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APPROXIMATION BY INTERPOLATING BLASCHKE PRODUCTS

KNUT ØYMA*

Abstract.

We give a new proof of the fact that every unimodular function on the unit circle can be uniformly approximated by the ratio of two interpolating Blaschke products. We also prove that every bounded analytic function of small norm is contained in the closed convex hull of the interpolating Blaschke products.

Let H^∞ be the set of bounded analytic functions on the unit disc D, and let $T = \partial D$. Every function $f(z) \in H^\infty$ has boundary values

$$f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta}) \text{ a.e.}$$

A Blaschke product $B(z)$ is a bounded analytic function of the form

$$z^m \prod_{n=1}^{\infty} \frac{-\overline{z_n}}{|z_n|} \cdot \frac{z - z_n}{1 - \overline{z_n} z} \quad \text{where} \quad \sum (1 - |z_n|) < \infty.$$

$$|B(e^{i\theta})| = 1 \text{ a.e.}$$

A sequence $\{z_n\} \subset D$ is called interpolating if every interpolation problem $f(z_n) = w_n$ where $\{w_n\} \in \ell^\infty$ has a solution in H^∞. A Blaschke product whose zero set is an interpolating sequence is called an interpolating Blaschke product. Interpolating Blaschke products play a central role in the theory of H^∞. A good reference is the book of Garnett [1]. In [3] Jones proved that every unimodular function in $L^\infty(T)$ çan be uniformly approximated by the ratio of two interpolating Blaschke products. We will give a new proof of this below. The original proof, however, gives a better geometric understanding of the problem. The ideas come from [2] where Garnett and Nicolau prove that every function in H^∞ can be uniformly approximated by a linear combination of interpolating Blaschke products. It is not known if every function in H^∞ of unit norm is contained in the

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* The author died July 18, 1996.

closed convex hull of the interpolating Blaschke products. Theorem 2 shows that this is true for all very small functions.

First we need sime definitions. By a square we mean a set of the form $Q = \{re^{i\theta}: 1 - h < r < 1, \theta_0 - \pi h < \theta < \theta_0 + \pi h\}$.

The top half of Q is $T(Q) = \left\{re^{i\theta} \in Q: r < 1 - \frac{h}{2}\right\}$ and the length of Q, $\ell(Q) = 2\pi h$. The pseudohyperbolic distance between two points in D is defined by $\rho(z, w) = \left|\frac{z - w}{1 - \bar{z}w}\right|$.

A sequence $\{z_v\}$ is interpolating if and only if

$$\inf_{n \neq m} \rho(z_n, z_m) > 0 \quad \text{and}$$

$$\sum_{z_v \in Q} (1 - |z_v|) < C\ell(Q) \quad \text{for all squares.}$$

A proof can be found in Chapter 6 of [1].

Below we consider only squares of the form

$$Q = \left\{re^{i\theta}: \frac{2\pi k}{2^n} \leqq \theta < \frac{2\pi(k+1)}{2^n}, 1 - 2^{-n} \leqq r < 1\right\}.$$

Define $m(r) = \min_{|z|=r} |f(z)|$. Several lemmas are needed.

LEMMA 1. *Let $B(z)$ be a Blaschke product. Then $B = B_1 B_2$ where* $\limsup_{r \to 1} m(r) = 1$ *for B_1 and B_2.*

PROOF. If $\{z_v\}$ is the zero set of B, consider the Blaschke product $b(z)$ with zero set $\{|z_v|\}$. Then $|B(z)| \geqq |b(|z|)|$ so we may assume that the seros of B are $\geqq 0$ and consider positive z only. Assume that $z_1 \leqq z_2 \leqq \ldots$. Let ${}^nB$ and ${}_mB$ have zeros $\{z_v\}, v \geqq n$ and $\{z_v\}$, $v \leqq m$ respectively. Assume that $a_k \in \langle 0, 1 \rangle$ and that $a_k \to 1$. Choose numbers $r_k \in \langle 0, 1 \rangle$ and natural numbers $m_k < n_k < m_{k+1}$ in the following way:

Let $m_1 = 1$. Choose $r_1 > z_{m1}$ such that $|{}_{m1}B(r_1)| > a_1$. Choose n_1 such that $z_{n1} > r_1$ and $|{}^{n1}B(r_1)| > a_1$. Choose $r_2 > z_{n1}$ such that $|{}_{n1}B(r_2)| > a_2$, and m_2 such that $z_{m2} > r_2$ and $|{}^{m2}B(r_2)| > a_2$.

