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AMENABILITY OF CONVOLUTION ALGEBRAS

A. T.-M. LAU and R. J. LOY

Abstract.

The amenability of the Banach algebra $\ell^1(G)$ of a discrete semigroup G, and its implications for the structure of G has been much studied over recent years. In this paper we investigate implications of amenability of the algebras $M(G)$, $M(G)^{**}$ and $\mathrm{LUC}(G)^*$ on the structure of G for locally compact G. The general thrust of the results is that taken together with mild algebraic hypotheses, such amenability necessitates that G satisfy some finiteness restrictions, and to be close to a group.

0. Introduction.

The amenability of the Banach algebra $\ell^1(G)$ of a discrete semigroup G has been considered by a number of authors, in particular [9], [16], [10]. Recently the second author, with others, [14], showed that $L^1(G)^{**}$, G a locally compact group, is amenable if and only if G is finite. Some results were also given there for certain discrete semigroups. The purpose of the present paper is to continue this investigation by considering the implications of amenability conditions on various algebras defined over locally compact groups and semigroups on the structure of the underlying groups and semigroups.

Amongst other results we show (Theorem 2.4) that for any connected locally compact group, $M(G)$ amenable necessitates the group be trivial; that a commutative, weakly cancellative, locally compact semigroup with $M(G)$ amenable is a finite lattice of groups, these groups necessarily finite if $M(G)^{**}$ is amenable (Theorem 3.3), and that a cancellative locally compact semigroup G with $M(G)$ amenable must be a topological group, necessarily finite if $M(G)^{**}$ is amenable (Theorem 4.7).

A readable account of many of the ideas used here can be found in the first two chapters of [27], see also the survey articles [1, 24].

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1. Notation and preliminaries.

Throughout this paper, unless otherwise stated, G will denote a locally compact semigroup, that is, a semigroup with a locally compact Hausdorff topology under which multiplication is separately continuous. If the multiplication is required to be jointly continuous, that is, as a function from $G \times G \to G$, G is a locally compact *topological* semigroup. Note that this distinction does not arise if G is a group: in a locally compact semigroup which is algebraically a group, inversion is necessarily continuous, and multiplication necessarily jointly continuous, [12]. We will denote by G_d the semigroup G taken with the discrete topology.

We shall require a hierarchy of cancellation properties. If the left version is defined, the right is analogous; one-sided will refer to either left or right. A semigroup G will be called

(i) *left cancellative* if for all $a, x, y \in G$, $ax = ay$ implies $x = y$;

(ii) *left weakly cancellative* if for all $x, y \in G$, $\{z \in G: zx = y\}$ is finite.

(iii) *compactly cancellative* if

$$C^{-1}D = \{x \in G: cx \in D \text{ for some } c \in C\}$$
$$DC^{-1} = \{x \in G: xc \in D \text{ for some } c \in C\}$$

are relatively compact for compact sets $C, D \subseteq G$.

For discrete semigroups property (iii) coincides with being weakly cancellative, and the latter term will be used in that case. We will not use the related notion of being topologically cancellative, see [1].

For a locally compact semigroup G, we write $C(G)$ for the algebra of continuous bounded complex-valued functions on G with the supremum norm; $C_0(G)$ for the subalgebra of functions in $C(G)$ vanishing at infinity; LUC(G) for the subalgebra of left uniformly continuous on G, that is, those functions in $C(G)$ such that $x \mapsto \ell_x f: G \to C(G)$ is continuous. Here $\ell_x f(y) = f(xy)$ is the left translate of the function f, $r_x f$ will be used for the right translate. The measure algebra $M(G)$ is the space of bounded regular Borel measures on G, with total variation norm and convolution product.

A closed translation invariant subspace X of $\ell^\infty(G)$ is *left introverted* if given $m \in X^*$, $f \in X$, and setting $m_\ell(f)(x) = m(\ell_x(f))$, then $m_\ell(f) \in X$. For such X, X^* with the dual norm is a Banach algebra under the product $n \cdot m(f) = \langle n, m_\ell(f) \rangle$. This is exactly the left Arens' product inherited from $\ell^1(G)^{**}$ via the adjoint map $\ell^1(G)^{**} \to X^*$, and $X^\perp$, as the kernel of this map, is a weak*-closed ideal of $\ell^1(G)^{**}$.

The algebra LUC(G) is always left introverted, so that LUC$(G)^*$ is a Banach algebra with the above product. The *topological centre* of LUC$(G)^*$ is

$$Z_t(\mathrm{LUC}(G)^*) = \{n \in \mathrm{LUC}(G)^*: m \mapsto n \cdot m \text{ is weak*-weak* continuous}\}.$$

If G is commutative this is the centre of $\mathrm{LUC}(G)^*$; it is generally strictly larger than the canonical image of $M(G)$ in $\mathrm{LUC}(G)^*$.

