Cartwright class which are of minimal exponential type.

In the case of an N-extremal measure μ we have thus seen that $L^2(\mu)$ consists of entire function of class $\mathscr{C}_0$. The function $F(f)$ given by (1.1) will be called the *canonical extension* of f.

The purpose of the present paper is to establish that also for certain determinate measures $\mu \in \mathscr{M}^*$ the space $L^2(\mu)$ consists of entire functions. A determinate measure μ with this property must necessarily be discrete, as we shall see below. It turns out that $L^2(\mu)$ consists of entire functions of class $\mathscr{C}_0$, if μ is a determinate measure of finite index, meaning that there exists a polynomial p such that the measure $|p|^2\mu$ is indeterminate. If k is the smallest possible degree of a polynomial p such that $|p|^2\mu$ is indeterminate, then $k-1$ is the index of μ. This concept was studied in previous papers of the authors, cf. [BD1], [BD2].

In the case of an N-extremal measure μ the canonical extension $F(f)$ of $f \in L^2(\mu)$ has the additional property that $F(p)(z) = p(z)$ for all $z \in \mathsf{C}$, when p is a polynomial. We shall see that this property cannot subsist in the determinate case. It will be replaced by a condition which involves discrete differential operators of the form

$$T = \sum_{l=1}^{N} \sum_{j=0}^{k_l} a_{l,j} \delta_{z_l}^{(j)}, \quad a_{l,j} \in \mathsf{C} \tag{1.2}$$

associated to a system (z_i, k_i), $i = 1, \ldots, N$ of mutually different points $z_i \in \mathsf{C}$ and multiplicities $k_i \in \mathsf{N}$. These operators act on entire functions F via the formula

$$T(F) = \sum_{l=1}^{N} \sum_{j=0}^{k_l} a_{l,j} F^{(j)}(z_l).$$

It is well-known that any T of the form (1.2) has a unique continuous extension from $\mathsf{C}[t]$ to $L^2(\mu)$ if μ is N-extremal. This extension $\widetilde{T}$ satisfies

$$\widetilde{T}(f) = T(F(f)), \; f \in L^2(\mu), \tag{1.3}$$

where $F(f)$ is the canonical extension of $f \in L^2(\mu)$. In fact, if $(q_n) \in \mathsf{C}[t]$ converges in $L^2(\mu)$ to $f \in L^2(\mu)$ then $q_n = F(q_n)$ converges in $\mathscr{H}(\mathsf{C})$ to $F(f)$ and hence $\lim_{n\to\infty} T(q_n) = T(F(f))$. We notice that $(T(p_n)) \in \ell^2$, and if $f \in L^2(\mu)$ has the Fourier expansion $\sum c_n p_n$ then

$$\widetilde{T}(f) = \sum_{n=0}^{\infty} c_n T(p_n). \tag{1.4}$$

If μ is determinate then T given by (1.2) has a (unique) continuous extension from $\mathsf{C}[t]$ to $L^2(\mu)$ if and only if $(T(p_n)) \in \ell^2$. Although $(p_n(z)) \notin \ell^2$ for $z \notin \operatorname{supp}(\mu)$, it is possible to characterize the differential operators T for which $(T(p_n)) \in \ell^2$. This was done in [BD2]. For determinate measures μ of finite index there are "many" of these operators, see below, and we shall prove the following:

THEOREM 1.1. *Let μ be a determinate measure of finite index. Then $L^2(\mu)$ consists of entire functions of class $\mathscr{C}_0$ via a continuous linear injection $E : L^2(\mu) \to \mathscr{H}(\mathsf{C})$ with the additional property that*

$$\widetilde{T}(f) = T(E(f)) \tag{1.5}$$

for all $f \in L^2(\mu)$ and all operators T of the form (1.2) for which $(T(p_n)) \in \ell^2$.

A realization $f \mapsto E(f)$ satisfying (1.5) is not uniquely determined. We give several different realizations, and to complete the paper, we characterize for given $f \in L^2(\mu)$ the set of entire functions F satisfying

$$\widetilde{T}(f) = T(F)$$

for all operators T such that $(T(p_n)) \in \ell^2$. All these functions F turn out to be of class $\mathscr{C}_0$.

