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BOUNDED LINEAR FUNCTIONALS ON VECTOR-VALUED H^∞ SPACES

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Introduction.

The theory of the space H^∞, its dual, and its predual L^1/H_0^1, is known to be difficult (see e.g. [Bo]). In this work I transfer some of the easier parts of this theory to a vector resp. operator-valued context.

Let X be a complex Banach space and $H^\infty(X)$ [$\mathsf{H}^\infty(X)$] the space of bounded holomorphic X-valued functions on the unit disc [admitting strong radial boundary values a.e.]. By the simple but for this paper crucial lemma 2.1 I can prove the F. and M. Riesz theorem for $\mathsf{H}^\infty(X)$ (2.2). Immediate consequences are that for reflexive X the spaces $H^1(X)$ and $L^1(X)/H_0^1(X)$ are L-embedded and $C(X)/A(X)$ is an M-ideal in the bidual (2.4–2.6). In §3, I study integral resp. strongly integral functionals on $\mathsf{H}^\infty(X)$, i.e. those given by Gel'fand integrable functions $g \in L^1(X', X)$ resp. Bochner integrable functions $g \in L^1(X')$. I found it interesting to see how the various characterizations of integral functionals in the scalar case [Kh 2; Ga] group exactly into two sets of conditions in the vector case: the first set, formulated on the circle and describing the integral functionals (3.1), and the second group, formulated on the disc and singling out the strongly integral functionals (3.5). Contrary to the scalar situation, some locally convex theory (Grothendieck's dual characterization of completeness) enters the proof of the latter theorem. The Banach space theoretic consequence of the former is that $L^1(X)/H_0^1(X)$ has property (X) if X is reflexive (3.2).

The paper concludes in §4 with spaces of operator-valued functions. Let K, N, B denote the spaces of compact resp. nuclear resp. bounded operators on separable Hilbert space. I prove an F. and M. Riesz theorem for $H^\infty(B)$ (4.5); consequently $H^1(N)$ and $L^1(N)/H_0^1(N)$ are L-embedded and $C(K)/A(K)$ is an M-ideal in the bidual (4.6, 4.7). The result on $L^1(N)/H_0^1(N)$ is related to a question of Pisier. Moreover, $L^1(N)/H_0^1(N)$ enjoys property (X) (4.8).

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§§1–3 of the paper are essentially a part of the author's habilitation thesis [He 3]. I would like to thank Professor A. V. Bukhvalov for his suggestions and stimulations to do this work. §4 results from a couple of stimulating discussions with Dr. H. Pfitzner who in particular pointed out Pisier's question to me. 4.8 answers a question of the referee.

§1. Preliminaries.

The basic theory of vector-valued Hardy spaces as developed mainly by Bukhvalov and Danilevich [Bu 1; BD] is assumed familiar. See [He 1; 3] for detailed and [LM] for a very coincise presentation. I fix some notation.

1.1. The basic measure space is the circle group T with normalized Haar measure λ. For $1 \leqq p \leqq \infty$, $L^p(\lambda; X) = L^p(X)$ is the X-valued Lebesgue-Bochner space and $L^p(\lambda; X', X) = L^p(X', X) (= L^p(X')$ iff X' has RNP) denotes the space of X-scalar equivalence classes of weak* measurable functions $f\colon \mathsf{T} \to X'$ with $|f| := L^0 - \sup_{x \in B_X} |\langle x, f\rangle| \in L^p$, normed by $\|f\|_p = \| |f| \|_p$. Such a function is Gel'fand integrable [DU, p. 53]. Under the canonical dual pairing, $L^\infty(X', X)$ is the dual of $L^1(X)$.

1.2. *The dual of* $L^\infty(X)$ has been described by various authors, see [CV, VII; BL]. Any member $g \in L^1(X', X)$ defines a functional $\varphi_g \in L^\infty(X)'$ by the formula $\varphi_g(f) := \int \langle f, g\rangle \, d\lambda$; moreover $\|\varphi_g\| = \|g\|_1$. (Proof for "$\geqq$": $\|g\|_1 =$ total variation of the vector measure $g \cdot \lambda$ [He 1, (0.8)2°] $\leqq \|\varphi_g\|$, by definition of the variation.) The functionals φ_g are called *integral* and the space of integral functionals is identified with $L^1(X', X)$.

A functional $\varphi \in L^\infty(X)'$ is called *concentrated* on a (measurable) set $E \subset \mathsf{T}$ if $\varphi(f) = \varphi(\mathbf{1}_E f) \, \forall f \in L^\infty(X)$, and *singular* if concentrated on sets of arbitrarily small λ-measure. The space of singular functionals is denoted by $L^\infty(X)'_s$.

