MATH. SCAND. 79 (1996), 237–254

SPECTRAL ESTIMATES FOR MAGNETIC OPERATORS

MICHAEL MELGAARD and GREGORY V. ROZENBLUM

Abstract.

The well-known CLR-estimate for the number of negative eigenvalues of the Schrödinger operator $-\Delta + V$ is carried over to a class of second order differential operators, generalizing the magnetic Schrödinger operator. The coefficients in the magnetic operators are variable, they may be non-smooth, unbounded and some degeneration is allowed.

I. Introduction.

Estimates of the spectrum of differential operators have been attracting attention for a long time both in physics and mathematics. They find applications, in particular, in studying the asymptotics of eigenvalues, some characteristics in the scattering theory and in the analysis of stability of matter (see e.g.[14], [9], [2], [3]).

An important result here is the so-called CLR-*estimate* for the number of negative eigenvalues $N_-(V)$ of the Schrödinger operator $-\Delta + V$ in R^d, $d \geqq 3$:

$$N_-(V) \leqq c \int_{\mathsf{R}^d} (V_-(x))^{d/2}\, dx, \tag{1.1}$$

where $V_-(x) = (|V(x)| - V(x))/2$. In this estimate, the constant c depends only on the dimension d of the space and holds for any potential V for which the right-hand side is finite. One of the reasons why (1.1) finds a number of applications is the fact that the integral in (1.1) is proportional to the volume of the region in the phase space where the classical Hamiltonian $H(p, x) = p^2 + V(x)$ is negative. This implies, in particular, that (1.1) behaves correctly when one includes the Planck constant h in the operator or introduces the coupling constant (replacing V by tV).

The estimate (1.1) has been proved by four, completely different methods. The original proof in [15] (the result was announced in [16]) was based on the piece-wise polynomial approximations in Sobolev spaces, refining the ideas by

Received February 8, 1995.

M. Birman and M. Solomyak [3]. The proof by E. Lieb [10] uses the representation of heat kernels by means of path integration (the Feynman-Kac formula). M. Cwikel's proof [5] was based on the ideas, actually belonging to the theory of interpolation of linear operators. The latest proof, by P. Li and S. T. Yau [8], probably the most simple one, uses some analysis of heat kernels.

Different methods of proof lead to different constants in (1.1) (for some applications, the value of the constant is crucial); in particular, for $d = 3$ the best c is obtained in [10]. On the other hand, the proofs admit different possibilities of generalization of (1.1). The methods in [10] and [8] seem to be the most specific; the method in [15] – the most general. In particular, the latter method provides one with estimates for differential operators of any order, probably degenerate, and also for some integral operators.

The Schrödinger operator describes behaviour of a particle in the electric field with potential V. The natural question in the case when a magnetic field is also present leads to spectral estimates for the "magnetic Schrödinger operator"

$$H_V = -(\nabla - i\alpha(x))^2 + V(x) \tag{1.2}$$

with magnetic vector potential $\boldsymbol{a} = (a_1, \dots, a_d)$. Phase space considerations hint that for the negative eigenvalues of (1.2), the same estimate (1.1) must hold, with a constant not depending on $\boldsymbol{a}$ (this by no means implies that the estimate cannot be improved for some classes of magnetic potentials). In fact, such an estimate has been proved. Usually, it is attributed to E. Lieb, but the first published version appeared in [17]. This proof uses the original approach by Lieb, representing the magnetic heat kernel by means of a path integral, this time, Itô's one. In discussing this, in [1], the question was asked whether it is possible to carry over other proofs of the CLR-estimate to the magnetic case. However, up till now no such proofs appeared. So, we have a somewhat unnatural situation: an analytical result lacks analytical proofs.

In the present paper we give an alternative proof of the estimate of the form (1.1) for a class of second order magnetic operators, generalizing (1.2). The coefficients are variable, they may be nonsmooth, unbounded, some degeneration of ellipticity is allowed. Making some speculations, one may say that our operators describe behaviour of quantum particles in a curved space, with a non-Euclidean metric, thus taking some general relativity effects into account. Here, a point of the degeneration of ellipticity of the operator corresponds to the zero speed of light. The light, having reached this point, canot leave it, so we have something like a "black hole". Thus, our results show that provided the degeneration is not too severe, in other words, if the black hole is "not too black", then, for any magnetic field, only a finite number of bound states of the particle (e.g. an electron) can be generated by a given electric potential. On the other hand, if the speed of light increases at infinity quickly enough then, for any magnetic field, the whole spectrum of the operator is discrete, as it is to be expected in the closed model of the Universe.

In Sec. 2 we give exact definition of our operators. We cannot follow the pattern of [18], [1], where operators (1.2) were considered, in our general conditions. In fact, for variable coefficients operators in dimension $d > 2$, the results on the coincidence of maximal and minimal Schrödinger forms and on essential self-adjointness do not hold (see, e.g. [11]), even without magnetic field. It is by the method of quadratic forms, that the magnetic operators are defined. Some approximation are used in obtaining the estimates, so we show strong resolvent continuous dependence of the operator on the magnetic potential. Our method is based on the combination of ideas of [15] and [8]. Since [15] was published in a quite obscure and virtually unaccessible Russian journal, in Sec. 3, we reproduce from [15] the original proof, as applied to second order operators. The spectral estimates are proved in Sec. 4.

We do not touch applications of our estimates. Nowadays, it is a matter of routine to derive results on the spectral asymptotics from spectral estimates, provided the latter ones are exact enough; see e.g., [14], [15], [2], [3], [12].

This paper had its origin in a M.Sc.Thesis by M.M. and R. Ravnstrup under the supervision of G.R., where the alternative proof of the magnetic CLR-estimate was found. Later, the general results were obtained by G.R. and M.M.

2. Magnetic Operators.

We define our operators by means of quadratic forms. Let $g(x) = \{g_{jk}(x)\}$, $1 \leqq j, k \leqq d$ be a (possibly, complex-valued) matrix function defined on R^d, positive-definite a.e.: there exists a function $\gamma(x) \geqq 0$, $\gamma > 0$ a.e. such that

$$\sum_{j=1}^{d} \sum_{k=1}^{d} g_{jk}(x)\xi_j\overline{\xi_k} \geqq \gamma(x)|\xi|^2 \tag{2.1}$$

for any $\xi = (\xi_1, \ldots, \xi_d) \in \mathsf{C}^d$; if $\gamma(x) \geqq \gamma_0 > 0$ the matrix (and, correspondingly, the operator) is called *uniformly elliptic*; otherwise we deal with *degeneration.*

The magnetic field will be described by the real magnetic vector potential $\boldsymbol{a}(x) = (a_1(x), \ldots, a_d(x))$. To be able to define the magnetic operator, we impose the following condition on the coefficients.

CONDITION A. $g_{jk} \in L^{\rho}_{\text{loc}}$, $\boldsymbol{a} \in L^{\sigma}_{\text{loc}}$, $\rho^{-1} + 2\sigma^{-1} = 1$.

Let Ω be some open set in R^d, and let $P_j = \partial/\partial x_j - ia_j$. We define

$$A_{g,\boldsymbol{a},\Omega}[\phi] = \int_\Omega \sum_{j=1}^{d} \sum_{k=1}^{d} g_{jk} P_j\phi \overline{P_k\phi}\, dx. \tag{2.2}$$

Condition A quarantees that the quadratic form (2.2) is defined on $C_0^\infty(\Omega)$. We shall omit some (or all) subscripts in the notation of this (and other) quadratic forms, provided this does not produce ambiguity.