A discrete semigroup G is *left amenable* if the space $\ell^\infty(G)$ admits a functional m such that $m(\mathbf{1}) = 1 = \|m\|$ and $m(\ell_x f) = m(f)$, $x \in G$, $f \in \ell^\infty(G)$. Similarly for *right amenable*. If G is both left and right amenable, it is *amenable*. In the case of a group, or even an inverse semigroup, left (or right) amenable implies amenable. These notions go back to [7].

A Banach algebra $\mathfrak{A}$ is *amenable* if every derivation $D: \mathfrak{A} \to X^*$ is inner, for every Banach $\mathfrak{A}$-bimodule X. If one only considers the bimodule $X = \mathfrak{A}$, one has the notion of *weak amenability*.

There are many alternative formulations of the notion of amenability, see [20, 2, 19, 6]. For our purposes here we only need some specific consequences of amenability. Suppose that $\mathfrak{A}$ is amenable, and $\mathscr{I}$ is a (closed, two-sided) ideal in $\mathfrak{A}$. Then

- $\mathfrak{A}$ has a bounded approximate identity;
- $\mathfrak{A}/\mathscr{I}$ is amenable; and
- if $\mathscr{I}$ has a Banach space complement in $\mathfrak{A}$, or has a bounded approximate identity, then $\mathscr{I}$ is itself amenable.

We shall need the following elementary result, [11, Corollary 8.7.4]. The authors had difficulty finding a simple direct proof in the literature, so we give one here for the reader's convenience.

LEMMA 1.1. *Let $T: X \to Y$ be a continuous linear surjection between Banach spaces X and Y. Then $T^{**}: X^{**} \to Y^{**}$ is surjective.*

PROOF. If $m \in Y^{**}$, Goldstine's theorem gives a bounded net (y_λ) in Y with $y_\lambda \xrightarrow{\text{weak*}} m$. By the open mapping theorem there is a bounded net (x_λ) in X with $T(x_\lambda) = y_\lambda$. If n is a weak*-cluster point of (x_λ), weak*-weak* continuity of T^{**} shows that $T^{**}(n) = m$.

LEMMA 1.2. (i) *For G a compactly cancellative locally compact topological semigroup, $C_0(G)$ is a translation invariant, left introverted subspace of* $\mathrm{LUC}(G)$.

(ii) *Conversely, the inclusion $C_0(G) \subseteq \mathrm{LUC}(G)$ always implies G is a topological semigroup.*

PROOF. (i) It suffices to show $\mathrm{LUC}(G)$ contains any continuous function f of compact support. Let F be the support of f, and suppose (x_α) is a net in G converging to x. If $\|\ell_{x_\alpha} f - \ell_x f\| \not\to 0$, we may suppose, by passing to a subnet if necessary, that there exist $\varepsilon > 0$ and a compact neighbourhood N of x such that $(x_\alpha) \subseteq N$ and $\|\ell_{x_\alpha} f - \ell_x f\| \geqq \varepsilon$ for all α. The function $t \mapsto \ell_{x_\alpha} f(t) - \ell_x f(t)$ vanishes outside the compact set $K = \overline{N^{-1}F}$, so that for each α there is $t_\alpha \in K$ such that

$$|\ell_{x_\alpha} f(t_\alpha) - \ell_x f(t_\alpha)| = \|\ell_{x_\alpha} f - \ell_x f\|.$$

By passing to a further subnet if necessary, suppose that $t_\alpha \to t$ for some $t \in K$. Since G is a topological semigroup, $x_\alpha t_\alpha \to xt$, so that

$$0 < \varepsilon \leqq |\ell_{x_\alpha} f(t_\alpha) - \ell_x f(t_\alpha)| \leqq |f(x_\alpha t_\alpha) - f(xt)| + |f(xt) - f(xt_\alpha)| \to 0$$

which is absurd.

Invariance of $C_0(G)$ follows from the assumption of compact cancellation. Now take $f \in C_0(G)$ with $\|f\| \leqq 1$, $m \in C_0(G)^*$ with $\|m\| \leqq 1$. Then $m_\ell(f) \in \mathrm{LUC}(G)$ since the latter is left introverted. Take $\mu \in M(G)$ representing m. Given $\varepsilon > 0$, take compact sets C, D such that $|\mu|(G \backslash C) < \varepsilon$, and $|f(x)| < \varepsilon$ for $x \notin D$. Then

$$|m_\ell(f)(y)| = \left|\int_G f(yx)\, d\mu(x)\right| \leqq \left|\int_C f(yx)\, d\mu(x)\right| + \varepsilon \leqq 2\varepsilon$$

for $y \notin DC^{-1}$. Thus $m_\ell(f) \in C_0(G)$.