Vector-valued Yosida-Hewitt decomposition:

$$L^\infty(X)' = L^1(X', X) \oplus_1 L^\infty(X)'_s. \ (\oplus_1\text{: } L\text{-decomposition.})$$

1.3. *The vector-valued Hardy spaces* used in this paper are: $\mathsf{H}^p(X) := \{f \in L^p(X) : \hat{f}(n) = 0 \, \forall n < 0\} \cong \{f\colon D \to X \text{ holomorphic: } \|f\|_p := \sup_{r<1} \|f_r\|_p < \infty$ and $\lim_{r \to 1} f(re^{i\vartheta})$ exists a.e.$\}$; $H^p(X') := \{f \in L^p(X', X) : \hat{f}(n) = 0 \ \ \forall n < 0\} \cong \{f\colon D \to X' \text{ holomorphic} : \|f\|_p := \sup_{r<1} \|f_r\|_p < \infty\}$ $(= \mathsf{H}^p(X')$ iff X' has ARNP); $A(X) := \{f \in C(\mathsf{T}; X) : \hat{f}(n) = 0 \, \forall n < 0\}$ (X-valued disc "algebra"). With the usual meaning of the subscript 0, the following canonical dualities hold:

$$(C(X)/A(X))' = H_0^1(X'); \ \mathsf{H}_0^1(X)' = L^\infty(X', X)/H^\infty(X'); \ (L^1(X)/\mathsf{H}_0^1(X))' = H^\infty(X').$$

§2. F. and M. Riesz Theorem for $\mathrm{H}^\infty(X)$.

The following lemma exploits the L^∞ module structure of $L^\infty(X)$ by "contracting" a functional with a function.

2.1. LEMMA. *Let $h \in L^\infty(X)$ be fixed. For $\varphi \in L^\infty(X)'$ define $\varphi h \in L^{\infty\prime}$ by the formula $(\varphi h)\ (u) := \varphi(uh)$, $u \in L^\infty$. If $\varphi = \varphi_i + \varphi_s$ is the vector-valued Yosida-Hewitt decomposition* (1.2) *of $\varphi \in L^\infty(X)'$ then $\varphi h = \varphi_i h + \varphi_s h$ is the (scalar) Yosida-Hewitt decomposition of $\varphi h \in L^{\infty\prime}$. (In other words, $(\varphi h)_i = \varphi_i h$ and $(\varphi h)_s = \varphi_s h$.)*

PROOF. Let $g \in L^1(X', X)$ be the function defining the integral part φ_i, i.e. $\varphi_i(f) = \int \langle f, g \rangle\, d\lambda \quad \forall f \in L^\infty(X)$. Then $(\varphi_i h)(u) = \int \langle uh, g \rangle\, d\lambda = \int u \langle h, g \rangle\, d\lambda$ $\forall u \in L^\infty$, so that $\varphi_i h$ is integral, given by $\langle h, g \rangle \in L^1$. On the other hand, for an arbitrary $\Psi \in L^\infty(X)'$, the contraction $\Psi h \in L^{\infty\prime}$ is concentrated on every set on which Ψ is concentrated. In particular, φ_sh is singular together with φ_s.

2.2. F. AND M. RIESZ THEOREM FOR $\mathrm{H}^\infty(X)$: *Let $\varphi = \varphi_i + \varphi_s$ be the vector-valued Yosida-Hewitt decomposition* (1.2) *of $\varphi \in L^\infty(X)'$*. If $\varphi | \mathrm{H}_0^\infty(X) = 0$ *then $\varphi_i | \mathrm{H}_0^\infty(X) = 0$ and $\varphi_s | \mathrm{H}^\infty(X) = 0$. In particular, the annihilator $\mathrm{H}^\infty(X)^\perp$ is invariant under the Yosida-Hewitt projection P: $L^\infty(X)' \twoheadrightarrow L^1(X', X)$, $\varphi \mapsto \varphi_i$.*

PROOF. In the scalar case, Gel'fand transformation translates the theorem into the well-known F.and M. Riesz theorem in logmodular algebras [Ho, p. 186; Ga, V. 4.4]. Turning to the vector case, let $\varphi \in L^\infty(X)'$ be given with $\varphi | \mathrm{H}_0^\infty(X) = 0$, and fix $h \in \mathrm{H}^\infty(X)$; one has to show $\varphi_s(h) = 0$. To this end, consider $\varphi h \in L^{\infty\prime}$ as in lemma 2.1 and note that $\varphi h | H_0^\infty = 0$, exploiting the fact that $u \in H_0^\infty$, $h \in \mathrm{H}^\infty(X)$ implies $uh \in \mathrm{H}_0^\infty(X)$. By the scalar case, $(\varphi h)_s | H^\infty = 0$, whence by the lemma, $\varphi_s(h) = \varphi_s h(1) = (\varphi h)_s(1) = 0$.

If one is interested only in the invariance result then, for reflexive X, a completely different approach is possible, discovered by Godefroy [Go 1] (for the scalar case). It is based on (a vector-valued version of) the Bukhvalov-Lozanovskii theorem [Bu 2; BL] and thus lastly on vector lattices instead of uniform algebras. Still another (scalar) proof, via the notion of "strictly convergent" (= *wuC*) series in L^∞, has been developed by Barbey and König [BK, VIII.]. See also Ando [A, 4.].

2.3. COROLLARY. *If $\varphi \in L^\infty(X)'$ is singular and $\varphi | \mathrm{H}^\infty(X) = 0$ then $\varphi | C(X) = 0$.*

PROOF. Apply the usual iteration [Ho, p. 186 f.]: The functional $f \mapsto \varphi(e^{-i\vartheta} f)$ on $L^\infty(X)$ vanishes on $\mathrm{H}_0^\infty(X)$ and is also singular, hence by 2.2: $\varphi(e^{-i\vartheta} x) = 0$ for all $x \in X$, etc. Thus φ vanishes on all trigonometric polynomials which are dense in $C(X)$.