For the form $A[\phi]$ to enjoy nice properties, we have to control the degeneration.

CONDITION B. $\gamma^{-1} \in L^p(\Omega)$, $p \geqq \max(d/2, 1)$.

2.1. *The Free Magnetic Operator.* Our aim is to associate a self-adjoint operator in $L^2(\mathrm{R}^d)$ to the form A. To do this, one has to know that the form is closable in $L^2(\mathrm{R}^d)$ (we remind that this means that if $\phi_n \to 0$ in $L^2(\mathrm{R}^d)$ and $A[\phi_n - \phi_m] \to 0$ then $A[\phi_n] \to 0$).

PROPOSITION 2.1. *Under conditions A and B, the form $A[\phi]$ is closable.*

PROOF. We begin by establishing an inequality which will be useful throughout the paper. It follows from (2.1) and the Hölder inequality that, for any $G \subset \Omega$,

$$(2.3)\quad A_G[\phi] \geqq \int_G \gamma(x) \sum_{k=1}^d |P_k\phi|^2\, dx \geqq \|\gamma^{-1}\|_{L_p(G)}^{-1} \| \|(\nabla - ia)\phi\|_{L_s(G)}^2, s = \frac{2p}{p+1}.$$

Consider the space H_g of vector-functions $v = (v_1, \ldots, v_d)$ on Ω such that

$$(2.4)\qquad a[v] \cong \int_\Omega \sum_{j=1}^d \sum_{k=1}^d g_{jk}(x) v_j(x) \overline{v_k(x)}\, dx < \infty.$$

Since g is a symmetric nonnegative matrix for all x, there exists a unitary matrix-valued function $\Phi(x)$ such that

$$g(x) \equiv \{g_{j,k}(x)\} = \Phi(x)^{-1} D(x) \Phi(x)$$

with a real diagonal matrix $D(x) = \operatorname{diag}(\delta_1(x), \ldots, \delta_d(x))$. Denote by $w(x)$ the vector function $\Phi(x)v(x)$, $w(x) = (w_1(x), \ldots, w_d(x))$. Then (2.4) takes the form

$$(2.5)\qquad a[v] = \int_\Omega \sum_{j=1}^d \delta_j(x) |w_j(x)|^2\, dx \equiv a_\Phi[w].$$

Now, suppose some sequence $v^n(x) \in H_g$ satisfies $a[v^n - v^m] \to 0$. Then, correspondingly, we have $a_\Phi[w^n - w^m] \to 0$ and, according to (2.5),

$$(2.6)\qquad \int_\Omega \delta_j |w_j^n - w_j^m|^2\, dx \to 0.$$

For any given j, the expression in (2.6) is the norm of $w_j^n - w_j^m$ in the L^2-space with respect to the measure $d\mu_j = \delta_j(x)\, dx$; since such L^2-spaces are complete, it follows that there exists limit w_j of w_j^n in this space, and therefore, due to the fact that $\delta_j(x) > 0$ almost everywhere, there exists some subsequence of w_j^n converging to w_j a.e. Such a subsequence may be taken the same for all j, and we keep the notation w_j^n for this subsequence. Moreover, we have $v^n = \Phi^{-1}w^n$, and therefore v^n converges to $v = \Phi^{-1}w$ a.e., and $a[v^n] \to a[v]$.

Now, let $A_\Omega[\phi_n - \phi_m] \to 0$, $\phi_n, \phi_m \in C_0^\infty(\Omega)$ and $\|\phi_n\|_{L^2} \to 0$. Take $v_j^n = P_j\phi_n$. The sequence $v^n \in H_g$ fits into the previous consideration. Therefore, for some subsequence and some element $v \in H_g$, we have

$$A_\Omega[\phi_n] \to a_\Omega[v]; \; P_j\phi_n \to v_j \text{ a.e.} \tag{2.7}$$

On the other hand, due to (2.3),

$$\|P_j(\phi_n - \phi_m)\|_{L_s(\Omega)} \to 0. \tag{2.8}$$

Introduce $\phi_n^j(x) = \exp(-i\mu_j(x))\phi_n$,

$$\mu_j = \int_{x_j^0}^{x_j} a_j(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_d)\, dy_j. \tag{2.9}$$

The later integral makes sense and defines a function in H^1_{loc} in the x_j variable for almost all $x_j' = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d)$. Using [18], we get

$$P_j\phi_n = e^{i\mu_j}\partial(e^{-i\mu_j}\phi_n)/\partial x_j = e^{i\mu_j}\partial\phi_n^j/\partial x_j,$$

and so, from (2.8), it follows that, for almost all x_j',

$$\left\|\frac{\partial}{\partial x_j}(\phi_n^j - \phi_m^j)\right\|_{L^s(\Omega_j(x_j'))} + \|\phi_n^j - \phi_m^j\|_{L^2(\Omega_j(x_j'))} \to 0,$$

where $\Omega_j(x_j')$ is the one-dimensional cross-section of Ω in the direction of x_j, for a given x_j'. Due to completeness of the Sobolev spaces there exists, for any j, a function w^j such that

$$\left\|\frac{\partial}{\partial x_j}(\phi_n^j - w^j)\right\|_{L^s(\Omega_j(x_j'))} + \|\phi_n^j - w^j\|_{L^2(\Omega_j(x_j'))} \to 0$$

for almost all x_j'. Moreover, since, for any n, we have $\exp(i\mu_j)\phi_n^j = \exp(i\mu_k)\phi_n^k$, the functions w^j are consistent in the same sense, $\exp(i\mu_j)w^j = \phi$. This implies, in particular, that (again for some subsequence) $P_j\phi_n \to P_j\phi$, $\phi_n \to \phi$ a.e. The latter relation, together with $\|\phi_n\|_{L^2} \to 0$, give that $\phi = 0$. So, we have $P_j\phi_n \to 0$ a.e. Now, (2.7) gives us that $v_j = 0$, $a[v] = 0$ and finally, $A_\Omega[\phi_n] \to 0$.

Due to Proposition 2.1, the form $A_{g,a,\Omega}$ generates a self-adjoint nonnegative operator $H_{0,g,a,\Omega}$ in $L^2(\Omega)$ (again, some or all subscripts may be omitted).

The following inequality is useful in obtaining estimates for the form A.

PROPOSITION 2.2. *Suppose that conditions* A *and* B *are satisfied. Then*

$$A_g[\phi] \geqq \int_\Omega \gamma(x) \sum_{k=1}^{d} \left| \frac{\partial}{\partial x_k} |\varphi| \right|^2 dx, \quad \phi \in C^\infty(\Omega). \tag{2.10}$$

PROOF. To show (2.10), one has to note that $A_g[\phi] \geqq A_\gamma[\phi] \equiv \int_\Omega \gamma(x) |P_j \phi|^2 dx$. Now (2.10) follows from the well-known fact that $|P_j \phi| \geqq \left| \frac{\partial}{\partial x_j} |\varphi| \right|$.

An immediate consequence of Proposition 2.2 is the following estimate.