2. Preliminary results.

As claimed in the introduction it imposes severe restrictions on a determinate measure μ, if $L^2(\mu)$ consists of entire functions.

PROPOSITION 2.1. *Let $\mu \in \mathscr{M}^*$ be determinate and assume that $E : L^2(\mu) \to \mathscr{H}(\mathbf{C})$ is a realization of $L^2(\mu)$ as entire functions. Then μ is a discrete measure, and for each $z \in \mathbf{C} \setminus \operatorname{supp}(\mu)$ there exists $p \in \mathbf{C}[t]$ such that $p(z) \neq E(p)(z)$.*

PROOF. If the support S of μ is non-discrete we can choose $x_0 \in S$ and a compact subset $F \subseteq S \setminus \{x_0\}$ having accumulation points. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function with compact support vanishing on F and such that $f(x_0) = 1$. The extension $E(f)$ of f to an entire function must necessarily vanish identically, but this is a contradiction.

For a discrete determinate measure μ it is known that $\sum |p_n(z)|^2 = \infty$ for all $z \notin \operatorname{supp}(\mu)$. Fix $z \notin \operatorname{supp}(\mu)$ and let us assume that the realization E has the property $E(p)(z) = p(z)$ for all $p \in \mathbf{C}[t]$. We define a sequence S_n of continuous linear functionals on ℓ^2 by

$$S_n(c) = \sum_{k=0}^{n} c_k p_k(z), \quad c = (c_n) \in \ell^2.$$

For any $c \in \ell^2$ there exists $f \in L^2(\mu)$ such that

$$\sum_{k=0}^{n} c_k p_k \to f \quad \text{in} \quad L^2(\mu),$$

and hence

$$S_n(c) = E\left(\sum_{k=0}^{n} c_k p_k\right)(z) \to E(f)(z).$$

By the Banach-Steinhaus Theorem this implies that

$$\sup_n \|S_n\| = \left(\sum_0^\infty |p_k(z)|^2\right)^{\frac{1}{2}} < \infty,$$

which is a contradiction.

The determinate measures of finite index are discrete, and we shall realize $L^2(\mu)$ as entire functions for this class of measures.

The index of determinacy of a determinate measure μ was introduced and studied by the authors in [BD1]. This index checks the determinacy under multiplication by even powers of $|t-z|$ for z a complex number, and it is defined as

$$\operatorname{ind}_z(\mu) = \sup\{k \in \mathbb{N} \mid |t-z|^{2k}\mu \text{ is determinate}\}. \tag{2.1}$$

Using the index of determinacy, determinate measures can be classified as follows:

If μ is constructed from an N-extremal measure by removing the mass at $k+1$ points in the support, then μ is determinate with

$$\operatorname{ind}_z(\mu) = \begin{cases} k, & \text{for } z \notin \operatorname{supp}(\mu) \\ k+1, & \text{for } z \in \operatorname{supp}(\mu). \end{cases} \tag{2.2}$$

For an arbitrary determinate measure μ the index of determinacy is either infinite for every z, or finite for every z. In the latter case the index has the form (2.2), and μ is derived from an N-extremal measure by removing the mass at $k+1$ points. Such an N-extremal measure is highly non-unique by a perturbation result of Berg and Christensen, cf. [BC, Theorem 8].

Using that the index of determinacy is constant at complex numbers outside of the support of μ, we define the index of determinacy of μ by

$$\operatorname{ind}(\mu) := \operatorname{ind}_z(\mu), \quad z \notin \operatorname{supp}(\mu). \tag{2.3}$$

We stress that a measure μ of finite index is discrete and $\operatorname{ind}(\mu)+1$ is the smallest degree of a polynomial p such that $|p|^2\mu$ is indeterminate.