2.4. PROPOSITION. *If X is reflexive then* $\mathrm{H}_0^1(X)'' = \mathrm{H}_0^1(X) \oplus_1 (L^\infty(X')'_s \cap \mathrm{H}^\infty(X')^\perp)$. *Thus* $\mathrm{H}_0^1(X)$ (*hence also* $\mathrm{H}^1(X)$ *is L-embedded in its bidual.*

PROOF. Using reflexivity all the time, 1.3 and 1.2 yield $\mathrm{H}_0^1(X)'' = (L^\infty(X')/\mathrm{H}^\infty(X'))' = \mathrm{H}^\infty(X')^\perp \subset L^\infty(X')' = L^1(X) \oplus_1 L^\infty(X')'_s$. Trivially, the restriction of an L-projection P to an invariant subspace Y is an L-projection in Y onto $Y \cap \mathrm{range}(P)$. With $Y := \mathrm{H}^\infty(X')^\perp$, 2.2 implies $\mathrm{H}_0^1(X)'' = (L^1(X) \cap \mathrm{H}^\infty(X')^\perp) \oplus_1 (L^\infty(X')'_s \cap \mathrm{H}^\infty(X')^\perp)$, and the first summand equals $\mathrm{H}_0^1(X)$.

2.5. PROPOSITION. *If X is reflexive then $C(X)/A(X)$ is an M-ideal in its bidual $L^\infty(X)/\mathrm{H}^\infty(X)$. (See* [HWW] *for definition and consequences* – e.g., *property* (u).)

PROOF. It is convenient to prove the assertion for X', so let $Z := C(X')/A(X')$; one has to show the L-nature of the canonical decomposition $Z''' = Z' \oplus Z^\perp$. Now by 1.3, $Z' = \mathrm{H}_0^1(X)$ and this decomposition reads $\mathrm{H}_0^1(X)'' = \mathrm{H}_0^1(X) \oplus Z^\perp$. On the other hand, 2.4 says $\mathrm{H}_0^1(X)'' = \mathrm{H}_0^1(X) \oplus_1 (L^\infty(X')'_s \cap \mathrm{H}^\infty(X')^\perp)$. I claim that this last L-summand is contained in $Z^\perp$ (then it will coincide with $Z^\perp$ by pure linear algebra): The annihilator $Z^\perp$ of $Z := C(X')/A(X')$ in $Z''' = \mathrm{H}_0^1(X)'' = \mathrm{H}^\infty(X')^\perp \subset L^\infty(X')'$ equals $\{\varphi \in \mathrm{H}^\infty(X')^\perp : \varphi | C(X') = 0\}$ and the claim follows from 2.3.

This proof differs from Luecking's (scalar) proof [HWW, p. 109] in the *use* of 2.4. If this is avoided as in [HWW, end of p. 109], one gets 2.4 as a consequence.

2.6. PROPOSITION. *If X is reflexive then $L^1(X)/\mathrm{H}_0^1(X)$ is L-embedded in its bidual.*

PROOF. By 1.3, 1.2 and reflexivity, $(L^1(X)/\mathrm{H}_0^1(X))'' = \mathrm{H}^\infty(X')' = L^\infty(X')'/\mathrm{H}^\infty(X')^\perp = (L^1(X) \oplus_1 L^\infty(X')'_s)/\mathrm{H}^\infty(X')^\perp$. It is well-known and not hard to prove (see e.g. [He 3, 2.3], [HWW, I.I.15]) that an L-projection P in a space Z canonically induces an L-projection $\bar{P}$ in Z/Y for a P-invariant subspace Y, with range $(\bar{P}) = \mathrm{range}(P)/(Y \cap \mathrm{range}(P))$ (as canonically isometrically embedded in Z/Y). With $Y := \mathrm{H}^\infty(X')^\perp$, 2.2 implies $(L^1(X)/\mathrm{H}_0^1(X))'' = L^1(X)/(L^1(X) \cap \mathrm{H}^\infty(X')^\perp) \oplus_1 \dots = L^1(X)/\mathrm{H}_0^1(X) \oplus_1 \dots$.

It follows from 2.6 and Godefroy's result [Go 1, 4; HWW, IV.2.2] that $L^1(X)/\mathrm{H}_0^1(X)$ is weakly sequentially complete for reflexive X. This fact (and probably 2.6) was obtained first by Petrenko [Bu 2, 5.3], cf. [He 2]. Recently, Pfitzner [Pf 1; HWW, IV.2.7] proved that spaces L-embedded in the bidual even enjoy Pelczyński's property (V^*). For the case of $L^1(X)/\mathrm{H}_0^1(X)$ with X reflexive, property (V^*) has been obtained earlier by this author [He 3, 3.6; He 4, 2.2] adapting Pelczyński's scalar proof [Pe, 7.1]. See also 3.2.

§3. Integral and strongly integral functionals.

For the scalar-valued origins of the following theorem, see [Kh 2, Theorem 1; BK, VIII.3.1; Ga, V.5.3; Kh 1, p. 300].

3.1. THEOREM (Characterization of integral functionals). *For* $\varphi \in \mathrm{H}^\infty(X)'$, *the following conditions are equivalent:*

1° φ *is integral*, i.e. $\exists g \in L^1(X', X)$ *with* $\varphi(f) = \int \langle f, g \rangle \, d\lambda \ \forall f \in \mathrm{H}^\infty(X)$

2° $f_n \in B_{\mathrm{H}^\infty(X)}$ *and* $f_n(e^{it}) \to 0$ a.e. *on* T *implies* $\varphi(f_n) \to 0$

3° $f_n \in \mathrm{H}^\infty(X)$, $\sum_{n=1}^\infty \|f_n(e^{it})\| \leqq 1$ a.e. *on* T (*whence* $\sum_{n=1}^\infty f_n(e^{it})$ *is a member of* $\mathrm{H}^\infty(X)$) *implies* $\varphi\left(\sum_{n=1}^\infty f_n\right) = \sum_{n=1}^\infty \varphi(f_n)$

4° $\varphi | B_{\mathrm{H}^\infty(X)}$ *has a point of continuity for* $\|\cdot\|_1$.