PROPOSITION 2.3. *Under conditions* A *and* B, *we have*

$$A_\Omega[\phi] \geqq C \, \|\gamma^{-1}\|_p^{-1} \, \|\phi\|_{q*}^2, \quad \phi \in C_0^\infty(\Omega), \tag{2.11}$$

where $q^{*-1} = 1/2 - 1/d = 1/2 - 1/d + 1/(2p)$, *with a constant* C *depending only on* d, p.

PROOF. To show (2.11), we combine (2.10), (2.3) and the Sobolev inequality $\|\nabla v\|_s \geqq C \, \|v\|_{q*}$ for $v = |\phi|$ with a compact support in Ω.

2.2 *The Magnetic Operator*. Let V be a real measurable function. We will define the operator $H_V = H_0 + V$ by the method of quadratic forms.

CONDITION C. $V \in L^1_{\mathrm{loc}}(\Omega)$; $(V + \tau)_- \equiv (|V + \tau| - (V + \tau))/2 \in L^q(\Omega)$ for some $\tau, q^{-1} + p^{-1} = 2/d$, $q > 1$.

PROPOSITION 2.4. *Let the conditions* A, B *and* C *be fulfilled. Then the form*

$$A_V[\phi] = \int_\Omega \sum_{j=1}^{d} \sum_{k=1}^{d} g_{j,k} P_j \phi \overline{P_k \phi} \, dx + \int_\Omega V |\phi|^2 \, dx \tag{2.12}$$

defined on $C_0^\infty(\Omega)$ *is closable.*

PROOF. We can suppose that $\tau = 0$, moreover, that the L^q-norm of V_- is small enough. Then, according to the Hölder inequality and Proposition 2.3, we have

$$\int_\Omega V_- \, |\phi|^2 \, dx \leqq \|V_-\|_q \, \|\phi\|_{q*}^2 \leqq C \, \|V_-\|_q \, \|\gamma^{-1}\|_p A_g[\phi], \tag{2.13}$$

for $\phi \in C_0^\infty(\Omega)$. So, if $\|V_-\|_q$ is small enough, then $\int_\Omega V_- \, |\phi|^2 \, dx \leqq \sigma A[\phi]$, with $\sigma < 1$, and this, according to the KLMN Theorem (see [13]), implies that the form $A_g[\phi] - \int V_- \, |\phi|^2$ is semi-bounded and closable. The positive form $\int V_+ \, |\phi|^2$, $(V_+ = (|V| + V)/2$ can be added to the later one with no trouble since $C_0^\infty(\Omega)$ is a common core for these forms.

As a result, we obtain a self-adjoint operator H_V.

2.3 *Operator Properties.* It is convenient to study the properties of the operators in question first for the case of some nice magnetic and electric fields. As in [18], this proves to be sufficient for general spectral estimates, due to the continuous dependence of the operators on the potentials.

PROPOSITION 2.5. *Let $\boldsymbol{a}_n$, $\boldsymbol{a} \in L^{\sigma}_{\mathrm{loc}}$, V_n, $V \in L^1_{\mathrm{loc}}$, and suppose $\boldsymbol{a}_n \xrightarrow{L^{\sigma}_{\mathrm{loc}}} \boldsymbol{a}$, $V_n \xrightarrow{L^1_{\mathrm{loc}}} V$. Then, the operators $H_{V_n, \boldsymbol{a}_n}$ converge to $H_{V,\boldsymbol{a}}$ in the strong resolvent sense.*

PROOF. The proof follows the pattern of the one in [18] for the case of the unit matrix g. We take $V_n = V$ fixed; the continuous dependence on V is proved exactly as in [18]. Let $f \in L^2$, $\phi_n = (H_n + i)^{-1} f$. Then $\|\phi_n\|_{L^2}$, and also

$$\|f\|_{L^2} \geqq (H_n \phi_n, \phi_n)_{L^2} = \|g^{1/2}(\nabla - i\boldsymbol{a}_n)\phi_n\|^2_{L_2} + \|V_+^{1/2}\phi_n\|^2_{L_2} - \|V_-^{1/2}\phi_n\|^2_{L_2}$$
$$\geqq \tfrac{1}{2}(\|g^{1/2}(\nabla - i\boldsymbol{a}_n)\phi_n\|^2_{L_2} + \||V|^{1/2}\phi_n\|^2_{L_2}).$$

Hence, we can take a subsequence (denoted still by ϕ_n) such that

$$\psi_n = g^{1/2}(\nabla - \mathrm{i}\boldsymbol{a}_n)\phi_n \xrightarrow{w} \psi,\ \phi_n \xrightarrow{w} \phi, \text{ and } |V|^{1/2}\phi_n \xrightarrow{w} |V|^{1/2}\phi.$$

Consider the space of vector functions $\mathscr{R} = g^{-1/2} C_0^\infty \cap L_V^2$; it is obviously dense in L^2. Take any $h \in \mathscr{R}$. Consider the operators $g^{1/2}(\nabla - \mathrm{i}\boldsymbol{a}_n)$, $g^{1/2}(\nabla - i\boldsymbol{a})$ on C_0^∞ (due to Proposition (2.1), they are closable). A standard calculation shows that

$$h \in D((g^{1/2}(\nabla - i\boldsymbol{a}_n))^*), h \in D((g^{1/2}(\nabla - i\boldsymbol{a}))^*)$$

and

$$(g^{1/2}(\nabla - i\boldsymbol{a}_n))^* \boldsymbol{h} = (-\nabla + i\boldsymbol{a}_n) g^{1/2} h, (g^{1/2}(\nabla - i\boldsymbol{a}))^* h = (-\nabla + i\boldsymbol{a}) g^{1/2} h.$$

Thus, we have strong convergence $(g^{1/2}(\nabla - i\boldsymbol{a}_n))^* h \xrightarrow{s} (g^{1/2}(\nabla - i\boldsymbol{a}))^* h$, and

$$((g^{1/2}(\nabla - i\boldsymbol{a}))^* h, \phi)_{L^2} = \lim_{n\to\infty} ((g^{1/2}(\nabla - i\boldsymbol{a}_n)^* h, \phi_n)_{L^2}$$
$$= \lim_{n\to\infty} (h, g^{1/2}(\nabla - i\boldsymbol{a}_n)\phi_n)\phi_n)_{L^2} = (h, \psi)_{L^2}.$$

This means that $\phi \in \mathrm{D}((\mathrm{g}^{1/2}(\nabla - \mathrm{i}\boldsymbol{a}))^{**}) = D(\overline{g^{1/2}(\nabla - i\boldsymbol{a})})$ and $\overline{g^{1/2}(\nabla - i\boldsymbol{a})\phi} = \psi$. Next, take $u \in C_0^\infty$. Then $g^{1/2}(\nabla - i\boldsymbol{a}_n)u \xrightarrow{s} g^{1/2}(\nabla - i\boldsymbol{a})u$, and so

$$(u, f)_{L^2} = (g^{1/2}(\nabla - i\boldsymbol{a}_n)u, g^{1/2}(\nabla - i\boldsymbol{a}_n)\phi_n)_{L^2} + (V^{1/2}u, |V|^{1/2}\phi_n)_{L^2} + i(u, \phi_n)_{L^2}$$
$$\to (g^{1/2}(\nabla - i\boldsymbol{a})u, \overline{g^{1/2}(\nabla - i\boldsymbol{a})\phi})_{L^2} + (V^{1/2}u, |V|^{1/2}\phi)_{L^2} + i(u, \phi_n)_{L^2}.$$

According to the definition of the operators by means of quadratic forms, this implies that $\phi \in D(H)$ and $(H + i)\phi = f$. Thus, any weak limit point of $\phi_n$is'

$(H + i)^{-1}f$. By weak compactness of the unit ball we see that $(H_n + i)^{-1}$ converges weakly to $(H + i)^{-1}$. The same holds for the operators $(H_n - i)^{-1}$. As usual, this implies strong resolvent convergence of H_n to H.