Continue inductively. If B_1 has zero set $\bigcup_{k} \bigcup_{m_k \leqq v < n_k} \{z_v\}$ and B_2 has zero set $\bigcup_{k} \bigcup_{n_k \leqq v < m_{k+1}}$ then $B = B_1 B_2$ solves the problem.

Let $B = B_1$ or $B = B_2$. Choose a small $\delta > 0$, and let $0 < \alpha < \beta < 1$ be constants to be chosen later. We may assume that $|B(0)| > \beta$.

Let $Q_1^1, Q_2^1, \ldots$ be the maximal squares such that $\inf_{T(Q)} |B(z)| < \alpha$. The squares $S_{p,k}^1$, $p = 1, \ldots, 128$ are the squares of length $\frac{1}{128}\ell(Q_k^1)$ contained in Q_k^1.

Choose a curve $\Gamma_k^1 = \{z: |z| = r\} \cap Q_k^1$ close to T such that $|B(z)| > \beta$ for $z \in \Gamma_k^1$ and such that $\Gamma_{p,k}^1 = \Gamma_k^1 \cap S_{p,k}^1 \neq \emptyset$.

This is possible by Lemma 1. Since $|B(z)| \geqq \alpha$ at some point in $T(Q_k^1)$ we may choose α so close to 1 that the zeros of B belonging to Q_k^1 are contained in $\bigcup_p S_{p,k}^1$. We define $R_{p,k}^1$ to be the component of $S_{p,k}^1 \backslash \Gamma_{p,k}^1$ closest to 0.

Let Q_n^2 be the maximal squares contained in some $S_{p,k}^1 \backslash R_{p,k}^1$ such that $\inf_{z \in T(Q)} |B(z)| < \alpha$. Construct $S_{p,k}^2, R_{p,k}^2, \Gamma_{p,k}^2$, $p = 1, \ldots, 128$ as before.

Continue the construction inductively. If $\varepsilon > 0$ is given we may choose β so close to 1 that

$$(*) \qquad \sum_{Q_n^m \subset S_{p,k}^{m-1}} \ell(Q_n^m) < \varepsilon \ell(S_{p,k}^{m-1})$$

See p. 332 of [1].

Let p be fixed. B_p isthe subproduct of B with zeros in $R_p = \bigcup_{n,k} R_{p,k}^n$. Every zero of B is contained in some R_p.

Since $|B_p(z)| \geqq \alpha$ at some point in $T(Q_n^k)$, we may assume that $|B_p|$ isclose to 1 in $T(S_{p,k}^n)$ by choosing α close to 1. We have

$$\sum_{\substack{B_p(z_v)=0 \\ z_v \in S_{p,k}^n}} (1 - |z_v|^2) \leqq C(\alpha)\ell(S_{p,k}^n) \text{ where } C(\alpha) \to 0 \text{ when } \alpha \to 1.$$

We need

LEMMA 2. *Assume that Q is a square and that $\delta > 0$ is small. If the curve $\Gamma = \{z: |z| = r\} \cap Q$ is sufficiently close to T, there exist points $z_n \in \Gamma$, equally separated such that*

(a) *$m \neq n$ implies $\rho(z_n, z_m) > 1/\log\frac{1}{\delta}$*

(b) *The associated Blaschke product B satisfies $|B| < \delta$ on Γ*

(c) $\sum(1 - |z_n|^2) < \frac{2}{3.1}\ell(Q)\log\frac{1}{\delta}$

For technical reasons we carry out the proof in the upper half plane and assume that $\Gamma = \{i + x, 0 \leqq x \leqq K)$ where $K > \left[\frac{1}{3.1}\log\frac{1}{\delta}\right] = N$. $B(z)$ has zeros

$$z_n = i + \frac{n}{N}, n = 1, \ldots, NK.$$

For $n \neq m$ $\rho(z_n, z_m) > \dfrac{\frac{1}{N}}{2} > \dfrac{1}{\log\frac{1}{\delta}}$ so (a) is satisfied.

In the proof of part (b) we need that $\prod_{n=1}^{\infty}\left(1+\frac{z^2}{n^2}\right) = \frac{e^{\pi z}-e^{-\pi z}}{2\pi z}$ which is easily obtained from the well known formula $\sin \pi z = \pi z \prod_1^{\infty}\left(1-\frac{z^2}{n^2}\right)$.

$$\operatorname*{Max}_{z\in\Gamma}|B(z)| = |B(i)| = \prod_{n=1}^{NK}\frac{\frac{n}{N}}{\sqrt{4+\frac{n^2}{N^2}}} = \prod_{n=1}^{NK}\frac{1}{\sqrt{1+\frac{(2N)^2}{n^2}}} \sim \prod_{n=1}^{\infty} =$$

$\sqrt{\dfrac{4\pi N}{e^{2\pi N}-e^{-2\pi N}}} < \delta$ if δ is small.