PROOF. 1° $\Rightarrow$ 2°: $|\langle f_n, g \rangle| \leqq \|f_n(\cdot)\| \, |g| \to 0$ a.e. on T (see 1.1 for notation and [He 1, (0.5) 5°] for the estimate), dominated by $|g| \in L^1$. Apply Lebesgue's theorem.

2° $\Rightarrow$ 3°: As regards the assertion in parentheses, clearly $\sum_{n=1}^\infty f_n(e^{it})$ converges in X a.e. on T, $\sum_{n=1}^\infty f_n \in L^\infty(X)$ and $\left(\sum_{n=1}^\infty f_n\right)^\wedge (k) = 0 \ \forall k < 0$ by bounded convergence. Now apply 1° to the remainders $\sum_{k=n}^\infty f_k(e^{it})$.

3° $\Rightarrow$ 1°: In the scalar case, various proofs have been developed: [Ga, V.5.3; Go 1, 33; Go 2, V.4(5); BK, VIII.3.1]. The vector case is handled by lemma 2.1. Let $\varphi \in L^\infty(X)'$ satisfy 3°. Since the integral part of φ (1.2) satisfies 3°, one can assume φ singular and has to prove $\varphi = 0$ on $\mathrm{H}^\infty(X)$. So let $h \in \mathrm{H}^\infty(X)$ be given; the contraction $\varphi h \in L^{\infty\prime}: u \mapsto \varphi(uh)$ is singular by 2.1 and respects the convergence of type 3° ($X = \mathrm{C}$): if $u_n \in H^\infty$, $\sum_{n=1}^\infty |u_n(e^{it})| \leqq 1$ a.e. then $u_n h \in \mathrm{H}^\infty(X)$, $\sum_{n=1}^\infty \|u_n(e^{it}) h(e^{it})\| \leqq \|h\|_\infty$ a.e., whence by assumption $\varphi h\left(\sum_{n=1}^\infty\right) = \varphi\left(\sum_{n=1}^\infty u_n h\right) = \sum_{n=1}^\infty \varphi(u_n h) = \sum_{n=1}^\infty \varphi h(u_n)$. By the scalar case $\varphi h = 0$ on H^∞ and $\varphi(h) = \varphi h(1) = 0$.

2° $\Rightarrow$ 4°: It is very easily seen that actually 2° $\Leftrightarrow \varphi | B_{\mathrm{H}^\infty(X)}$ is continuous for $\|\cdot\|_1$.

4° $\Rightarrow$ 1°: This rests on the Khavin lemma, with the aid of which Khavin [Kh 1, p. 299 f.] proved that if $\varphi \in L^{\infty\prime}$ and $\varphi | B_{H^\infty}$ has a point of continuity for $\|\cdot\|_1$ then the singular part of φ (1.2) vanishes on H^∞. For the vector-valued case, the same proof works, and is given in detail in [He 2, 3.2].

NOTE. As in the scalar case, 2° $\Rightarrow$ 1° has an easier proof, transferring [Kh 2, p. 88] or [Ga, V Exercise 19] to the vector situation.

3.2. COROLLARY. *If X is reflexive then $L^1(X)/\mathrm{H}_0^1(X)$ has property* (X) *of Godefroy and Talagrand. Consequently, $L^1(X)/\mathrm{H}_0^1(X)$ is strongly unique predual of* $\mathrm{H}^{\infty}(X')$.

PROOF. By definition, a Banach space Z has property (X) if $z'' \in Z''$, $z''\left(\sum_{n=1}^{\infty} z'_n\right) = \sum_{n=1}^{\infty} z''(z'_n)$ $\forall$ wuC series $\sum z'_n$ in Z' implies $z'' \in Z$ ($\sum_{n=1}^{\infty} z'_n$ is taken for $\sigma(Z', Z)$). Let $\varphi \in (L^1(X)/\mathrm{H}_0^1(X))'' = \mathrm{H}^{\infty}(X')'$ be given, and suppose $\varphi\left(\sum_{n=1}^{\infty} f_n\right) = \sum_{n=1}^{\infty} \varphi(f_n)$ $\forall$ wuC series $\sum f_n$ in $\mathrm{H}^{\infty}(X')$, summed for $\sigma(\mathrm{H}^{\infty}(X'), L^1(X)/\mathrm{H}_0^1(X))$. The "test series" of 3.1.3° (X') are wuC (since wuC = w*uC in dual spces), and of course the pointwise limit considered there coincides with the weak* limit. Thus 3.1 yields that φ is given by integration against a function $g \in L^1(X)$, i.e. $\varphi \in L^1(X)/\mathrm{H}_0^1(X)$. The consequence is well-known [GT, 5; Go 2, V.3].

NOTES. i) (X) entails (V*) [Go 2, V.5; GS, III.2; E, 13] providing still another proof of (V*) for $L^1(X)/\mathrm{H}_0^1(X)$ if X is reflexive.

ii) In the scalar case, the unique predual result is due to Ando [A] and Wojtaszczyk [W], and property (X) of L^1/H_0^1 goes back to Barbey [BK, VIII.].