As a result of our proposition, it will be sufficient to prove the spectral estimates for smooth and/or bounded electric and magnetic potentials, provided the estimates would imply only itegral norms of the data of the operators.

A special role in our considerations will be played by operators with matrix g having a special form, namely

$$g(x) = \gamma(x)I. \tag{2.14}$$

It is for these matrices and smooth magnetic fields, that we obtain a nice representation for the heat semigroup. Let $g(x)$ have the form (2.14). Then for each k, $1 \leqq k \leqq d$, for $\Omega = Q$ (a cube) we can define a nonnegative quadratic form

$$h_a^k[u] = \int_\Omega \gamma(x)\left|\frac{\partial u}{\partial x_k} - ia_k u\right|^2 dx$$

with $D(h_a^k) = C_0^\infty(Q)$. This form is closable and defines a selfadjoint operator H_a^k in L^2. In the sense of quadratic forms, we have $h_a = \sum_k h_a^k$.

PROPOSITION 2.6. *Let $a_k \in C_0^\infty(\Omega)$ and let*

$$\mu_k(x) = \int_{-\infty}^{x_k} a_k(x_1, \ldots, y_k, \ldots, x_d)\, dy_k.$$

Then

$$e^{-i\mu_k(x)}D(H_a^k) = D(H_0^k) \tag{2.15}$$

and

$$H_a^k e^{-i\mu_k(x)} = e^{-i\mu_k(x)}H_0^k. \tag{2.16}$$

REMARK. For $\gamma(x) = 1$, this is the result of [18]. We must use another way of. proving, since, as in many instances above, we do not have any essentially selfadjoint operators and so we can only manipulate with quadratic forms.

PROOF. Let $\psi \in D(H_a^k)$ and $\psi_k = \partial\psi/\partial x_k$. This means that $\mathscr{L}(u, \psi) = (\gamma_k(x)(\partial u - ia_k u), (\psi_k - ia_k\psi))$ is a linear bounded functional in L^2 with respect to the variable $u \in C_0^\infty(Q)$. Taking $\phi = e^{-i\mu_k(x)}\psi$, we have

$$\begin{aligned}\mathscr{L}(u, \psi) &= \int \gamma\left(\frac{\partial u}{\partial x_k} - ia_k u\right)\overline{e^{i\mu_k}\frac{\partial \phi}{\partial x_k}}\, dx \\ &= \int \gamma e^{-i\mu_k}\left(\frac{\partial u}{\partial x_k} - ia_k u\right)\overline{\frac{\partial \phi}{\partial x_k}}\, dx = \int \gamma \frac{\partial v}{\partial x_k}\overline{\frac{\partial \phi}{\partial x_k}}\, dx,\end{aligned}$$

where $v = e^{-i\mu_k}u$. Now (and this is the *only* place we use it), for smooth a_k, the functions u and v belong to $C_0^\infty(Q)$ simultaneously, and so, $\mathscr{L}$, considered as a functional in $v \in C_0^\infty$, is continuous which means that $\phi \in D(H_0^k)$. The opposite inclusion of domains and the equality (2.16) follow now in an elementary way.

3. Non-Magnetic Spectral Estimates.

In this section we reproduce, with some minor modifications, the original proof of the CLR-estimate, for the particular case of second-order operators.

3.1 *The Spectral Estimate.* Let $\Omega \subset \mathsf{R}^d$ and the function γ satisfy condition B. Let $b(x)$ be a real function,

$$b \in L^q(\Omega),\ q^{-1} + p^{-1} = 2d^{-1}. \tag{3.1}$$

For any set $G \subset \Omega$ we introduce the quadratic form

$$B[u] = \int_\Omega b(x)\,|u(x)|^2\,dx. \tag{3.2}$$

Sometimes, G and b will be included in the notation, as subscripts of B.

To the quadratic form $A_\Omega[\phi]$ given by (2.2), with $\boldsymbol{a} = 0$, we associate the space $H_g = H_g(\Omega)$, the closure of $C_0^\infty(\Omega)$ with respect to the norm $(A[\phi])^{1/2}$. It follows from (2.13) that the form B is bounded with respect to A and thus B generates a bounded self-adjoint operator $T = T_{A,B}$ in H_g. We denote by $n_\pm(\lambda, T)$ the number of points of spectrum of $\pm T$ in (λ, ∞); $\lambda > 0$.

THEOREM 3.1. *Let the condition* B *hold with* $1 \leqq p, q < \infty$; *if* $q = 1$, *it is possible that* $p = \infty$. *Let* $\theta^{-1} = 2d^{-1}$. *Then*

$$n_\pm(\lambda, T) \leqq c\lambda^{-\theta}\,\|\gamma^{-1}\|_p^\theta\,\|b_\pm\|_q^\theta, \tag{3.3}$$

with a constant c depending on p, q, d but not on b, g, Ω.

REMARK. Theorem 3.1 is generalized to the case of forms (and operators) defined in the space of vector-functions in Ω. The formulation is obvious: the coefficient $g_{j,k}$, b become $s \times s$ matrices; the ellipticity condition takes the form:

$$\langle \sum g_{j,k}\zeta_j\overline{\zeta_k}\eta, \eta\rangle \geqq \gamma(x)\,|\zeta|^2\,|\eta|^2$$

for $\zeta_j \in \mathbf{C}^d$, $\eta \in \mathbf{C}^s$, and it is the norm of $g(x)^{-1}$ which is denoted by $\|\gamma^{-1}\|$. The proof goes through without any changes.

PROOF. We start by stating an inequality for the forms A_G and B_G for the case when G has a rather special form. We shall call a parallelepiped in R^d a *brick* if the ratio of the longest and shortest edges is not greater than 2. Then, for G being a brick and for any function $u \in C^\infty(\bar{G})$ satisfying $\int_G u(x)\,dx = 0$, we have

$$B_G[u] \leqq c_1 \|\gamma^{-1}\|_{p,G} \|b\|_{q,G} A_G[u] \tag{3.4}$$

with some $c_1 = c_1(p, q, d)$. To show (3.4), recall the inequality (2.3) (this time, with $\boldsymbol{a} = 0$). However, instead of using the Sobolev inequality to obtain (3.4) (as it was used in proving (2.13)), we apply the fact (actually, a form of the Poincaré inequality) that for G being a *unit* cube, the expression $\|\nabla u\|_s$ is, on functions with zero mean value over G, equivalent to the usual norm in the Sobolev space $W_s^1(G)$. On the other hand, there is a continuous embedding of $W_s^1(G)$ into $L_{2q}(G)$; the latter two results and (2.3), taken together, give (3.4) for the unit cube, the constant c_1 being the product of constants in the Poincaré inequality and the embedding theorem. Under dilations ($x \mapsto tx$) both parts in (3.4) have the same order of homogenity in t; this provides us with (3.4) for any cube, with the same constant. Finally, to pass to a brick G, we make non-uniform dilation $x_j \mapsto t_j x_j$, $1 \leqq t_j \leqq 2$, of a cube to G. This gives us (3.4), with c_1 getting an additional factor $2^{(d-1)/s}$.