(ii) Finally, suppose that $C_0(G) \subseteq \mathrm{LUC}(G)$, let $x_\alpha \to x$, $y_\beta \to y$ in G, and suppose that $f \in C_0(G)$. Then $\|\ell_{x_\alpha} - \ell_x f\| \to 0$, so that

$$\begin{aligned} |f(x_\alpha y_\beta) - f(xy)| &\leqq |\ell_{x_\alpha} f(y_\beta) - \ell_x f(y_\beta)| + |\ell_x f(y_\beta) - \ell_x f(y)| \\ &\leqq \|\ell_{x_\alpha} f - \ell_x f\| + |f(xy_\beta) - f(xy)| \\ &\to 0 \end{aligned}$$

If $x_\alpha y_\beta \not\to xy$ then a subnet stays outside a compact neighbourhood of xy, and choosing a nonzero $f \in C(G)$ vanishing outside this neighbourhood gives a contradiction.

We remark that use of (i) greatly simplifies the proof of Lemma 4 of [23]. Note also that if G is itself compact, (i) implies that $\mathrm{LUC}(G) = C(G)$, (see [26]).

For $\mu \in M(G)$, define $\bar{\mu} \in \mathrm{LUC}(G)^*$ by $\bar{\mu}: f \mapsto \int f(x)\, d\mu(x)$, $f \in \mathrm{LUC}(G)$.

LEMMA 1.3. *Let G be a compactly cancellative locally compact topological semigroup. Suppose that G satisfies*

(i) *there exists a non-zero $\nu \in M(G)$ with $\nu \geqq 0$ and $x \mapsto \nu * \delta_x$* continuous;
(ii) $Z_t(\mathrm{LUC}(G)^*) = \{\bar{\mu}: \mu \in M(G)\}$.
Then if LUC$(G)^*$ *is amenable, G is compact and $M(G)$ is amenable.*

PROOF. By [23, Lemma 4(c)]

$$\mathrm{LUC}(G)^* = \{\bar{\mu}: \mu \in M(G)\} \oplus C_0(G)^\perp$$

where, by earlier remarks, $C_0(G)^\perp$ is a weak*-closed ideal in LUC$(G)^*$. This latter being amenable, $C_0(G)^\perp$ is itself amenable. In particular $C_0(G)^\perp$ has a bounded approximate identity $\{e_i\}$, and without loss of generality we may suppose that $\{e_i\}$ converges weak* to some $E \in$ LUC$(G)^*$. Necessarily $E \in C_0(G)^\perp$. For $x \in C_0(G)^\perp$ we have

$$e_i x \xrightarrow{\|\cdot\|} x \quad \text{and} \quad e_i x \xrightarrow{\text{weak*}} Ex,$$

the second by weak*-continuity of product in the first variable. Thus E is a left identity for $C_0(G)^\perp$. But then for $x \in C_0(G)^\perp$, $xE = \lim(xE)e_i = \lim x(Ee_i) = x$, so that E is a two-sided identity for $C_0(G)^\perp$. Thus for $n \in$ LUC$(G)^*$, $En = (En)E = E(nE) = nE$, since $En, nE \subset C_0(G)^\perp$, and it follows that E is central in LUC$(G)^*$. By hypothesis (ii), $E \in \{\bar{\mu}\colon \mu \in M(G)\}$, so $E = 0$ and hence $X^\perp = \{0\}$. Thus LUC$(G) \subseteq C_0(G)$, so that $1 \in C_0(G)$ and so G is compact. In that case $M(G) = C_0(G)^* =$ LUC$(G)^*$ is amenable.

REMARKS. The hypotheses (i) and (ii) here are satisfied if G is a group, or a weakly cancellative discrete semigroup, see [23]. The identity argument is analogous to [14, Theorem 1.3].

LEMMA 1.4. *Suppose that G is a weakly cancellative locally compact semigroup with $M(G)$ amenable. Then $\ell^1(G)$ is amenable, and G_d is amenable.*

PROOF. We have $M(G) = M_d(G) \oplus M_c(G)$ as Banach spaces, where $M_d(G)$ is the closed subalgebra of discrete measures and $M_c(G)$ is the closed subspace of continuous measures on G; $M_d(G)$ is isometrically isomorphic to $\ell^1(G)$. Thus it suffices to show that $M_c(G)$ is an ideal, for then $\ell^1(G) \cong M(G)/M_c(G)$ is amenable, whence G_d is amenable by [9, Lemma 3]. Now for $\mu, \nu \in M(G)$, then

$$(\mu * \nu)(A) = \int \mu(Ax^{-1})\, d\nu(x) = \int \nu(y^{-1}A)\, d\mu(y)$$

[29]. So for $\mu \in M_c(G)$ and $z \in G$,

$$(\mu * \nu)(\{z\}) = \int \mu(\{z\}x^{-1})\, d\nu(x) = 0, (\nu * \mu)(\{z\}) = \int \mu(x^{-1}\{z\})\, d\nu(y) = 0,$$

by weak cancellation, so that $\mu * \nu$, $\nu * \mu \in M_c(G)$.