3.3. REMARK. Let μ be a finite measure and $\varphi \in L^{\infty}(\mu; X)'$. Then the criterion analogous to 3.1.3° holds for φ to be integral. Namely, for $X = \mathrm{C}$ this is the well-known property (X) of $L^1(\mu)$ [Go 1, 30]; the general case is reduced to the scalar case by lemma 2.1 as before. Now the same proof as in 3.2 shows that *for X reflexive, $L^1(\mu; X)$ has property* (X) and is hence strongly unique predual of $L^{\infty}(\mu; X')$. The latter conclusion follows also from [CG 2, Theorem 1] in connection with [CG 1, Remark 1].

3.4. DEFINITION. a) [Kh2, 5.] A continuous function $w: [0, 1] \to \mathrm{R}_+$ with $w(r) > 0$ if $r < 1$, $w(1) = 0$ is called a *weight*. For $f: D \to X$, put $\|f\|_w := \sup_{x \in D} \|f(z)\|\, w(z)$ where $w(z) := w(|z|)$.

b) The locally convex topology on $\mathrm{H}^{\infty}(X)$ induced by the family of norms $\|\cdot\|_w$ is the *strict topology β*.

It is easy to see that this is the vector-valued analogue of the strict topology on H^{∞} studied in [RR; RS; Co, V.; BS].

3.5. THEOREM (Characterization of strongly integral functionals). *For $\varphi \in \mathrm{H}^{\infty}(X)'$, the following conditions are equivalent:*

1° *φ is strongly integral,* i.e. *$\exists g \in L^1(\mathrm{T}, \lambda; X')$ with $\varphi(f) = \int_{\mathrm{T}} \langle f, g \rangle \, d\lambda$ $\forall f \in \mathrm{H}^{\infty}(X)$*

2° $\exists g \in L^1(D, \lambda_2; X')$ *with* $\varphi(f) = \iint_D \langle f, g\rangle \, d\lambda_2 \; \forall f \in \mathsf{H}^\infty(X)$ (λ_2: *area measure*)

3° $\exists$ *weight* w $\exists C \in \mathsf{R}_+$ *with* $|\varphi(f)| \leqq C \, \|f\|_w \; \forall f \in \mathsf{H}^\infty(X)$ (i.o.w., φ *is* β*-continuous*)

4° $\exists m \in M(D; X')$ *with* $\varphi(f) = \iint_D \langle f, dm\rangle \; \forall f \in \mathsf{H}^\infty(X)$ (M: vector measures [σ-additive] of bounded variation)

5° $f_n \in B_{\mathsf{H}^\infty(X)}$ *and* $f_n(z) \to 0 \; \forall z \in D$ *implies* $\varphi(f_n) \to 0$.

Proof. First I show the equivalence of 1° through 4°, following [Kh 2, 5.] (cf. also [RS, 4.]). The proof rests on the following lemma on balayage whose proof is omitted (see [He 3, Lemma 2.8]).

Lemma. *For* $m \in M(D; X)$ *put* $B[m]\colon \mathsf{T} \to X$, $B[m](e^{i\vartheta}) := \iint_D P_z(\vartheta) m(dz)$ ($P_z(\vartheta)$: *Poisson kernel*).

i) $B[m](e^{i\vartheta})$ *is defined* a.e. *on* T *and* $B\colon M(D; X) \to L^1(\mathsf{T}; X)$ *is an operator of norm* $\leqq 1$

ii) *For* $m \in M(D; X)$ *and* $f \in H^\infty(X')$: $\int_{\mathsf{T}} \langle B[m], f\rangle \, d\lambda = \iint_D \langle dm, f\rangle$

iii) $B \,|\, L^1(D, \lambda_2; X) \to L^1(\mathsf{T}, \lambda; X)$ *is surjective.*

1° $\Rightarrow$ 2°: For $g \in L^1(\mathsf{T}; X')$ representing φ as in 1°, the lemma (iii) yields some $\tilde{g} \in L^1(D; X')$ with $B[\tilde{g}] = g$. And by (ii), for $f \in \mathsf{H}^\infty(X) \subset H^\infty(X'')$:

$$\varphi(f) = \int_{\mathsf{T}} \langle f, g\rangle \, d\lambda = \int_{\mathsf{T}} \langle f, B[\tilde{g}]\rangle \, d\lambda = \iint_D \langle f, \tilde{g}\rangle \, d\lambda_2.$$

2° $\Rightarrow$ 3°: For $g \in L^1(D; X')$ representing φ as in 2°, since $\int_0^1 r \int_0^{2\pi} \|g(re^{i\vartheta})\| \, d\vartheta dr < \infty$, there exists $u\colon [0,1] \to]0, \infty[$ continuous with $u(r) \uparrow \infty$ as $r \uparrow 1$ and $C := \int_0^1 u(r) r \int_0^{2\pi} \|g(re^{i\vartheta})\| \, d\vartheta dr < \infty$. Then $w(r) := \dfrac{1}{\mathrm{u(r)}}$ is a weight and for $f \in \mathsf{H}^\infty(X)$: $|\varphi(f)| = \left| \int_0^1 \int_0^{2\pi} \langle w(r) f(re^{i\vartheta}), u(r) g(re^{i\vartheta})\rangle \, r d\vartheta dr \right| \leqq C \, \|f\|_w$.