In this proof and in the next section, we shall use the min-max principle for the eigenvalues of self-adjoint operators (see e.g. [14] [Th. XIII.1]). As applied to our operator $T = T_{A,B}$, this principle gives

$$n_\pm(\lambda, T) = \min \operatorname{codim}\{\mathscr{K} \subset C_0^\infty(\Omega),\ \pm B[u] < \lambda A[u],\quad u \in \mathscr{K} \setminus \{0\}\}, \tag{3.5}$$

or in the usual formulation

$$n_\pm(\lambda, T) = \max \dim\{\mathscr{L} \subset C_0^\infty(\Omega),\ \pm B[u] \geqq \lambda A[u],\quad u \in \mathscr{L}\}. \tag{3.6}$$

So, to prove (3.3), it suffices to find, for any given λ, such a subspace $\mathscr{K} = \mathscr{K}_N$ for which (3.5) holds and which has a codimension less than $c\lambda^{-\theta} \|\gamma^{-1}\|_p^\theta \|b\|_q^\theta$.

Introduce the function of sets

$$J(G) = \left(\int_G |g^{-1}|^p \, dx\right)^{\theta/p} \left(\int_G |b|^q\right)^{\theta/q}. \tag{3.7}$$

This function, as it easily follows from the Hölder inequality, is *lower semi-additive*, i.e. for disjoint sets G_1, G_2:

$$J(G_1 \cup G_2) \geqq J(G_1) + J(G_2).$$

Also, this function is *continuous* in the sense that continuity of the Lebesgue measure of a monotone family G_t implies continuity of $J(G_t)$. The proof of the following lemma will be given in the end of this section.

LEMMA 3.2. *Let on Borel subsets in a cube Q a lower semi-additive and continuous function of sets $J(G)$ be given. Then, for any $n \geqq 1$, there exists a covering Ξ of Q by bricks $\Delta \subset Q$ such that the number $|\Xi|$ of bricks is not greater than n, each point of the cube Q belongs to not more than d bricks and for any $\Delta \in \Xi$*

(3.8) $$J(\Delta) \leqq \kappa n^{-1} J(Q), \quad \kappa = 2^{d+1}.$$

We apply Lemma 3.2 to the function $J(G)$ defined in (3.7). As a result, we obtain, for a given cube Q, the covering Ξ. Introduce the space $\mathscr{K}_N$ consisting of functions in $C_0^\infty(Q)$ for which the integral of u over each of $\Delta_k \in \Xi$ equals zero. In other words, on $\mathscr{K}_N$ exactly N functionals vanish, so the codimension of $\mathscr{K}_N$ equals N. Next, we show that for $u \in \mathscr{K}_N$, we have

$$B_Q[u] \leqq c\lambda^{-\theta} J(Q) A_Q[u].$$

Enumerate the bricks in Ξ: $\Delta_1, \ldots, \Delta_N$ and define $G_1 = \Delta_1$, $G_k = \Delta_k \backslash \cup_{j<k} \Delta_j$. Denote by $u_k \in C^\infty(\Delta_k)$ the restriction of u onto Δ_k, $b^{(k)} = \chi_{G_k} b \in L^p(\Delta_k)$, where χ is the characteristic function. We have

(3.9) $$B_Q[u] = \sum \int_{\Delta_k} b^{(k)}(x) |u(x)|^2 \, dx = \sum B_{\Delta_k, b^{(k)}}[u_k].$$

Next, since any point in Q belongs to not more than μ bricks Δ_k,

(3.10) $$A_Q[u] \geqq \mu^{-1} \sum A_{\Delta_k}[u_k].$$

According to (3.4), $B_{\Delta_k}[u_k] \leqq c_1 J(\Delta_k)^{1/\theta} A_{\Delta_k}[u_k]$. Summing this over k and using (3.9) and (3.10), we come to

$$AB_Q[u] \leqq c_1 d J(Q)^{1/\theta} (\kappa N^{-1})^{1/\theta} A_Q[u].$$

So, for $\lambda = c_1 d J(Q)^{1/\theta} (\kappa N^{-1})^{1/\theta}$, we have constructed the subspace of codimension N where the inequality $B[u] \leqq \lambda A[u]$ holds. According to (3.5), this implies that $n_+(\lambda, T) \leqq N$ and this takes care of our theorem for the case $\Omega = Q$. As for the general case, note first that the min-max principle allows one to consider only the case $\Omega = \mathrm{R}^d$. We use it in the form (3.6). Suppose that on some finite-dimensional subspace $\mathscr{L} \subset C_0^\infty(\mathrm{R}^d)$ the inequality in (3.6) holds. Then supports of all functions in $\mathscr{L}$ are contained in some cube Q, therefore, as it is already proved, its dimension is not greater than $n_\pm(\lambda, T_{A,B,Q}) \leqq c \, \|\gamma^{-1}\|_{p,Q}^\theta \, \|b\|_{q,Q}^\theta \lambda^{-\theta}$, which gives us the required estimate for $n_\pm(\lambda, T_{\mathrm{R}^d})$. Finally, replacement of b by $b_\pm$ in the estimates is implied by the fact that $-B_{b_-}[u] \leqq B_b[u] \leqq B_{b_+}[u]$.

3.2 *The Covering Lemma.* Now, we prove Lemma 3.2. There are several ways of establishing this lemma. It may be derived from the general covering principle by M. de Guzman [7], or from a covering theorem due to Besicovich. We give a somewhat simplified original proof from [15].

Proof. Set $J(Q) = 1$. We call the set $G \subset Q$ "poor" if $J(G) \leqq \kappa n^{-1}$ and „rich" if $J(G) \geqq \kappa n^{-1}$ (in the case of equality, G is both rich and poor). Our aim is to cover Q by not more than n poor bricks.

Suppose Δ is some rich cube and let its edge have length l. Then at least one of the following possibilities takes place:

(i) Δ can be covered by not more than 2^d poor bricks $\Delta_j \subset \Delta$;

(ii) Δ can be cut into m_1 poor cubes and m_2 rich cubes, where $m_1 < 2^d$, $m_2 \geqq 2$;

(iii) there exists a rich cube $\Delta' \subset \Delta$ such that the set $\Delta \backslash \Delta'$ is also rich but can be covered by not more than d poor bricks.

To justify this assertion, we cut Δ into 2^d equal cubes with edge $l/2$. If they are all poor, we have the case (i); if at least two of them are rich, we have the case (ii). Hence, suppose that only one of small cubes, namely Δ_1, is rich and all the other ones are poor. If the set $S_1 = \Delta \backslash \Delta_1$ is rich, we have the case (iii). Otherwise, introduce the family of cubes Δ_t, with edge $tl/2$, which are cut out of the same corner of Δ, as Δ_1; $S_t = \Delta \backslash \Delta_t$. For any $0 < t \leqq 1$, as one can easily see, S_t can be covered by d bricks, lying in S_t. When t decreases from 1 to 0, the function $J(\Delta_t)$ decreases continuously to 0 and the function $J(S_t)$ grows. Set $t_1 = \sup\{t, \Delta_t \text{ is poor}\}$, $t_2 = \inf\{t, D_t \text{ is poor}\}$. If $t_1 \geqq t_2$, then Δ_{t_1}, S_{t_1} are both poor, and we have the case (i), since all bricks covering S_{t_1} are also poor. If $t_1 \leqq t_2$ then Δ_{t_2} and S_{t_2} are both rich and we have the case (iii).