To each measure μ which is either N-extremal or determinate of finite index we associate an entire function F_μ with simple zeros at the points of $\operatorname{supp}(\mu)$. We recall from [BD1] that

$$F_\mu(w) = \exp\left(-w\sum_{n=0}^\infty \frac{1}{x_n}\right)\prod_{n=0}^\infty\left(1-\frac{w}{x_n}\right)\exp\left(\frac{w}{x_n}\right), \tag{2.4}$$

where $\{x_n : n \in \mathbb{N}\}$ is the support of μ. This function F_μ is the uniquely determined entire function of minimal exponential type having $\operatorname{supp}(\mu)$ as its set of zeros and satisfying $F_\mu(0) = 1$. In the above formulation we tacitly assume $0 \notin \operatorname{supp}(\mu)$. If however $0 \in \operatorname{supp}(\mu)$, the above expression for F_μ shall be multiplied with w and $\{x_n : n \in \mathbb{N}\} = \operatorname{supp}(\mu) \setminus \{0\}$.

That F_μ is of minimal exponential type follows by a theorem of M. Riesz [R], according to which the entire functions in the Nevanlinna matrix for an indeterminate moment problem are of minimal exponential type. The function F_μ is also in the Cartwright class.

THEOREM 2.2. *Let μ be N-extremal. For each $f \in L^2(\mu)$ we have*

$$F(f)(z) = \sum_{x \in \operatorname{supp}(\mu)} \frac{F_\mu(z)}{F'_\mu(x)(z-x)} f(x)\,, \quad z \in \mathbb{C},$$

where the series converges uniformly on compact subsets of $\mathbb{C}$.

PROOF. Without loss of generality we may assume that $0 \in \operatorname{supp}(\mu)$, so F_μ is proportional to the function D from the Nevanlinna matrix, cf. [A], and it is well known that

$$\sum_{n=0}^{\infty} p_n(z)p_n(x) = \frac{B(z)D(x) - B(x)D(z)}{z-x},$$

cf. [BD1], [BuCa], where

$$B(z) = -1 + z\sum_{n=0}^{\infty} q_n(0)p_n(z).$$

Here (q_n) denotes the sequence of polynomials of the second kind given by

$$q_n(z) = \int \frac{p_n(z) - p_n(x)}{z-x}\, d\mu(x).$$

Since D vanishes on $\operatorname{supp}(\mu)$ we get

$$F(f)(z) = \int \left(\sum_{n=0}^{\infty} p_n(z)p_n(x) \right) f(x)\,d\mu(x) = -D(z) \int \frac{B(x)f(x)}{z-x}\, d\mu(x),$$

and

$$\frac{B(x)f(x)}{z-x} = -\frac{f(x)}{z-x} + \frac{xf(x)}{z-x} \sum_{n=0}^{\infty} q_n(0)p_n(x)$$

belongs to $L^1(\mu)$ because $\sum q_n(0)p_n(x) \in L^2(\mu)$.

The mass at $x \in \operatorname{supp}(\mu)$ is given by ([A, p. 114])

$$\mu(\{x\}) = \frac{-1}{B(x)D'(x)}$$

showing that

$$F(f)(z) = \sum_{x \in \mathrm{supp}(\mu)} \frac{D(z)}{D'(x)(z-x)} f(x)$$

and the series converges uniformly on compact subsets of C. Since D and F_μ are proportional the result follows.

From Theorem 2.2 it is easy to verify that the realization $F(L^2(\mu))$ is a Hilbert space of entire functions in the sense of de Branges, see [Br, p. 57]. For details see Corollary 3.3 below.

In [BD2] we obtained the following result:

THEOREM 2.3. *Let $\mu \in \mathscr{M}^*$ be determinate and let (p_n) be the sequence of orthonormal polynomials corresponding to μ. Let $(z_1, k_1), \ldots, (z_N, k_N)$ be given, where the z's are different complex numbers and the k's are nonnegative integers. Putting $M = \sum_{l=1}^{N}(k_l + 1)$ and*

$$\mathscr{T} = \{T = \sum_{l=1}^{N} \sum_{j=0}^{k_l} a_{l,j} \delta_{z_l}^{(j)} \mid a_{l,j} \in \mathbf{C}\},$$

we have

(i) *If*

$$\mathrm{ind}(\mu) \geq \left(\sum_{l:\mu(\{z_l\})>0} k_l + \sum_{l:\mu(\{z_l\})=0} (k_l+1) \right) - 1,$$

then the sequence $(T(p_n))$ belongs to ℓ^2 only in the trivial cases, i.e., if and only if T is a linear combination of Dirac deltas evaluated at points z_l which are mass points of the measure μ.