3° $\Rightarrow$ 4°: With w as in 3°, the map $f \mapsto wf$ is an isometric embedding ($\mathsf{H}^\infty(X)$,

$\|\cdot\|_w) \hookrightarrow C_0(D;X) \subset C(\bar{D};X)$ and φ is $\|\cdot\|_w$-continuous. By Singer's theorem [Si] (and Hahn-Banach), there exists $m \in M(\bar{D};X')$ vanishing on T (i.e. $m \in M(D;X')$ with $\varphi(f) = \iint_D \langle wf, dm\rangle = \iint_D \langle f, w\,dm\rangle$, $f \in \mathrm{H}^\infty(X)$.

$4° \Rightarrow 1°$: For $m \in M(D;X')$ representing φ as in 4°, the lemma (ii) yields $\varphi(f) = \iint_D \langle f, dm\rangle = \int_{\mathrm{T}} \langle f, B(m)\rangle\, d\lambda\ \forall f \in \mathrm{H}^\infty(X) \subset H^\infty(X'')$, where $B[m] \in L^1(\mathrm{T};X')$.

The proof of the equivalence of 1°–4° is complete.

$2° \Rightarrow 5°$: Bounded convergence.

$5° \Rightarrow 3°$: Putting $B := B_{\mathrm{H}^\infty(X)}$, condition 5° trivially implies that $\varphi|B$ is sequentially continuous for β. The topology κ of uniform convergence on compact subsets of D is certainly weaker than β, and coincides with β on B, as is easy to see [RS, 3.7]. κ being metrizable, the same holds for $\beta|B$ so that $\varphi|B$ is actually continuous for β. Now I apply Grothendieck's dual characterization of completeness [Sc, IV.6.2] for the locally convex space $(\mathrm{H}^\infty(X), \beta)$ and the topology of uniform convergence on norm-bounded sets (that is, on B) on $(\mathrm{H}^\infty(X), \beta)'$. By the equivalence of 1° and 3° already established, as a vector space $(\mathrm{H}^\infty(X), \beta)' = L^1(X')/\mathrm{H}_0^1(X')$, and the topology in question is given by the quotient norm, as proved below. In particular, this topology is complete, and Grothendieck's theorem yields that φ is continuous for β, i.e. 3°. It remains to establish the

Claim. *For* $\Psi = g + \mathrm{H}_0^1(X') \in L^1(X')/\mathrm{H}_0^1(X')$ $(g \in L^1(X'))$, $\|\Psi\|_{L^1(X')/\mathrm{H}_0^1(X')} = \|\Psi\|_{L^1(X',X)/\mathrm{H}_0^1(X')} = \|\Psi\|_{\mathrm{H}^\infty(X)'}$.

Proof. i) $\geqq$ clear; $\leqq$: Let $\varepsilon > 0$ and $h \in H_0^1(X')$ such that $d(g, H_0^1(X')) \geqq \|g - h\|_1 - \varepsilon$. Now $h_r := P_r * h \in \mathrm{H}_0^1(X')$, whence $d(g, \mathrm{H}_0^1(X')) \leqq \overline{\lim}_{r\to 1} \|g - h_r\|_1 \leqq \lim_{r\to 1} \|g - g_r\|_1 + \sup_{r<1} \|g_r - h_r\|_1 = 0 + \|g - h\|_1 \leqq d(g, H_0^1(X')) + \varepsilon$.

ii) $\mathrm{H}^\infty(X)' = L^\infty(X)'/\mathrm{H}^\infty(X)^\perp = L^1(X',X)/H_0^1(X') \oplus_1 \dots$ by 1.2, 2.2 and the lemma on quotients of L-decompositions used already in the proof of 2.6.

3.6. Remark. The theorem persists if $\mathrm{H}^\infty(X)$ is replaced by $H^\infty(X)$ throughout.

3.7. Remark. Every integral functional on $\mathrm{H}^\infty(X)$ is strongly integral if X' has RNP.

Proof. The space of integral functionals $L^1(X',X)/H_0^1(X')$ equals the space of strongly integral functionals $L^1(X')/\mathrm{H}_0^1(X')$ iff $L^1(X',X) = L^1(X') + H_0^1(X')$. Now if X' has RNP then $L^1(X',X) = L^1(X')$ and the equality holds. The reverse implication is rather deep [LM, IV b] (take $q = 1$ there).

§4. F. and M. Riesz Theorem for $H^{\infty}(B(H))$.

Recall from 2.4, 2.6, 3.2 that for reflexive X the spaces $\mathsf{H}_0^1(X)$ and $L^1(X)/\mathsf{H}_0^1(X)$ are L-complemented in their biduals and have property (X). Now I establish the same for $X = N(H)$, the space of nuclear linear operators on separable Hilbert space. Since $N(H)' = B(H)$ lacks RNP, the dual $L^{\infty}(B, N)$ of $L^1(N)$ does *not* coincide with $L^{\infty}(B)$ [DU, IV.1.1] and the Yosida-Hewitt decomposition 1.2 is of no (direct) use. The strategy is rather to exploit the Takesaki L-decomposition [T 3, III.2.14] of the dual of the von Neumann algebra $L^{\infty}(B, N)$. In order to devolve this decomposition on the quotient $(L^1(N)/\mathsf{H}_0^1(N))'' = L^{\infty}(B, N)'/H^{\infty}(B)^{\perp}$ as in 2.6, an F. and M. Riesz theorem for $H^{\infty}(B)$ is needed.