To conclude the proof of the Lemma, assume $n \geqq \kappa$ (otherwise the trivial covering of Q by itself can be taken), and suppose that we already have a partition of Q into cubes (both rich and poor) and rich sets of the form S, as in (iii). We apply the previously described construction to those rich cubes for which the cases (ii) or (iii) take place, getting a new partition, the continuation of the previous one. So, starting with the trivial partition, we apply the continuation procedure successively. At each step, the quantity of rich elements of the partition increases, and due to semi-additiveness of $\mathscr{J}$, it cannot exceed $\kappa^{-1}n$. This means that after several steps, in our partition there will be only poor cubes (n_1 in number), n_2 rich cubes satisfying (1), and n_3 rich sets S_k, as in (iii).

We have $n_1 \leqq (2^d - 1)n_2$, since in our procedure, poor cubes are produced only in the case (ii), and one new rich cube produces no more than $2^d - 2$ poor ones. So, the inequality $n_2 \leqq \kappa^{-1}n$ implies $n_1 \leqq (2^d - 2)\kappa^{-1}n$. Finally, we cover all remaining rich cubes and sets S_k by poor bricks, which produces no more than $2^d(n_2 + n_3) \leqq 2^d\kappa^{-1}n$ poor bricks; there are no more than $2^{d+1}\kappa^{-1}n \leqq n$ bricks altogether, giving us the desired covering. Our bricks start overlapping only on the final stage (we dealt only with partitions before), so, any point of Q belongs to no more than d bricks.

4. Estimates for Magnetic Schrödinger Operators.

Let $H_0 = A_0 + V$ be the Schrödinger operator defined in Sec. 2, $H_{\boldsymbol{a}}$ denotes the operator with magnetic potential $\boldsymbol{a}$. We consider first the case of operators defined in a bounded domain Ω (e.g. a cube) in $\mathbf{R}^d$, and then pass to the general case.

4.1 *Bounded Domains.* Suppose $V = -W$, with smooth bounded W, $W \geqq \tau_p > 0$, so the operator of multiplication by W is bounded and invertible in L^2. According to the Birman-Schwinger principle, (see e.g. [19]), the number of negative eigenvalues of the operators H_0 equals the number $n(1, T_0(V))$ of the eigenvalues greater than 1 for the operator $T_0(V)$ defined by quadratic forms $A_0[u]$, $B[u] = \int_\Omega W(x)|u|^2\,dx$, which is the same as the number $N(1, X_{0,V})$ of the eigenvalues in $(0,1)$ for the operator $X_{0,V} = W^{-1/2}A_0W^{-1/2}$ (the latter operator is similar to $T_0(V)^{-1}$). We denote the eigenvalues of $X_{0,V}$ by $\lambda_j(X_{0,V})$. Similar relations hold for the eigenvalues of the magnetic operator H_a, with corresponding operator $X_{a,V}$.

Consider the heat equation

$$\frac{\partial u}{\partial t} = -X_{0,V}u, \tag{4.1}$$

with self-adjoint operator $X_{0,V}$. We can represent the solution of (4.1) as $u(t) = e^{-X_{0,V}t}u(0)$ with $G_0(t) = e^{-X_{0,V}t}$ being the *heat semigroup*. Suppose that the conditions of Theorem 3.1 are satisfied for the form $A_0[u]$, $B[u]$. Then, in particular, since the eigenvalues of $X_{0,V}$ have at least power order of growth, the operator $G_0(t)$ is Hilbert-Schmidt for any $t > 0$ and, therefore, an integral operator. These facts are stated more precisely in Proposition 4.1.

PROPOSITION 4.1. *For the Hilbert-Schmidt norm $|G_0(t)|_2$, $t > 0$, we have*

$$|G_0(t)|_2^2 \leqq Ct^{-d/2}\,\|\gamma^{-1}\|_p^{d/2}\,\|W\|_q^{d/2}. \tag{4.2}$$

PROOF. $\lambda_j(X_{0,V})$ are inverses of eigenvalues of $T_{0,V}$. So, due to Theorem 3.1,

$$\lambda_j(X_{0,V}) \geqq cj^{2/d}\,\|\gamma^{-1}\|_p^{-1}\,\|W\|_q^{-1}. \tag{4.3}$$

Therefore

$$\begin{aligned}|G_0(t)|_2^2 &= \sum e^{-2t\lambda_j(X_{0,V})} \leqq \sum \exp(-2ctj^{2/d}\,\|\gamma^{-1}\|_p^{-1}\,\|W\|_q^{-1})\\ &\leqq C(t^{-1}\,\|\gamma^{-1}\|_p\,\|W\|_q)^{d/2}\int_0^\infty e^{-s^{2/d}}\,ds\end{aligned}$$

(the latter equality is the result of the change of variables in the integral).

This result, in particular, implies that $G_0(t)$ has an integral kernel $G_0(t,x,y)$ and

$$|G_0(t)|_2^2 = \int_{\Omega\times\Omega} |G_0(t,x,y)|^2\,dxdy$$

Along with H_a, H_0, we consider operators H_a^γ, H_0^γ, which are defined in the same way, with matrix $g(x)$ replaced by the diagonal matrix $\gamma(x)I$ where $\gamma(x)$ is the ellipticity constant at the point $x \in \Omega$ (see Sec. 2). Of course, the estimate (4.2) remains valid for the heat operator corresponding to H_0^γ. Next, consider the magnetic operator H_a. From (2.1), it follows that $H_a \geqq H_a^\gamma$, and similarly, $X_{a,V} \geqq X_{a,V}^\gamma$. We are going to obtain spectral estimates for operators of the form H_a^γ, and this will imply similar estimates for H_a.

Consider the *magnetic heat operator* $G_a(t) = \exp(-tX_{a,V}^\gamma)$. Apriori we know very little about it, since we even do not know that $(X_{a,V}^\gamma)^{-1}$ is compact. The main step in proving this and then the spectral estimates for $H_{a,V}^\gamma$ is the following:

THEOREM 4.2. *Let $a \in L^2_{\mathrm{loc}}$, γ^{-1}, g, V, $V^{-1} \in L^\infty$. Then the operator $G_a(t)$ is Hilbert-Schmidt and for its integral kernel $G_a(t,x,y)$ the estimate*

$$|G_a(t,x,y)| \leqq G_0(t,x,y) \tag{4.4}$$

holds for almost all $(x,y) \in \Omega \times \Omega$.

PROOF. Take $a \in C_0^\infty$ first. The inclusion $G_a \in B_2$ is clear: the term corresponding to the magnetic field in the quadratic form of the operator $X_{a,V}^\gamma$ is a form-compact perturbation – here it is crucial that $\gamma^{-1} \in L^\infty$ – and so not only the order of the eigenvalues of $X_{a,V}^\gamma$ and $X_{0,V}^\gamma$ is the same but even the asymptotic behaviour is the same (see e.g. [2], [3]). This means, in particular, that G_a is Hilbert-Schmidt. So here one has only to prove the estimate (4.4).