(ii) *If*

$$0 \leq \mathrm{ind}(\mu) \leq \left(\sum_{l:\mu(\{z_l\})>0} k_l + \sum_{l:\mu(\{z_l\})=0} (k_l+1) \right) - 2,$$

then,

$$\dim\{T \in \mathscr{T} \mid (T(p_n)) \in \ell^2\} = M - \mathrm{ind}(\mu) - 1 \geq 1.$$

Furthermore, $(T(p_n)) \in \ell^2$ if and only if $T(z^k F_\mu(z)) = 0$ for $k = 0, 1, \ldots, \mathrm{ind}(\mu)$.

COROLLARY 2.4. *Let $\mu \in \mathcal{M}^*$ be a determinate measure of finite index. For an operator $T \in \mathcal{T}$ we have $(T(p_n)) \in \ell^2$ if and only if $T(z^k F_\mu(z)) = 0$ for* $k = 0, 1, \ldots, \operatorname{ind}(\mu)$.

PROOF. It is enough to consider the case (i), and to prove that the equations $T(z^k F_\mu(z)) = 0$ for $k \le \operatorname{ind}(\mu)$ imply that T is a linear combination of Dirac deltas at mass points of μ. To simplify the notation we assume that the system is ordered such that there exist positive integers $0 \le N_1 \le N_2 \le N$ for which

$$\begin{cases} \mu(\{z_l\}) > 0 \text{ and } k_l = 0 \text{ for } l = 1, \ldots, N_1 \\ \mu(\{z_l\}) > 0 \text{ and } k_l > 0 \text{ for } l = N_1 + 1, \ldots, N_2 \\ \mu(\{z_l\}) = 0 \text{ for } l = N_2 + 1, \ldots, N . \end{cases}$$

Using $F_\mu(z_l) = 0$ for $l = 1, \ldots, N_2$, the equations $T(z^k F_\mu(z)) = 0$ can be written

$$\sum_{l=N_1+1}^{N_2} \sum_{j=1}^{k_l} a_{l,j} \delta_{z_l}^{(j)}(z^k F_\mu(z)) + \sum_{l=N_2+1}^{N} \sum_{j=0}^{k_l} a_{l,j} \delta_{z_l}^{(j)}(z^k F_\mu(z)) = 0.$$

This system has

$$p := \sum_{l=N_1+1}^{N_2} k_l + \sum_{l=N_2+1}^{N} (k_l + 1)$$

variables $a_{l,j}$ and $\operatorname{ind}(\mu) + 1$ equations, and $p \le \operatorname{ind}(\mu) + 1$ since we consider the case (i). We claim that the system of equations with $k \le p - 1$ ($\le \operatorname{ind}(\mu)$) has a non-singular matrix, and therefore the variables involved are 0, i.e.

$$T = \sum_{l=1}^{N_2} a_{l,0} \delta_{z_l} .$$

The columns of the matrix can be put together in blocks

$$\left\{ \delta_{z_l}^{(j)}(z^k F_\mu(z)) \right\}_{\substack{k=0,\ldots,p-1, \\ j=1,\ldots,k_l}} \quad l = N_1 + 1, \ldots, N_2$$

and

$$\left\{ \delta_{z_l}^{(j)}(z^k F_\mu(z)) \right\}_{\substack{k=0,\ldots,p-1, \\ j=0,\ldots,k_l}} \quad l = N_2 + 1, \ldots, N.$$

Since $F_\mu(z_l) = 0$, $F'_\mu(z_l) \ne 0$ for $l = N_1 + 1, \ldots, N_2$ and $F_\mu(z_l) \ne 0$ for $l = N_2 + 1, \ldots, N$, column operations show that these blocks are equivalent to the blocks

$$\left\{\delta_{z_l}^{(j)}(z^k)\right\}_{\substack{k=0,\dots,p-1\\ j=0,\dots,k_l-1}},\quad \left\{\delta_{z_l}^{(j)}(z^k)\right\}_{\substack{k=0,\dots,p-1\\ j=0,\dots,k_l}}$$

The determinant of the matrix formed by these blocks is a variant of Vandermondes determinant and is non-zero.