4.1. So let $(H,(\cdot,\cdot))$ be a separable Hilbert space, $B = B(H)$ resp. $N = N(H)$ the space of bounded resp. nuclear (= trace class) operators on H. As is well-known, N is the (unique) predual of B under the dual pairing $\langle a, b\rangle := \operatorname{tr}(ab)$, $a \in N$, $b \in B$ [Pd, 3.4.13; T 3, III.3.9]. Therefore, by general vector-valued theory (1.1) $L^1(N)' = L^{\infty}(B, N)$, but in this operator situation the latter space has a more familiar description: Let $L_w^{\infty}(B) := \{f\colon \mathsf{T} \to B \text{ bounded: } \forall x,\ y \in H\colon (f(\cdot)x, y) \text{ measurable}\}/\{f = 0 \text{ a.e.}\}$, equipped with the essential sup norm. Invoking the separability of H (hence of $N(H)$) and [Pd, 4.6.11] it is routine to establish that $L_w^{\infty}(B) = L^{\infty}(B, N)$ canonically. If an orthornormal basis (e_n) of H is fixed, then also $L_w^{\infty}(B) = \{f\colon \mathsf{T} \to B \text{ bounded: } \forall i, j\colon (f(\cdot)e_j, e_i) =: f_{ij} \in L^{\infty}\}/\{f = 0 \text{ a.e.}\}$. In this description, $H^{\infty}(B) = \{f \in L_w^{\infty}(B)\colon \forall i,\ j\colon f_{ij} \in H^{\infty}\}$. (Use that generally $f \in L^{\infty}(X', X)$ is in $H^{\infty}(X')$ iff $\langle x, f\rangle \in H^{\infty}\ \forall x \in X$ and [Pd, 4.6.11] again.)

4.2. It is known that $L_w^{\infty}(B)$ acts naturally as a von Neumann algebra on the Hilbert space $L^2(H)$; these operators on $L^2(H)$ are called decomposable [Pd, E 4.7.5; T 3, IV.7]. So $L^1(N)$ is the unique predual of $L_w^{\infty}(B)$; in fact, $L^1(N)$ has property (X) [Go 2, V.4(4)].

4.3. *Takesaki decomposition* [T1; T3, III.2]: This is the noncommutative analogue of the Yosida-Hewitt decomposition. Let A be an arbitrary von Neumann algebra, $A_* =$ predual of $A =$ space of σ-weakly continuous ("normal") functionals on A. There is a central projection $z \in A''$ (identified with the universal enveloping von Neumann algebra of A) such that $A_* = A'z$, where $a'z \in A'$, $(a'z)(a) := (az)(a')$ $(a \in A,\ a' \in A')$. z is called the support projection of $A_* \subset A'$. Then $A'_{\perp} := A'(1 - z)$ is the space of "singular" functionals on A.

THEOREM [T 3, III.2.14]. $A' = A_* \oplus_1 A'_{\perp}$.

The corresponding decomposition of $\varphi \in A'$ is denoted $\varphi = \varphi_* + \varphi_{\perp}$; $T\colon A' \twoheadrightarrow A_*$, $T\varphi := \varphi_* = \varphi z$ is called Takesaki L-projection.

4.4. LEMMA. *Let $p \in A$ be a projection, $\varphi \in A'$. The Takesaki decomposition of $\varphi | pAp$ is induced by that of φ.*

This lemma is analogous to 2.1. Note that pAp can be identified with a von Neumann algebra on range (p).

PROOF. This is known [T 2, p. 365]. The argument is that if $z \in A''$ is the support projection of $A_* \subset A'$ then $pz = zp = pzp \in pA''p = (pAp)''$ is the support projection of $pA_*p = (pAp)_* \subset (pAp)' = pA'p$.

4.5. F. AND M. RIESZ THEOREM FOR $H^\infty(B)$: *Let $B = B(H)$ as in 5.1, $K = K(H) \subset B$ be the space of compact operators, and $\varphi = \varphi_* + \varphi_\perp$ be the Takesaki decomposition (5.3) of a member $\varphi \in L^\infty_w(B)'$. If $\varphi | H^\infty(B) = 0$ then separately $\varphi_* | H^\infty(B) = 0 = \varphi_\perp | H^\infty(B)$. Moreover, $\varphi_\perp$ also vanishes on $C(K)$.*

PROOF. First case: $\dim H < \infty$. Then of course $L^\infty_w(B) = L^\infty(B)$, and both the Takesaki (5.3) and the Yosida-Hewitt (2.2) L-projections in $L^\infty(B)'$ have range $L^1(N)$. By uniqueness of the L-complement [HWW, 1.1.2], both decompositions coincide and the theorem is a special case of 2.2 resp. 2.3.