This proves (b).

In the unit disc $1-|z_n|$ corresponds to $\operatorname{Im} z_n$ in the upper half plane so we obtain $\sum(1-|z_n|^2 < 2\sum(1-|z_n|) < \frac{2}{3.1}\ell(Q)\log\frac{1}{\delta}$.

We may now choose $\Gamma^n_{p,k}$ close to T and finitely many points $z_m = z_m{}^{,p,k,n}$ on $\Gamma^n_{p,k}$ such that (a), (b) and (c) are satisfied. Denote the associated Blaschke product by $B_{p,k,n}$. Using the subharmonicity of $\log|B_{p,k,n}|$ and an easy harmonic measure estimate we obtain $|B_{p,k,n}| < \delta^{\frac{1}{4}}$ on $R^n_{p,k}$.

Let $B^*_p = \prod_{k,n} B_{p,k,n}$. By Lemma 2 and (*) B^*_p is interpolating and $|B^*_p| < \delta^{\frac{1}{4}}$ on R_p.

$$\sum_{\substack{B_pB^*_p(z_v)=0\\ z_v\in Q^n_k}}(1-|z_v|^2) \leqq C(\alpha)\ell(Q^n_k) + \frac{2}{3.1}\ell(S^n_{p,k})\log\frac{1}{\delta}(1+\varepsilon+\varepsilon^2+\dots)$$

$$+\,127\ell(S^n_{p,k})\log\frac{1}{\delta}(\varepsilon+\varepsilon^2+\dots)$$

The first term comes from the zeros of B_p, the second term takes care of the zeros of B^*_p in $S^n_{p,k}$ and the last term comes from the zeros of B^*_p in $S^n_{r,k}$ for $r \neq p$.

Note that $\ell(S^n_{p,k}) = \frac{1}{128}\ell(Q^n_k)$. If ε is small and α is close to 1 we obtain $\sum_{\substack{B_pB^*_p(z_v)=0\\ z_v\in Q^n_k}}(1-|z_v|^2) < \frac{1}{192}\log\frac{1}{\delta}\ell(Q^n_k)$.

If $z = re^{i\theta}$ let $Q_z = \{\rho e^{i\theta}: 1-4(1-r) < \rho < 1, |\varphi-\theta| \leqq 2(1-r)\}$.

Assume that z satisfies

$$(\ast) \qquad Q_n^k \cap Q_z \neq \emptyset \Rightarrow \ell(Q_k^n) < \frac{1-|z|}{1000}$$

Then $\sum_{\substack{B_pB_p^*(z_v)=0 \\ z_v \in Q_z}} (1-|z_v|^2) < (4(1-|z|) + \frac{2}{1000}(1-|z|))\frac{1}{192}\log\frac{1}{\delta} < \frac{1}{47}(1-|z|)\log\frac{1}{\delta}$.

An easy calculation proves that if $z_v \notin Q_z$ then $\arg \dfrac{z_v - z}{z_v - \frac{1}{\bar{z}}} = \theta_v$ satisfies $\cos\theta_v > \frac{7}{25}$.

Define $\rho(z, R_p) = \inf_{w \in R_p} \rho(z, w)$

LEMMA 3. *There exists a constant $A = A(\delta) < 1$ such that if $\rho(z, R_p) \geqq A$ and $|B_pB_p^*(z)| = \delta^{\frac{1}{5}}$ then $(1-|z|^2)\,|(B_pB_p^*)'(z)| > \dfrac{\delta^{\frac{1}{5}}}{50}\log\dfrac{1}{\delta}$.*

PROOF. If A is large then $(\ast)$ is satisfied.

$\frac{2}{5}\log\frac{1}{\delta} = \log|B_pB_p^*(z)|^{-2} \sim \sum_v \frac{((1-|z|^2)(1-|z_v|^2)}{|1-\overline{z_v}z|^2} = \sum_1 + \sum_2$ where $\{z_v\}$ are the zeros of $B_pB_p^*$, $\sum_1 = \sum_{Q_z}$ and $\sim$ is close to equality when $A \to 1$.

$$\sum_{-} < \frac{2}{1-|z|}\sum_{z_v \in Q_z}(1-|z_v|^2) < \frac{2}{47}\log\frac{1}{\delta}.$$

Therefore $\sum_2 > (\frac{2}{5} - \frac{2}{47})\log\frac{1}{\delta} = \frac{84}{235}\log\frac{1}{\delta}$.