3. The determinate case.

For a given measure $\mu \in \mathcal{M}^*$ of finite index of determinacy we denote by $\mathcal{D}(\mu)$ the set of operators of the form (1.2) for which $(T(p_n)) \in \ell^2$, allowing the system (z_i, k_i) and N to vary. It is an infinite dimensional vector space. Any $T \in \mathcal{D}(\mu)$ can be extended from $\mathrm{C}[t]$ to a continuous linear operator $\tilde{T}$ in the space $L^2(\mu)$ via Fourier expansions:

$$\tilde{T}(f) = \sum_n \left(\int_R f(t)p_n(t)d\mu(t) \right) T(p_n), \quad \text{for} \quad f \in L^2(\mu).$$

We choose different real numbers $x_0, \dots, x_{\mathrm{ind}(\mu)}$ outside of the support of μ and consider the measure

$$\sigma = \mu + \sum_{i=0}^{\mathrm{ind}(\mu)} \delta_{x_i}. \tag{3.1}$$

From the above, cf. Theorem 3.9 (1) in [BD1], it follows that the measure σ is N-extremal.

Given a function $f \in L^2(\mu)$, we extend it to a function $\tilde{f}$ in the space $L^2(\sigma)$ in the following way

$$\tilde{f}(t) = \begin{cases} f(t), & \text{for } t \in \mathrm{supp}(\mu) \\ 0, & \text{for } t = x_i, i = 0, \dots, \mathrm{ind}(\mu). \end{cases} \tag{3.2}$$

Clearly, $f \mapsto \tilde{f}$ is a linear isometry of $L^2(\mu)$ into $L^2(\sigma)$.

Since σ is N-extremal, $\tilde{f}$ has a canonical extension to an entire function of class $\mathscr{C}_0$ given by

$$F(\tilde{f})(z) = \sum_n \left(\int_R \tilde{f}(t)q_n(t)d\sigma(t) \right) q_n(z), \tag{3.3}$$

where (q_n) is the sequence of orthonormal polynomials with respect to σ. We can now formulate:

THEOREM 3.1. *Let μ be a determinate measure with finite index of determinacy $\mathrm{ind}(\mu)$. The mapping $E(f) := F(\tilde{f})$ given by (3.3) is a realization of $L^2(\mu)$ as entire functions of class $\mathscr{C}_0$ such that for any operator $T \in \mathcal{D}(\mu)$*

$$\widetilde{T}(f) = T(E(f)), \quad f \in L^2(\mu). \tag{3.4}$$

PROOF. It is clear that $E(f) = F(\tilde{f})$ is a realization of $L^2(\mu)$ as entire functions of class $\mathscr{C}_0$.

The set of functions $f \in L^2(\mu)$ for which (3.4) holds is a closed subspace, and therefore it suffices to prove (3.4) for $f = \chi_{\{x\}}$, $x \in \operatorname{supp}(\mu)$, where χ_A denotes the indicator function of the set A. This is a consequence of the following result:

PROPOSITION 3.2. *For $x \in \operatorname{supp}(\mu)$ we have*

$$E(\chi_{\{x\}})(z) = \frac{F_\mu(z)p(z)}{F'_\mu(x)p(x)(z-x)}, \quad z \in \mathbf{C},$$

where p is the unique monic polynomial of degree $\operatorname{ind}(\mu)+1$ which vanishes at $x_0, \ldots, x_{\operatorname{ind}(\mu)}$.

The function

$$\frac{F_\mu(z)}{F'_\mu(x)(z-x)}$$

is an entire function of class $\mathscr{C}_0$ equal to $\chi_{\{x\}}$ on $\operatorname{supp}(\mu)$ and we have

$$\widetilde{T}(\chi_{\{x\}}) = T(E(\chi_{\{x\}})) = T\left(\frac{F_\mu(z)}{F'_\mu(x)(z-x)}\right) \quad \text{for} \quad T \in \mathscr{D}(\mu).$$

PROOF. For $f = \chi_{\{x\}}$ we find

$$\begin{aligned}
\tilde{f}(t) &= \begin{cases} f(t), & \text{if } t \in \operatorname{supp}(\mu) \\ 0, & \text{for } t = x_i, i = 0, \ldots, \operatorname{ind}(\mu) \end{cases} \\
&= \begin{cases} 1, & \text{for } t = x, \\ 0, & \text{otherwise} \end{cases} \\
&= \chi_{\{x\}}(t).
\end{aligned}$$