2. Locally compact groups.

Throughout this section G will denote a locally compact group. For $L^1(G)$, the situation is well understood, $L^1(G)$ is amenable if and only if G is amenable, [20], and is always weakly amenable, [21, 8]. However the implications of amenability of other algebras on groups are much more fragmentary. It has been known for some time that $M(G)$, G a non-discrete locally compact abelian group, admits point derivations, [3], and so is not even weakly amenable. The situation for non-abelian groups is unresolved in general; however recent results [22] have shown the Fourier algebra need not be amenable for compact G. The major thrust of the results in this section is towards the conjecture that $M(G)$ is amenable only if G is discrete and amenable, the converse being trivial.

Note that if N is a closed normal subgroup of G, then $\mathrm{LUC}(G/N)$ can be identified with the closed subspace of $\mathrm{LUC}(G)$ consisting of those functions which are constant on the cosets of N.

THEOREM 2.1. *Let N be a closed normal subgroup of G. Then if $T\mu$ is the restriction of $\bar{\mu}$ to* $\mathrm{LUC}(G/N)$, *the map $\mu \mapsto T\mu$ is a continuous homomorphism of $M(G)$ onto $M(G/N)$.*

PROOF. It is clear that T is a norm decreasing homomorphism of $M(G)$ into $\mathrm{LUC}(G/N)^*$. Let $q\colon G \to G/N$ be the quotient map. For $\mu \in M(G)$ positive with compact support C, $T(\mu)$ defines a positive Borel measure on $\Delta(G/N)$, the spectrum of the commutative C^*-algebra $\mathrm{LUC}(G/N)$. Since G/N is open in $\Delta(G/N)$, Borel subsets of G/N are Borel subsets of $\Delta(G/N)$. Thus

$$\|T(\mu)\| \geqq T(\mu)(q(C)) = \int_{CN} du = \int_C d\mu \geqq \|T(\mu)\|$$

whence $q(C)$ contains the support of $T(\mu)$. Thus $T(\mu) \in M(G/N)$. Since such measures μ have dense span in $M(G)$, continuity shows that $T(M(G)) \subseteq M(G/N)$.

Now let $v \subseteq G$ be positive with compact support X. Choose a compact set $Y \subseteq G$ with $q(Y) = X$. With $q' = q|Y$, consider the subalgebra $\mathfrak{A} = \{f \circ q'\colon f \in C(X)\}$ of $C(Y)$. As the range of a *-homomorphism $C(X) \to C(Y)$, $\mathfrak{A}$ is closed. Define a functional ϕ on $\mathfrak{A}$ by

$$\phi(f \circ q') = \int_X f(x)\, dv(x)$$

Then ϕ is positive, so $\phi(\mathbf{1}) = \|\phi\|$. By the Hahn-Banach theorem, there is ψ on $C(Y)$ extending ϕ, such that $\|\psi\| = \|\phi\|$. Since $\psi(\mathbf{1}) = \phi(\mathbf{1})$, $\|\psi\| = \psi(\mathbf{1})$ so that ψ is positive.

If $\mu \in M(Y)$ is the Borel measure implementing ψ, μ can be considered as a measure on G, and for $f \in \mathrm{LUC}(G/N)$ we have

$$\langle T(\mu), f\rangle = \int_G f(y)\, d\mu(y) = \int_Y f(y)\, d\mu(y) = \int_X f(x)\, dv(x) = \int_{G/N} f(x)\, dv(x),$$

that is, $T(\mu) = v$. Further, $\|\mu\| = \|\psi\| = \|\phi\| = \|v\|$.

For general positive $v \in M(G/N)$, regularity shows there is a countable pairwise disjoint sequence (X_n) of compact subsets of G/N such that for any Borel set $E \subseteq G/N$,

$$\nu(E) = \sum_{n=1}^{\infty} \nu(E \cap X_n),$$

Defining measures by $\nu_n(E) = \nu(E \cap X_n)$, let μ_n be the measure on G constructed above for ν_n. Then

$$\sum_{n=1}^{\infty} \|\mu_n\| = \sum_{n=1}^{\infty} \|\nu_n\| = \|\nu\| < \infty$$

so that $\mu = \sum_{n=1}^{\infty} \mu_n$ is defined, and $T(\mu) = \nu$ is clear.