We have that

$$(1-|z|^2)\frac{(B_pB_p^*)'}{B_pB_p^*} = \sum_v \frac{\bar{z}(1-|z|^2)(1-|z_v|^2)}{|1-\overline{z_v}z|^2}\cdot\frac{\frac{1}{\bar{z}} - z_v}{z - z_v} = \sum_1{}' + \sum_2{}'$$

where
$$\sum_1{}' = \sum_{z_v \in Q_z^*}$$

$$\left|\sum_1{}'\right| < \frac{1}{A}\sum_1 < \frac{1}{A}\frac{2}{47}\log\frac{1}{\delta}$$

$$\left|\sum_2{}'\right| \geqq \frac{7}{25}\sum_2 > \frac{7}{25}\cdot\frac{84}{235}\log\frac{1}{\delta}.$$

Therefore

$$(1-|z|^2)\,|(B_pB_p^*)'| = \delta^{\frac{1}{5}}\left|\sum_1{}' + \sum_2{}'\right| \geqq \delta^{\frac{1}{5}}\left(\frac{7\cdot 84}{25\cdot 235} - \frac{1}{A}\cdot\tfrac{2}{47}\right)\log\frac{1}{\delta} > \frac{\delta^{\frac{1}{5}}}{\frac{7}{8}\frac{3}{7}}\log\frac{1}{\delta}$$

when A is close to 1. This completes the proof of Lemma 3.

By a theorem of Frostman, see [1] p. 79, there exists γ such that $|\gamma| = \delta^{\frac{1}{5}}$ and $C_p = \dfrac{B_pB_p^* - \gamma}{1 - \bar{\gamma}B_pB_p^*}$ is a Blaschke product. Assume that $C_p(z) = 0$. Then $|B_pB_p^*| = \delta^{\frac{1}{5}}$. Lemma 3 and an easy calculation proves that $(1-|z|^2)\,|C_p'(z)| > \dfrac{\delta^{\frac{1}{5}}(1-\delta^{\frac{4}{5}})}{200}\log\dfrac{1}{\delta}$ if $\rho(z, R_p) \geqq A$. If $\rho(z, R_p) < A$ there exists $w \in R_p$ such that $\rho(w, z) < A$ and $|B_pB_p^*(w)| < \delta^{\frac{1}{4}}$. Hence there exists w' such that $\rho(w', z) < A$ and $(1-|w'|^2)\,|C_p'(w')| > \eta(A, \delta) > 0$.

By Lemma 1 in [2] C_p is a finite product of interpolating Blaschke products. Therefore we have on the unit circle $B_pB_p^* = C_p(1 + \gamma_p(e^{i\theta}))$ where $|\gamma_p| < 3|\gamma|$ if γ is small.

Let $\prod_p B_p^* = B^*$ and $\prod_p C_p = C$. Then we obtain $BB^* = C(1 + \varepsilon(e^{i\theta}))$ where ε is small when γ is small.

C and B^* can be uniformly approximated by interpolating Blaschke products, since finite products of interpolating Blaschke products have this property, see [4]. Hence every Blaschke product can be uniformly approximated on the unit circle by a quotient of two interpolating Blaschke products. By a theorem of Rudin and Douglas, [1] p. 192, every unimodular function on T can be approximated by a quotient of two Blaschke products. Hence we obtain

THEOREM 1 (Jones). *Every unimodular function in $L^\infty(T)$ can be uniformly approximated by a quotient of two interpolating Blaschke products.*

If λ is a unimodular constant and B is an interpolating Blaschke product we include λ and λB among the interpolating Blaschke products. By K we mean the closed convex hull of the interpolating Blaschke products. K is closed under multiplication. In [2] Garnett and Nicolau proved that for any $f \in H^\infty$ there exists a constant c_f such that $f \in c_f \cdot K$, but they did not give any estimate of c_f in terms of f. We will prove

THEOREM 2. *Any $f \in H^\infty$ of sufficiently small norm is contained in K.*

Our proof is essentially a repetition of the argument in [2], but we prove that the parameter σ there depends on δ only.

In the proof of Theorem 1 we found a constant γ, $|\gamma| = \delta^{\frac{1}{5}}$ such that $C_p = \dfrac{B_pB_p^* - \gamma}{1 - \bar{\gamma}B_pB_p^*} \in K$. Therefore

$$B_pB_p^* = \frac{C_p+\gamma}{1+\bar{\gamma}C_p} = (C_p+\gamma)\sum_0^\infty(-\bar{\gamma}C_p)^n \in \frac{1+|\gamma|}{1-|\gamma|}K = c_1(\delta)K.$$

By Lemma 1.4, p. 404 in [1] $B_p^{**} = \dfrac{B_p^*-\sigma}{1-\bar{\sigma}B_p^*}$ is an interpolating Blaschke product if σ is sufficiently small. We claim that σ depends on δ only. The proof of this is postponed. The zeros of B_p^{**} are close to the zeros of B_p^* in the pseudohyperbolic metric.