For $T \in \mathscr{D}(\mu)$ we denote by $\widetilde{T}$ and $\widetilde{T}_\sigma$ the continuous extensions of T from $\mathbf{C}[t]$ to $L^2(\mu)$ and $L^2(\sigma)$ respectively. We then have $\widetilde{T}(f) = \widetilde{T}_\sigma(\tilde{f})$ for $f \in L^2(\mu)$ because $\|f - p\|_{L^2(\mu)} \leq \|\tilde{f} - p\|_{L^2(\sigma)}$ when $p \in \mathbf{C}[t]$, and in particular $\widetilde{T}(\chi_{\{x\}}) = \widetilde{T}_\sigma(\chi_{\{x\}})$ when $x \in \operatorname{supp}(\mu)$.

By Theorem 2.2 we have

$$F(\tilde{f})(z) = \frac{F_\sigma(z)}{F'_\sigma(x)(z-x)} = \frac{F_\mu(z)p(z)}{F'_\mu(x)p(x)(z-x)},$$

because $F_\sigma(z) = \beta p(z)F_\mu(z)$ for a certain constant β, and hence $F'_\sigma(x) = \beta p'(x)F_\mu(x) + \beta p(x)F'_\mu(x) = \beta p(x)F'_\mu(x)$. This gives by (1.3)

$$\widetilde{T}(\chi_{\{x\}}) = T\left(\frac{F_\mu(z)p(z)}{F'_\mu(x)p(x)(z-x)}\right),$$

but since

$$\frac{F_\mu(z)p(z)}{F'_\mu(x)p(x)(z-x)} = \frac{F_\mu(z)}{F'_\mu(x)(z-x)} + q(z)F_\mu(z),$$

where

$$q(z) = \frac{p(z)-p(x)}{F'_\mu(x)(z-x)p(x)}$$

is a polynomial of degree $\operatorname{ind}(\mu)$, we have $T(qF_\mu) = 0$ by Corollary 2.4, and the second assertion follows.

COROLLARY 3.3. *With the notation above we have*

$$E(f)(z) = \sum_{x\in\operatorname{supp}(\mu)} \frac{F_\mu(z)p(z)}{F'_\mu(x)p(x)(z-x)} f(x) \quad \text{for } f \in L^2(\mu), \tag{3.5}$$

where the series converges uniformly on compact subsets of $\mathbf{C}$.

The realization $E(L^2(\mu)) \subseteq \mathscr{H}(\mathbf{C})$ *is a Hilbert space of entire functions in the sense of de Branges.*

PROOF. Formula (3.5) follows immediately from Theorem 2.2 and Proposition 3.2. To see that $E(L^2(\mu))$ is a Hilbert space of entire functions in the sense of de Branges we shall verify the properties (H1)–(H3) from [Br, p. 57]. We shall only comment on (H1): If $w \in \mathbf{C}\setminus\mathbf{R}$ is a zero of $E(f)$ we have

$$\sum_{x\in\operatorname{supp}(\mu)} \frac{f(x)}{F'_\mu(x)p(x)(w-x)} = 0,$$

and hence for $z \neq w$

$$\begin{aligned} E\left(f(x)\frac{x-\overline{w}}{x-w}\right)(z) &= F_\mu(z)p(z) \sum_{x\in\operatorname{supp}(\mu)} \frac{f(x)}{F'_\mu(x)p(x)(z-x)}\left(1+\frac{w-\overline{w}}{x-w}\right) \\ &= E(f)(z) + F_\mu(z)p(z)(w-\overline{w})S(z), \end{aligned}$$

where

$$S(z) = \sum_{x\in\mathrm{supp}(\mu)} \frac{f(x)}{F'_\mu(x)p(x)}\left(\frac{1}{(z-x)(x-w)} + \frac{1}{(z-w)(w-x)}\right).$$

Therefore we get

$$E\left(f(x)\frac{x-\overline{w}}{x-w}\right)(z) = E(f)(z)\frac{z-\overline{w}}{z-w},$$

which shows (H1).