The operator $X_{0,V}^\gamma$ is the form sum of operators: $\mathscr{D}_k = -W^{-1/2}\dfrac{\partial}{\partial x_k}\gamma\dfrac{\partial}{\partial x_k}W^{-1/2}$, so, according to the Trotter-Kato formula,

$$e^{-tX_{0,V}^\gamma} = \operatorname*{s-lim}_{n\to\infty}\,(e^{-\frac{t}{n}\mathscr{D}_1}\dots e^{-\frac{t}{n}\mathscr{D}_d})^n. \tag{4.5}$$

On the other hand, the operator $X_{a,V}^\gamma$ is the form sum of

$$\mathscr{D}_{k,a} = -W^{-1/2}(\partial/\partial x_k - ia_k)\gamma(\partial/\partial x_k - ia_k)W^{-1/2}$$

and therefore

$$e^{-tx_{a,V}^\gamma} = \operatorname*{s-lim}_{n\to\infty}\,(e^{-\frac{t}{n}\mathscr{D}_{1,a}}\dots e^{-\frac{t}{n}\mathscr{D}_{d,a}})^n. \tag{4.6}$$

Now, according to Proposition 2.6, we have $e^{-\frac{t}{n}\mathscr{D}_{k,a}} = e^{i\mu_k(x)}e^{-\frac{t}{n}\mathscr{D}_k}e^{-i\mu_k(x)}$, and hence

$$e^{-tX_{a,V}^\gamma} = \operatorname*{s-lim}_{n\to\infty}\,(e^{i\lambda_1}e^{-\frac{t}{n}\mathscr{D}_1}e^{i(\lambda_2-\lambda_1)}e^{-\frac{t}{n}\mathscr{D}_2}\dots e^{-\frac{t}{n}\mathscr{D}_d}e^{-i\lambda_d}).$$

It remains to note that the operators $e^{-\mathscr{D}_k t/n}$ are integral operators with positive kernels in x_k variable (cf. [6]) and the operators $e^{\pm i\mu_k}$ are multiplications by functions with absolute value 1. This gives us the inequality

$$|G_a^n(t,x,y)| \leqq G_0^n(t,x,y), \tag{4.7}$$

where G_a^n and G_0^n are operators under the limit sign in (4.5) and (4.6). As in [17] strong convergence of integral operators implies convergence of their integral kernels almost everywhere, and this gives us (4.5).

Finally, we dispose of the condition $a \in C_0^\infty(\Omega)$. Take a sequence $a^\varepsilon \in C_0^\infty(\Omega)$, $a^\varepsilon \to a$ in L^2_{loc}. Then, due to Proposition 2.5, there is strong resolvent convergence of the operators $X^\gamma_{a^\varepsilon, V}$ to $X^\gamma_{a,V}$ which implies strong converence of corresponsing exponents and therefore convergence of heat kernels a.e.

The first consequence of Theorem 4.2 is the spectral estimate for $\gamma^{-1} \in L^\infty$.

PROPOSITION 4.3. *Let $\gamma^{-1} \in L^\infty$ and the conditions of Theorem* 3.1 *hold. Then*

$$\lambda_j(X^\gamma_{a,V}) \geqq cj^{2/d} \|\gamma^{-1}\|_p^{-1} \|W\|_q^{-1} \tag{4.8}$$

with constant c depending only on p, q, d.

PROOF. Due to the pointwise majoration of integral kernels,

$$|G_a(t)|_2^2 = \int_{\Omega\times\Omega} |G_a(t,x,y)|^2\,dxdy \leqq \int_{\Omega\times\Omega} |G(t,x,y)|^2\,dxdy = |G_0(t)|_2^2. \tag{4.9}$$

So, according to Proposition 4.1,

$$|G_a(t)|_2^2 = \sum \exp(-2t\lambda_j(X^\gamma_{a,V})) \leqq ct^{-d/2} \|\gamma^{-1}\|_p^{d/2} \|W\|_q^{d/2}. \tag{4.10}$$

Setting, for given j, $t = (2\lambda_j)^{-1}$, we get $e^{-1}j \leqq c\lambda_j^{d/2} \|\gamma^{-1}\|_p^{d/2} \|W\|_q^{d/2}$, which is equivalent to (4.8).

REMARK. We could not use here the pointwise domination of integral kernels to estimate the trace norm of $G_a(t)$ in a similar way to [12]. The reason is that pointwise domination of kernels is sufficient for domination of trace norms of operators only provided the operators are positive and the kernels are continuous. The former condition is satisfied in our case but the latter one cannot be guaranteed for a discontinuous γ (discontinuities of a and V can be dealt with, as in [17]).

Our next step will be disposing of the condition $\gamma^{-1} \in L^\infty$. Let $\gamma_\varepsilon = \max(\varepsilon, \gamma)$. Of course, $\gamma_\varepsilon \to \gamma$ in L^∞ and $\gamma_\varepsilon^{-1} \to \gamma^{-1}$ in L^p. We were not able to prove strong resolvent convergence of corresponding operators, so the trick just used cannot be applied here. However, we can use another approach.

PROPOSITION 4.4. *Let $\gamma^{-1} \in L^p(\Omega)$, $\gamma \in L^\infty_{\mathrm{loc}}(\Omega)$, $a \in L^2_{\mathrm{loc}}$, $V < 0$, V^{-1}, $V \in L^\infty$, $p \geqq d/2$. Then for the operator $X^\gamma_{a,V}$ the estimate* (4.8) *holds.*

PROOF. According to the min-max principle, for the distribution function $N(\lambda)$ of the operator $X^{\gamma}_{a,V}$, we have

$$N(\lambda_j) = \max\dim\{\mathscr{L} \subset C_0^\infty(\Omega), A[u] < \lambda B_W[u], u \in \mathscr{L}\backslash\{0\}\}. \tag{4.11}$$

Let, for given λ, $\mathscr{L}$ be some finite-dimensional subspace where the inequality in (4.11) holds. Consider the form $A_\varepsilon[u]$ corresponding to γ_ε. As $\varepsilon \to 0$, for any fixed $u \in C_0^\infty(\Omega)$. the integrand in $A_\varepsilon[u]$ converges pointwise and monotonely to the integrand in $A[u]$. Therefore, we can pass to the limit under the integral sign, so, for $\delta > 0$ fixed and ε small enough, $A_\varepsilon[u] \leqq \lambda(1+\delta)B_W[u]$. Since $\mathscr{L}$ is finite-dimensional, one can find common ε, serving all $u \in \mathscr{L}$. This means that, for ε small enough, on the subspace $\mathscr{L}$ the inequality $A_\varepsilon[u] \leqq \lambda(1+\delta)B_W[u]$ holds, so the dimension of $\mathscr{L}$ cannot be greater than $N(\lambda(1+\delta), X^{\gamma_\varepsilon}_{a,V})$. For the latter quantity, Proposition 4.3 provides us with an estimate (equivalent to (4.8)):

$$N(\lambda, X^{\gamma_\varepsilon}_{a,V}) \leqq c(\lambda(1+\delta))^{d/2} \|\gamma_\varepsilon^{-1}\|_p^{d/2} \|V\|_q^{d/2} \leqq c(\lambda(1+\delta))^{d/2} \|\gamma^{-1}\|_p^{d/2} \|V\|_q^{d/2}.$$

Therefore the same estimate holds for $N(\lambda, X^{\gamma}_{a,V})$.