Second case: $\dim H = \infty$. The following finite-dimensional approximation is inspired by [Pi, 3.4]. Fix an orthonormal basis $(e_n)_{n \in \mathrm{N}}$ of H and let $p_n \in B$ be the orthogonal projection onto $H_n := \operatorname{lin}\{e_1, \ldots, e_n\}$. The space $B_n := p_n B p_n$ consists of matrices with 0 outside the upper left $n \times n$-corner and can thus be identified with $B(H_n)$; also, $L^\infty(B_n) = p_n L^\infty_w(B) p_n$, where $p_n \in L^\infty_w(B)$ is considered now as the constant function p_n (these are the identifications after lemma 4.4). Let $\varphi_n := \varphi | L^\infty(B_n)$, then $\varphi | H^\infty(B) = 0$ obviously implies $\varphi_n | H^\infty(B_n) = 0$, hence $\varphi_{n,*} | H^\infty(B_n) = 0$ and $\varphi_{n,\perp} | C(B_n) = 0$ by the first case of the proof. Lemma 4.4 says $\varphi_{n,*} = \varphi_{*,n} := \varphi_* | L^\infty(B_n)$ and $\varphi_{n,\perp} = \varphi_{\perp,n} := \varphi_\perp | L^\infty(B_n)$.

i) Let $\varphi_* \in L^\infty_w(B)_*$ be given by $g \in L^1(N)$, and let $f \in H^\infty(B)$. In order to prove $\varphi_*(f) = \int \langle g, f \rangle \, d\lambda = 0$, consider $0 = \varphi_*(p_n f(\cdot) p_n) = \int \langle g(t), p_n f(t) p_n \rangle \frac{dt}{2\pi}$. Now pointwise $p_n f(t) p_n \xrightarrow{n \to \infty} f(t)$ in the weak operator topology, hence [Pd, 4.6.14] for $\sigma(B, N)$. Thus $\langle g(t), p_n f(t) p_n \rangle \xrightarrow{n \to \infty} \langle g(t), f(t) \rangle$ pointwise and dominated by $\|f\|_{L^\infty_\bullet(B)} \|g(\cdot)\|_N \in L^1$. By Lebesgue's theorem, $\int \langle g, f \rangle \, d\lambda = \lim_{n \to \infty} \int \langle g, p_n f p_n \rangle \, d\lambda = 0$.

ii) Now let $f \in C(K)$. In order to prove $\varphi_\perp(f) = 0$ it suffices to observe that $\lim_{n \to \infty} \|f(e^{it}) - p_n f(e^{it}) p_n\|_{K(H)} = 0$ uniformly in $e^{it} \in \mathrm{T}$. Then norm continuity of $\varphi_\perp$ suffices to conclude $\varphi_\perp(f) = \lim_{n \to \infty} \varphi_\perp(p_n f p_n) = 0$.

For the next corollary, note that $N = K'$, hence $\mathrm{H}^1_0(N) = H^1_0(N) = (C(K)/A(K))'$.

4.6. COROLLARY 1. $\mathrm{H}_0^1(N)'' = \mathrm{H}_0^1(N) \oplus_1 (L_w^\infty(B)'_\perp \cap H^\infty(B)^\perp)$. *$C(K)/A(K)$ is an M-ideal in its bidual $L_w^\infty(B)/H^\infty(B)$.*

PROOF. The decomposition follows from theorems 4.3 and 4.5 (cf. 2.4). The second assertion follows from the last part of theorem 4.5: $L_w^\infty(B)'_\perp \cap H^\infty(B)^\perp \subset C(K)^\perp$ (cf. 2.5).

REMARK. The M-embedded nature of $C(K)/A(K)$ can also be derived from Page's theorem [Pa, Theorem 6], according to which $C(K(H))/A(K(H))$ is isometrically isomorphic to the space of compact Hankel operators on the Hilbert space $H^2(H)$, cf. [HWW, III.1.8].

4.7. COROLLARY 2. *$L^1(N)/\mathrm{H}_0^1(N)$ is L-embedded in its bidual.*

PROOF. Theorems 4.3 and 4.5 (cf. 2.6).

As noted in section 2.6, it follows that $L^1(N)/\mathrm{H}_0^1(N)$ is weakly sequentially complete and even enjoys property (V*). This points into the direction of a positive answer to a question of Pisier: $H^\infty(B)$ is it a Grothendieck space? This question arises because both H^∞ and B are Grothendieck spaces and even have property (V) [Bo; Pf 2].

4.8. PROPOSITION. *$L^1(N)/\mathrm{H}_0^1(N)$ has property* (X).

PROOF. Similar to that of 4.5. Let $\varphi = \varphi_* + \varphi_\perp \in L_w^\infty(B)'$ respect the w^*-convergence of wuC series in $H^\infty(B)$; since φ_* does the same I can assume from the outset that $\varphi = \varphi_\perp$ is singular and have to show that φ vanishes on $H^\infty(B)$. In the first case ($\dim H < \infty$) this follows again by the argument "Takesaki = Yosida-Hewitt" and 3.2. For the second case ($\dim H = \infty$) note first that $\sum_{n=1}^{\infty} (p_n - p_{n-1})$ (where $p_0 := 0$) is a wuC series in B, so that for every $f \in H^\infty(B)$ both series $\sum_{n=1}^{\infty} (p_n - p_{n-1})f$ and $\sum_{n=1}^{\infty} f(p_n - p_{n-1})$ are wuC series in $H^\infty(B)$ with w^*-limit f. Now fix $f \in H^\infty(B)$; in order to prove $\varphi(f) = 0$ it therefore suffices to prove $\varphi(p_m f) = 0$ for every *fixed* m, and now it suffices to prove $\varphi(p_m f p_n) = 0$ for every $n \geqq m$. But $p_m f p_n = p_n p_m f p_n \in H^\infty(B_n)$; $\varphi | L^\infty(B_n)$ is singular (4.4) and respects the w^*-convergence of wuC series in $H^\infty(B_n)$, hence vanishes on $H^\infty(B_n)$ by the first case of the proof which is now complete.

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