In Theorem 3.1, to get an extension of $f \in L^2(\mu)$ to an entire function, we add mass points to the measure μ until we reach an N-extremal measure σ. We next extend f by zero to a function in $L^2(\sigma)$, and use its canonical extension to an entire function. However, there is a different way to obtain N-extremal measures from a determinate measure μ having finite index of determinacy. We prove that this approach can also be used to find entire extensions of functions in $L^2(\mu)$, such that (3.4) holds.

For a determinate measure μ with finite index of determinacy $\mathrm{ind}(\mu)$, we take a polynomial

$$R(t) = \prod_{l=1}^{N}(t-z_l)^{k_l+1}, \quad \text{with} \quad \sum_{l=1}^{N}(k_l+1) = \mathrm{ind}(\mu)+1,$$

where $z_l \notin \mathrm{supp}(\mu)$, $l = 1, \dots, N$.

It follows from Lemma 2.1 in [BD2] that $\sigma = |R|^2\mu$ is an indeterminate measure, but the measure $|R(t)/(t-z_1)|^2\mu$ is determinate. According to Lemma A in Section 3 of [BD1], we conclude that the measure $\sigma = |R|^2\mu$ is N-extremal.

Given a function $f \in L^2(\mu)$, we define $f^\natural \in L^2(\sigma)$ by $f^\natural = f/R$. Since σ is N-extremal, $f^\natural$ has a canonical extension $F(f^\natural)$ and we define

$$E(f)(z) := R(z)F(f^\natural)(z). \tag{3.6}$$

THEOREM 3.4. *Let μ be a determinate measure of finite index and let R be as above. Then $L^2(\mu)$ is realized as entire functions of class $\mathscr{C}_0$ via* (3.6), *and it has the property*

$$\widetilde{T}(f) = T(E(f)), \quad f \in L^2(\mu) \tag{3.7}$$

for any discrete differential operator $T \in \mathscr{D}(\mu)$.

PROOF. The set of functions $f \in L^2(\mu)$ for which (3.7) holds is a closed subspace, and therefore it suffices to prove (3.7) for $f = \chi_{\{x\}}$, $x \in \mathrm{supp}(\mu)$.

In this case $f^\natural(t) = (1/R(x))\chi_{\{x\}}(t)$, and since $F_\mu = F_\sigma$ we get

$$F(f^\natural)(z) = \frac{F_\mu(z)}{R(x)F'_\mu(x)(z-x)},$$

hence

$$R(z)F(f^\natural)(z) = \frac{F_\mu(z)}{F'_\mu(x)(z-x)} + r(z)F_\mu(z),$$

where

$$r(z) = \frac{1}{R(x)F'_\mu(x)}\frac{R(z)-R(x)}{z-x}$$

is a polynomial of degree $\mathrm{ind}(\mu)$. Now formula (3.7) follows from Corollary 2.4 and Proposition 3.2.

Like in Corollary 3.3 we have

$$E(f)(z) = \sum_{x\in\mathrm{supp}(\mu)} \frac{F_\mu(z)R(z)}{F'_\mu(x)R(x)(z-x)}f(x) \quad \text{for } f \in L^2(\mu).$$

The realization $E(L^2(\mu))$ is a Hilbert space in the sense of de Branges if R is a real polynomial.

For given $f \in L^2(\mu)$ we shall now describe the set of all entire functions F satisfying

$$\tilde{T}(f) = T(F) \quad \text{for all} \quad T \in \mathscr{D}(\mu). \tag{3.8}$$

THEOREM 3.5. *Let μ be a determinate measure of finite index and let $f \in L^2(\mu)$.*

(i) *Given $(z_1,k_1),\ldots,(z_N,k_N)$, where $z_1,\ldots,z_N$ are different points of $\mathsf{C}, k_1,\ldots,k_N \in \mathsf{N}$, and assume that $0 \le N_2 \le N$ exists such that $z_l \in \mathrm{supp}(\mu)$ and $k_l > 0$ for $l = 1,\ldots,N_2$ and $z_l \notin \mathrm{supp}(\mu)$ for $l = N_2+1,\ldots,N$ and that*

$$\sum_{l=1}^{N_2} k_l + \sum_{l=N_2+1}^{N} (k_l+1) = \mathrm{ind}(\mu)+1, \tag{3.9}$$

then there exists a unique entire function F satisfying (3.8) and the interpolation conditions

$$F^{(j)}(z_l) = \alpha_{l,j} \quad \begin{cases} j = 1,\ldots,k_l\,, & l = 1,\ldots,N_2 \\ j = 0,\ldots,k_l\,, & l = N_2+1,\ldots,N \end{cases} \tag{3.10}$$

where $\alpha_{l,j}$ are arbitrarily given. This entire function F is of class $\mathscr{C}_0$.