COROLLARY 2.2. *Let N be a closed normal subgroup of G. Then if $M(G)$ is amenable, so is $M(G/N)$.*

COROLLARY 2.3. *Let N be a closed normal subgroup of G such that G/N is abelian. Then if $M(G)$ is amenable, N is open.*

PROOF. By Corollary 2.2, $M(G/N)$ is amenable, and so by [3] G/N must be discrete, that is, N is open.

THEOREM 2.4. *Suppose that G is connected. Then $M(G)$ is amenable if and only if G is trivial.*

PROOF. First consider the case that G is a connected Lie group. If $M(G)$ is amenable, then G_d is amenable by Lemma 1.4. So by [27, Theorem 3.9], G is solvable. Let

$$G = G_1 \rhd G_2 \rhd \ldots \rhd G_k = \{e\}$$

be a normal series for G with each G_i/G_{i+1} abelian. Then certainly G_1/G_2 is abelian, so by Corollary 2.3 G_2 is open in G. But G is connected, so that $G_1 = G_2$. That $G = \{e\}$ now follows by induction.

For a general connected group G, there is a directed set (N_i) of compact normal subgroups of G such that each G/N_i is a connected Lie group, and $N_i \downarrow \{e\}$, see [25, §4.7]. But by Corollary 2.2 each $M(G/N_i)$ is amenable, hence trivial by Corollary 2.3. Thus $G = N_i$ for each i, and so G is trivial.

COROLLARY 2.5. *Suppose that* LUC$(G)^*$ *is amenable. Then G is compact and G_d is amenable. If, further, G is connected, then G is trivial.*

PROOF. The first follows from Lemmas 1.3 & 1.4 and [23, Theorem 1]. Since LUC$(G) = C(G)$ when G is compact, Therem 2.4 completes the proof.

We conjecture that LUC$(G)^*$ is amenable if and only if G is finite. Using Lemma 1.3 and Theorems 2.1 and 2.4, this would include the result that $M(G)$ amenable implies G finite in the case where G is compact and totally disconnected. And even this we have been unable to answer.

Although of somewhat different nature, we conclude this section with the following result.

THEOREM 2.6. *G is compact if and only if $L^1(G)$ is a left (or right) ideal in* LUC$(G)^*$.

PROOF. If G is compact then LUC$(G)^* = M(G)$, and $L^1(G)$ is even a two-sided ideal in $M(G)$.

For the converse, we have

$$\mathrm{LUC}(G)^* = M(G) \oplus C_0(G)^{\perp}$$

where $C_0(G)^{\perp}$ is an ideal, [13, Lemma 1.1]. If G fais to be compact, let (K_α) be the net of compact sets of G directed by set inclusion. Take $x_\alpha \notin K_\alpha$, and let ϕ be a weak*-cluster point of the net (δ_{x_α}). Then $\phi \in C_0(G)^{\perp}$, $\phi \in 0$, $\|\phi\| = 1$. For any $f \in L^1(G)$, $f\phi$, $\phi f \in C_0(G)^{\perp}$, and if $L^1(G)$ is either a left or right ideal, then one must also lie in $L^1(G)$, and hence be zero. But for $f \geqq 0$, $\|f\| = 1$ both ϕf, $f\phi$ are also of norm one.

We remark that it is well known that $L^1(G)$ is a left or right ideal in $L^1(G)^{**}$ if and only if G is compact. See, for example, [5, 18].

3. Semigroups: the commutative case.

Let G be a commutative semigroup, such that $\ell^1(G)^{**}$ is amenable. Then $\ell^1(G)$ is also amenable, [14, Theorem 1.8], so by [17, Theorem 2.7], G is a finite semilattice of groups, that is, there is a finite semigroup of idempotents S such that

$$G = \bigcup_{\alpha \in S} G_\alpha$$

where, for each α, $\beta \in S$, $G_\alpha G_\beta \subseteq G_{\alpha\beta}$.

The first result of this section uses this structure, but does not presuppose commutativity.

THEOREM 3.1. *Let G be a finite semilattice of semigroups, each of which is either weakly cancellative, or one-sided cancellative with identity. Then G is finite if $\ell^1(G)^{**}$ is amenable.*

PROOF. Let S be the semilattice, and set $z = \prod\{s : s \in S\}$. Then $sz = z$ for all $s \in S$, so that z is a zero for S.