Repeating the argument with B_p^{**} we obtain $C_p^* = \dfrac{B_pB_p^{**}-\gamma^*}{1-\overline{\gamma^*}B_pB_p^{**}} \in K$ where $|\gamma^*|$ depends on δ and σ and hence δ only. As above we obtain $B_pB_p^{**} \in c_2(\delta)K$. Subtracting $B_pB_p^{**}$ from $B_pB_p^*$ leads to $B_p \in \left(\dfrac{1+|\sigma|}{|\sigma|}c_2(\delta)+c_1(\delta)\right)K = c_3(\delta)K$. Hence $B = \prod_{p=1}^{128} B_p \in c_3^{128}K$ for each of the two subproducts of Lemma 1.

Therefore every Blaschke product belongs to $c_3^{256}K$. This proves Theorem 2 since every $f \in H^\infty$ of norm 1 belongs to the closed convex hull of the Blaschke products by a theorem of Marshall, [1], p. 196.

It remains to prove the claim. Let $\{z_v\}$ be the zeros of B_p^* and let ${}_mB_p^*$ be the Blaschke product with zeros $\{z_v\}$, $v \neq m$.

By Lemma 1.4, p. 404 in [1], it sufficies to prove that

$$(\ast) \qquad |{}_mB_p^*(z_m)| \geqq \kappa(\delta) > 0.$$

The zeros of B_p^* lying on $\Gamma_{p,k}^n$ consist of K consequtive groups each consisting of N zeros. See the proof of Lemma 2. Consider such a group $z_m, \ldots, z_{m+N-1}$. Let L be the part of $\Gamma_{p,k}^n$ lying between z_m and z_{m+N-1}. I is the radial projection of L onto T and Q is the set of points in D between L and I.

By the construction of B_p^*

$$\sum_{\substack{k\\ Q_k^{n+1}\cap Q \neq \emptyset}} \ell(Q_k^{n+1}) < \varepsilon\ell(I)$$

Therefore we can find N disjoint sets $E_m, \ldots, E_{m+N-1}$ contained in $I \setminus \bigcup_k Q_k^{n+1}$ of measure $> \dfrac{|I|}{2N}$. In this way we can associate disjoint sets $E_v \subset T$ to each z_v. Let $\omega_v(z)$ be the harmonic measure of E_v with respect to D. An estimate of the Poisson kernel shows that $\omega(z_v) \geqq a(N) = a(\delta) > 0$ since N depends on δ only.

Let $\arg z_v = h_v$. We have that

$$a(\delta) \leqq \omega_v(z_v) = \frac{1}{2\pi}\int_{E_v}\frac{1-|z_v|^2}{1-2|z_v|\cos(\theta_v-t)+|z_v|^2}\,dt \leqq \frac{|E_v|}{\pi(1-|z_v|)}.$$

Therefore $1 - |z_v| \leqq \dfrac{|E_v|}{\pi a(\delta)}$. Consequently $\sum(1 - |z_v|) \leqq \dfrac{2}{a(\delta)}$ since the sets E_v are disjoint.

The statement ($\ast\atop\ast$) is conformal invariant, so we may assume that $z_m = 0$. By Lemma 2(a) $|z_v| \geqq \dfrac{1}{\log \frac{1}{\delta}} = b$ for $v \neq m$. Finally

$$\prod_{v \neq m} |z_v| = \exp\left(\sum_{v \neq m} \log|z_v|\right) \geqq \exp\left(\frac{\log b}{1-b}\sum(1 - |z_v|)\right)$$

$$\geqq \exp\left(\frac{\log b}{1-b}\frac{2}{a(\delta)}\right) = \kappa(\delta) > 0$$

REMARK. A long and technical numerical calculation shows that $f(z)$ belongs to K if $\|f(z)\| < 10^{-1000}$.

REFERENCES

1. J. Garnett, *Bounded Analytic Functions*, Academic Press 1981.
2. J. Garnett and A. Nicolau, *Interpolation Blaschke products generate H^∞*, Pacific J. Math. 173, 501–511
3. P. Jones, *Ratios of interpolating Blaschke products*, Pacific J. Math. 95 (1981), 311–321.
4. D. Marshall and A. Stray, *Interpolating Blaschke products*, Pacific J. Math. 173, 491–500

AGDER COLLEGE
4604 KRISTIANSAND
NORWAY