(ii) *If F is an entire function satisfying (3.8), then $F+pF_\mu$, where p is any polynomial of degree not bigger than* $\mathrm{ind}(\mu)$, *are the only entire functions satisfying* (3.8). *All of them are of class $\mathscr{C}_0$.*

PROOF. (i) We first prove the existence. Assume that F is an entire function satisfying (3.8). From the hypothesis on the z_l's and since F_μ has simple zeros, we deduce that $F'_\mu(z_l) \neq 0$ for $l = 1, \dots, N_2$ and $F_\mu(z_l) \neq 0$ for $l = N_2 + 1, \dots, N$. Hence, if p denotes a polynomial, the equations

$$\delta_{z_l}^{(j)}(p(z)F_\mu(z)) = F^{(j)}(z_l) - \alpha_{l,j}\,, \begin{cases} j = 1, \dots, k_l,\ l = 1, \dots, N_2 \\ j = 0, \dots, k_l\,,\ l = N_2 + 1, \dots, N \end{cases}$$

determine the quantities $p^{(j)}(z_l)$ uniquely for $j = 0, \dots, k_l - 1, l = 1, \dots, N_2$ and for $j = 0, \dots, k_l, l = N_2 + 1, \dots, N$. The hypothesis (3.9) guarantees that p is uniquely determined as a polynomial of degree $\leq \operatorname{ind}(\mu)$. This means that $F - pF_\mu$ satisfies the interpolation conditions (3.10), and $F - pF_\mu$ still satisfies (3.8) by Corollary 2.4.

To prove uniqueness, assume that F and G are entire functions satisfying (3.8) and (3.10). We shall prove that $F(x) = G(x)$ for all $x \in \mathbf{C} \setminus (\operatorname{supp}(\mu) \cup \{z_{N_2+1}, \dots, z_N\})$. This clearly implies $F \equiv G$. For x as above we consider the linear system

$$\sum_{l=1}^{N_2} \sum_{j=1}^{k_l} a_{l,j} \delta_{z_l}^{(j)}\left(z^k F_\mu(z)\right) + \sum_{l=N_2+1}^{N} \sum_{j=0}^{k_l} a_{l,j} \delta_{z_l}^{(j)}\left(z^k F_\mu(z)\right) = x^k F_\mu(x)$$

where $0 \leq k \leq \operatorname{ind}(\mu)$. The system is quadratic by (3.9), and it has a unique solution $(a_{l,j})$, cf. the proof of Corollary 2.4. This means that the operator

$$T := \sum_{l=1}^{N_2} \sum_{j=1}^{k_l} a_{l,j} \delta_{z_l}^{(j)} + \sum_{l=N_2+1}^{N} \sum_{j=0}^{k_l} a_{l,j} \delta_{z_l}^{(j)} - \delta_x$$

belongs to $\mathscr{D}(\mu)$, so $T(F) = T(G) = \widetilde{T}(f)$ by (3.8), but since F and G both satisfy (3.10) we conclude that $F(x) = G(x)$.

Since (3.8) has a solution F which is of class $\mathscr{C}_0$, the solution $F - pF_\mu$ from the existence part is again of class $\mathscr{C}_0$.

(ii) Let F, G be entire functions satisfying (3.8). The method in (i) shows that it is possible to find a polynomial p of degree $\leq \mathit{ind}(\mu)$ such that $G - pF_\mu$ satisfies the interpolation conditions

$$\delta_{z_l}^{(j)}(G - pF_\mu) = F^{(j)}(z_l)$$

with l,j as in (3.10). By the uniqueness assertion $G - pF_\mu = F$.

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