It follows that G_z is an ideal in G. Thus as Banach spaces,

$$\ell^1(G)^{**} = \ell^1(G_z)^{**} \oplus \ell^1\left(\bigcup_{\alpha \neq z} G_\alpha\right)^{**},$$

where $\ell^1(G_z)^{**}$ is a complemented ideal in the amenable algebra $\ell^1(G)^{**}$, and so is itself amenable. Thus G_z is finite by [14, Corollary 1.9].

Suppose inductively that G is finite whenever $|S| \leqq k$; we know this is true for $k = 1$. If $|S| = k + 1$, define

$$T = \{\alpha \in S\colon |\alpha G| \leqq k\},$$

Necessarily $z \in T$, so certainly $T \neq \emptyset$.

For any $\alpha \in T$, $\beta \in S$, $|\alpha\beta S| \leqq |\alpha S| \leqq k$, so that $\alpha\beta \in T$. Thus T is an ideal in S. Further, $S \backslash T$ is clearly closed under products, and so is a subsemigroup of S. Thus $S \backslash T$ is a finite semilattice with at most k elements.

For $\alpha \in T$, $|\alpha S| \leqq k$, so that αS is a proper ideal in S. Thus $\cup \{G_\beta\colon \beta \in \alpha S\}$ is the union of at most k semigroups, and is an ideal in G. But as Banach spaces,

$$\ell^1(G)^{**} = \ell^1\left(\bigcup_{\beta \in \alpha S} G_\beta\right)^{**} \oplus \ell^1\left(\bigcup_{\beta \in S \backslash \alpha S} G_\beta\right)^{**}$$

so that $\ell^1(\cup_{\beta \in \alpha S} G_\beta)^{**}$ is a complemented ideal in $\ell^1(G)^{**}$, and so is amenable. But then by the inductive hypothesis, we have G_β is finite for $\beta \in \alpha S$. Since $\alpha \in \alpha S$ we have G_α finite, and this holds for all $\alpha \in T$.

Finally,

$$\ell^1\left(\bigcup_{\alpha \in S \backslash T} G_\alpha\right)^{**} \cong \ell^1(G)^{**}/\ell^1\left(\bigcup_{\alpha \in T} G_\alpha\right)^{**}$$

is amenable, so by inductive hypothesis again $\cup_{\alpha \in S \backslash T} G_\alpha$ is finite. We thus have G_α finite for each $\alpha \in S$.

COROLLARY 3.2. *Let G be a commutative semigroup such that $\ell^1(G)^{**}$ is amenable. Then G is a finite semilattice of finite groups.*

PROOF. As noted earlier G satisfies the hypothesis of the theorem with the semigroups being groups.

THEOREM 3.3. *Suppose that G is a commutative, weakly cancellative locally compact semigroup. If $M(G)$ is amenable then G is discrete, and is a finite semilattice of groups. If $M(G)^{**}$ is amenable then G is a finite semilattice of finite groups.*

PROOF. By Lemma 1.4, $\ell^1(G)$ is amenable, so as above G is a finite semilattice of groups $G_1, \ldots, G_n$, where we may suppose that G_1 is an ideal of G. Let V be a compact neighbourhood in G of the identity e of G_1, and let U be the interior of V. Then

$$e \in e(U \cap G_1) = U \cap G_1 \subseteq V \cap G_1 = e(V \cap G_1) \subseteq eV \subseteq G_1$$

so that, by separate continuity, eV is a compact neighbourhood of e in G_1. Thus G_1 is a locally compact group. Since $M(G_1)$ is a unital ideal in an amenable algebra, it is itself amenable, whence G_1 is discrete.

Now for each $2 \leqq j \leqq n$, $G_j = \cup_{x \in G_1} x^{-1}\{e\} \cap G_j$. If $x \in G_1$ is such that $x^{-1}\{e\} \cap G_j \neq \emptyset$, such x necessarily existing for each j, then this is a non-empty open finite set in G_j, so that G_j is discrete.

If $M(G)^{**}$ is amenable then by [14, Theorem 1.8] so is $M(G)$, hence G is a finite semilattice of discrete groups. These are then finite by Corollary 3.2.

EXAMPLE 3.4. Let R be the additive group of real numbers with the usual topology, and take $G = \mathsf{R} \cup \{\infty\}$ to be the one-point compactification if R, with ∞ an absorbing element. Then G is a commutative locally compact semigroup, which is a semilattice of the groups R and $\{\infty\}$. However $M(G)$ is not amenable, since otherwise $M(G)/\mathsf{C} \cong M(\mathsf{R})$ would be amenable.

4. Semigroups: the general case.

For the non-commutative case there is no longer any characterization of amenability of $\ell^1(G)$, in terms of G, except in the case of inverse semigroups [9]. See also [10].

We recall some further standard notions from semigroup theory, for more details see [4]. Again only the left versions will be defined.