4.2. *The General Case.* After Proposition 4.4 is proved, only quite standard steps lead to the proof of the result generalizing Theorem 3.1.

THEOREM 4.5. *Let $\gamma^{-1} \in L^p(\mathsf{R}^d)$, $\gamma \in L^\infty_{\mathrm{loc}}$, $W \in L^q$, $\boldsymbol{a} \in L^2_{\mathrm{loc}}$, $p^{-1} + q^{-1} = 2d^{-1}$, $p, q > 1$; $p \geqq 1$ if $q = \infty$. Then the estimate*

$$n_\pm(\mu, T) \leqq c\mu^{-d/2} \|\gamma^{-1}\|_p^{d/2} \|W\|_q^{d/2} \tag{4.12}$$

holds for the operator $T = T_{A,B}$ defined by the quadratic forms

$$A_{j,k}[u] = \int g_{j,k}\left(\frac{\partial u}{\partial x_k} - ia_k u\right)\overline{\left(\frac{\partial u}{\partial x_j} - ia_j u\right)} dx,$$

$$B[u] = \int W|u|^2\, dx.$$

PROOF. To prove (4.12) having Proposition 4.4 already at our disposal, we must first go over from bounded invertible W to general $W \in L^p$ and secondly from the cube Ω to the whole space R^d.

The first step is taken care of by the fact that only L^q- norm of W enters in the estimate (4.8), so we can approximate in L^q a given nonnegative $W \in L^q$ by bounded invertible functions and pass to the limit in the spectral estimate, just as if was made when proving Proposition 4.4. If W changes sign, the min-max principle implies that since the form B_+, with W_+ replacing W in B, majorates the form B, so $n(\mu, T_{A,B_+}) \geqq n(\mu, T_{A,B})$, and for T_{A,B_+} we already have the estimate involving only norm of W_+.

The second step, passing from the cube to R^d, is performed exactly as when we were proving Theorem 3.1, again by min-max principle.

Now, the generalization of the CLR-estimate follows from Theorem 4.5 by a standard application of the Birman-Schwinger principle, as e.g. in [14].

THEOREM 4.6. *Let the conditions* A, B *and* C *be satisfied. Then, for the number of negative eigenvalues of the operator* $H_{g,\mathbf{a},V}$, *the estimate*

$$N_-(H_{g,\mathbf{a},V}) \leqq c \,\|\gamma^{-1}\|_p^\theta \,\|V_-\|_q^\theta \tag{4.13}$$

holds, with constant c depending only on p, q and d.

In particular cases, one can choose suitable exponents p, q. In the case of a uniformly elliptic operator, (in particular, for the unit matrix g) we take $p = \infty$, $q = d/2$ and obtain

$$N_-(H_{\mathbf{a},V}) \leqq c \int V_-^{d/2}\, dx, \quad d \geqq 3,$$

i.e. the usual CLR-estimate. The other interesting case is $q = \infty$, $p = d/2$ (note, that $d = 2$ is admissible here). Take, in particular, $V = -\lambda$. Then $N_-(H_{g,\mathbf{a},V})$ equals the number $N(\lambda, H_{g,\mathbf{a},0})$ of eigenvalues of the operator $H_{g,\mathbf{a},0}$ (without electric potential!) in $(-\infty, \lambda)$. Now, the estimate (4.13) takes the form

$$N(\lambda, H_{g,\mathbf{a},0}) \leqq c\lambda^\theta \int |\gamma^{-1}|^{d/2}\, dx. \tag{4.14}$$

It follows from (4.14) that if the degeneration is not too severe and the ellipticity constant γ grows at infinity quickly enough, then the spectrum of $H_{g,\mathbf{a},0}$ in the Euclidean space is discrete and, moreover, has the same properties as operators on compact manifolds. For operators without magnetic fields, this was found out in [15]; (4.14) shows that no magnetic field can destroy this property.

REFERENCES

1. J. Avron, I. Herbst and B. Simon, *Schrödinger operators with magnetic fields.* I. *General Interactions*, Duke Math. J. 45 (1978), 847–883.
2. M. Birman and M. Solomyak, *Spectral asymptotic behaviour of nonsmooth elliptic operators*, I, Tr. Mosk. Mat. Ob-va 27 (1972), 3–52 (in Russian; English translation in Trans. Moscow Math. Soc. 27 (1972)).
3. M. Birman and M. Solomyak, *Qualitative analysis in Sobolev embedding theorems and applications to spectral theory*, Amer. Math. Soc. Transl. (2), 114 (1980).
4. M. Birman and M. Solomyak, *Piecewise-polynomial approximations of functions of the class* W_p^d, Mat. Sb. 73 (115) (3) (1967), 331–355 (in Russian).
5. M. Cwikel, *Weak type estimates for singular values and the number of bound states of Schrödinger operators*, Ann. Math. 206 (1977), 93–100.

6. E. B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Univ. Press, 1990.
7. M. de Guzman, *A covering lemma with applications*, Studia Math. 34 (1970), 299–317.
8. P. Li and S. T. Yau, *On the Schrödinger equation and the eigenvalue problem*, Comm. Math. Phys. 88 (1983), 309–318.
9. E. Lieb, *Stability of Matter: From Atoms to Stars*, Bull. Amer. Math. Soc. 22 (1980), 1–49.
10. E. Lieb, *Bounds on the eigenvalues of the Laplace and Schrödinger operators*, Bull. Amer. Math. Soc. 82 (5) (1976), 751–753.
11. V. Maz'ya, *Sobolev Spaces*, Springer-Verlag, Berlin-New York, 1985.
12. M. Melgaard and R. Ravnstrup, *The* CLR-*bound for the One-Body Magnetic Schrödinger Operator*, Master Thesis, Aalborg University, Spring 1994.
13. M. Reed and B. Simon, *Methods of Modern Mathematical Physics* II, *Fourier Analysis, Self-Adjointness*, Academic Press, New York, 1975.
14. M. Reed and B. Simon, *Methods of Modern Mathematical Physics* IV, *Analysis of Operators*, Academic Press, New York, 1978.
15. G. Rozenblum, *Distribution of the discrete spectrum of singular differential operators*, Izvestia Vuz, Mathematika 20 (1) (1976), 75–86. (In Russian; English translation in Soviet Mathematics. Izv. VUZ, 20 (1) (1976), 63–71).
16. G. Rozenblum, *Distribution of the discrete spectrum of singular differential operators*, Dokl. Akad. Nauk SSSR 202 (5) (1972), 1012–1015. (In Russian; English translation in Soviet Math. Dokl. 13 (1972), 245–249.)
17. B. Simon, *Functional Integration and Quantum Physics*, Academic Press, NY, 1979.
18. B. Simon, *Maximal and minimal Schrödinger forms*, J. Operator Theory 1 (1979), 37–47.
19. B. Simon, *Trace Ideals and its Applications*, Cambridge Univ. Press, 1979.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
INSTITUTE FOR ELECTRONIC SYSTEMS
AALBORG UNIVERSITY
FREDERIKS BAJERS VEJ 7E
9220 AALBORG Ø
DENMARK