(a) A semigroup G is *left reversible* if for all $x, y \in G$, $xG \cap yG \neq \emptyset$;

(b) $H \subseteq G$ is a *left ideal group* if H is a left ideal in G, as well as being a group under the semigroup operation.

For future reference we summarize some known structural implications of amenability.

LEMMA 4.1. *Let G be a semigroup with $\ell^1(G)$ amenable. Then*

(i) *G is amenable*, [9, *Lemma* 3],

(ii) *G is (left and right) reversible*, [15; 28, *Lemma* 1],

(iii) *G is regular and has a finite (and non-empty) set of idempotents*, [10, *Theorem* 2].

If G is left reversible, define a relation σ on G by $x\sigma y$ if there exists $z \in G$ such that $xz = yz$. This relation is a congruence, and the semigroup G/σ is right cancellative. Similarly, one defines a congruence ρ on a right reversible semigroup such that G/ρ is left cancellative.

Finally we note that if H is a semigroup, and θ: $G \to H$ preserves products and maps onto H, then the map $\delta_s \mapsto \delta_{\theta(s)}$ extends in the obvious way to a continuous homomorphism Θ of $\ell^1(G)$ onto $\ell^1(H)$, and then $\Theta^{**}: \ell^1(G)^{**} \to \ell^1(H)^{**}$ is an epimorphism by Lemma 1.1.

The first result is well known for inverse semigroups, [9, Therem 8], and in fact can be obtained from Theorem 2.3 and Corollary 1.3 of [16].

LEMMA 4.2 (Grønbæk). *Let G be a one-sided cancellative semigroup with $\ell^1(G)$ amenable. Then G is an amenable group.*

PROOF. Assume that G is left cancellative. Let $\mathscr{E} = \{e_1, \ldots, e_n\}$ be the set of idempotents of G. Since G is left cancellative it is a right group, [4, §1.11]. But then by [4, Therem 1.27], G is a direct product $G_0 \times \mathscr{E}$ of a group G_0 and the right zero semigroup $\mathscr{E}$. Since $\mathscr{E}$ is a homomorphic image of G, it is amenable, and so must be a singleton. Thus G is a group.

In particular, this shows that if G is a semigroup with $\ell^1(G)$ amenable, then G/ρ, G/σ, being one-sided cancellative, are each amenable groups.

LEMMA 4.3. *Let G be a semigroup with $\ell^1(G)$ amenable, $\mathscr{E}$ its (finite) set of idempotents. Then there exists $z \in \mathscr{E}$ such that for any $a, b \in G$, $a\sigma b$ if and only if $az = bz$.*

PROOF. By definition, if $x, y \in G$ and $x\sigma x$, there is $z \in G$ with $xz = yz$. Regularity ensures there is z^* with $zz^* \in \mathscr{E}$, so that $x\sigma y$ precisely when $xe = ye$ for some $e \in \mathscr{E}$. Let k be the cardinality of a minimal set $\mathscr{F} = \{e_1, \ldots, e_k\}$ of idempotents implementing σ in this manner.

The result is clear if $\mathscr{F}$ is a singleton, so suppose $k > 1$. Since G/σ is a group, the image of any e_j is the identity of the group, whence the elements of $\mathscr{E}$ are all σ-equivalent. Thus given $i \neq j$, choose e_ℓ such that $e_i e_\ell = e_j e_\ell$. Now $e_i e_\ell$ also maps to the group identity, so there exists e_m such that $e_\ell e_m = (e_i e_\ell) e_m$, whence $e_i s = e_j s = s$ where $s = e_\ell e_m$.

Thus if $x\sigma y$ in G is implemented by either e_i or e_j, it is also implemented by s, and hence by some single idempotent $e_r \in \mathscr{F}$. Hence σ is implemented by $\mathscr{F}' = \mathscr{F} \cup \{e_r\} \backslash \{e_i, e_j\}$ which has $k - 1$ elements, contrary to the minimality of $\mathscr{F}$.

THEOREM 4.4. *Let G be a semigroup with $\ell^1(G)$ amenable. Then G contains exactly one left ideal group G_0. Furthermore G_0 is amenable, it is the unique right ideal group, $G_0 \cong G/\sigma \cong G/\rho$, and there is an idempotent $z \in G$ such that*

$$G = z^{-1}G_0 = G_0 z^{-1}.$$

PROOF. By Lemma 4.3, take an idempotent $z \in G$ which implements σ. Let $H = G/\sigma$, so by Lemma 4.2 H is a group. Choose a set of coset representatives F in G, and define $G_0 = Fz$. Then G_0 is a left ideal group in G.

To see this, first note that for $x \in G$, $f \in F$, $xfz\sigma f'$ for some $f' \in F$, whence $xfz^2 = f'z$, and so $xfz \in G_0$. Thus $xG_0 \subseteq G_0$ for any $x \in G$, so that G_0 is a left ideal.

Further, suppose $y = fz \in G_0$, where $f \in F$, and take any $w \in G_0$. Since H is a group, left multiplication by the class $\bar{w}$ of w is a surjection on H, and hence there is $f' \in G_0$ with $\bar{w} \cdot \bar{f}' = \bar{f}$. But then $wf'z = fz$ so that $wG_0 \supseteq G_0$. Thus $wG_0 = G_0$.

Also, $w = gz$ for some $g \in F$. Taking $f' \in F$ with $f'zg\sigma g$ (using right multiplication in H) we have $f'zgz = fz$, so that $y \in G_0 w$.

Thus $wG_0 = G_0 w = G_0$ for all $w \in G_0$, and this means that G_0 is a group.

Now suppose that G' is another left ideal group in G. Then $G_0 \cap G' \neq \emptyset$ by right reversibility, and is a left ideal contained in both groups, whence $G_0 = G'$. Since $G_0 \cong G/\sigma$ we have the "left" statement of the lemma.

Similar consideration of G/ρ shows that G contains a unique right ideal group H. Taking $x \in HG \subseteq G_0 \cap H$, we have $H = Hx \subseteq G_0$ because G_0 is a left ideal, and $G_0 = xG_0 \subseteq H$ because H is a right ideal, whence $H = G_0$.

THEOREM 4.5. *Let G be a semigroup with $\ell^1(G)^{**}$ amenable.*

(i) *If G is one-sided cancellative, then G is a finite group.*

(ii) *G/ρ, G/σ are finite groups.*

(iii) *G has exactly one left ideal group G, this is also a right ideal in G, and is finite.*

(iv) *If G is one-sided weakly cancellative, then G is finite.*

PROOF. (i) Theorem 1.8 of [14] shows $\ell^1(G)$ is amenable, so if G is left cancellative, G is a group by Lemma 4.2. But then Theorem 1.3 of [14], shows G is finite.

(ii) As noted earlier, G/ρ, G/σ are amenable groups, which are finite by (i).

(iii) Theorem 4.4 and (ii).

(iv) Left weakly cancellative means that the σ-equivalence classes are finite, and we are done by (ii).

Part (iv) of Theorem 4.5 cannot be proved by the same techniques as the two-sided cancellative case, Lemma 1.3 above and [14], because the topological centre of $\ell^1(G)^{**}$ need not be $\ell^1(G)$. Indeed it may be all of $\ell^1(G)^{**}$, [23]. Since any finite semigroup is weakly cancellative, we certainly cannot conclude that G is a group in (iv). And certainly weakly cancellative semigroups need not be amenable.

EXAMPLE 4.6. Let G be a finite set of cardinality at least 2, $z \in G$ a fixed element, and define a product on G by $st = z$. Then G is commutative, and weakly cancellative with $G_0 = \{z\}$ the only ideal group in G, however $\ell^1(G) = \ell^1(G)^{**}$ is not amenable since G is not regular. Since for finite dimensional algebras amenability is the same as semisimplicity, $\ell^1(G)^{**}$ is not semisimple. Indeed, writing $G = \{s_1, \ldots, s_n\}$, define $\psi_i \in \ell^1(G)^{**}$ by $\psi_i(f) = f(s_{i+1}) - f(s_i)$ for

$f \in \ell^\infty(G)$, $1 \leqq i < n$. Let $\mathscr{I}$ be the linear span of $\psi_1, \ldots, \psi_n$. Then for $\phi \in \ell^1(G)^{**}$, $\phi * \psi_i = \psi_i * \phi = 0$, $1 \leqq i < n$ so that $\mathscr{I}$ is a nil ideal in $\ell^1(G)^{**}$.

THEOREM 4.7. *Suppose that G is a cancellative locally compact semigroup. If $M(G)$ is amenable then G is a locally compact topological group, hence trivial if connected. If $M(G)^{**}$ is amenable, G is a finite group.*

PROOF. It first follows from Lemma 1.4 that $\ell^1(G)$ is amenable, then Theorem 4.4 and cancellation shows G is a group, hence a topological group by an earlier observation. Now apply Theorem 2.4. The second statement then follows from [14, Corollary 1.4].

Theorems 3.3 and 4.7 together give a partial answer to Problem 23 of [24].

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALBERTA
EDMONTON T6G 2G1
CANADA
email: tlau@vega.math.ualberta.ca

DEPARTMENT OF MATHEMATICS
AUSTRALIAN NATIONAL UNIVERSITY
ACT 0200
AUSTRALIA
email: vick.loy@maths.anu.edu.au