MATH. SCAND. 79 (1996), 297–319

THE POISSON TRANSFORM FOR SPLIT REDUCTIVE SYMMETRIC SPACES

H. THORLEIFSSON

1. Introduction.

Let G be a real reductive Lie group in Harish-Chandra's class, and σ an involution on G commuting with a Cartan involution θ. Let H be an open subgroup of the subgroup G^σ of σ-fixed points in G. Then G/H is a reductive symmetric space. Let $\mathfrak{q}_0$ (resp. $\mathfrak{p}_0$) be the -1-eigenspace of the involution σ (resp. θ). Then we have a decomposition $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{q}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ of the Lie algebra $\mathfrak{g}_0$ of G ($\mathfrak{h}_0$ (resp. $\mathfrak{k}_0$) is the Lie algebra of H (resp. K)). G/H is called split if a maximal abelian subspace of $\mathfrak{q}_0 \cap \mathfrak{p}_0$ is also maximal abelian in $\mathfrak{q}_0$. Let $\mathscr{E}(G/H)$ be the Fréchetspace of smooth functions on G/H. The Poisson transform maps certain parabolically induced representations equivarianly into $\mathscr{E}(G/H)$. I am interested in describing the image of the Poisson transform for split spaces G/H.

Before I describe the problem handled here and the results given more precisely, let me recall the definition of the Poisson transform and give some motivation for the problem handled. Let $P = MAN$ be the Langlands decomposition of a minimal $\sigma\theta$-stable parabolic subgroup of G, and $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ the corresponding decomposition of its complexified Lie algebra. Identify $(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*$ with the subspace of $\mathfrak{a}^*$ of elements ν trivial on $\mathfrak{a} \cap \mathfrak{h}$. For $\nu \in (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*$ let $A \to \mathbf{C}$, $a \mapsto a^\nu$ be the one-dimensional character with differential ν and let $(\pi_{P,\nu}, I_P^\infty(\nu))$ be the representation of G smoothly induced from the one-dimensional character $(M \cap H)AN \to \mathbf{C}$, $man \mapsto a^\nu$. The Poisson transform $\mathscr{P}$: $I_P^\infty(\nu) \to \mathscr{E}(G/H)$ is given by $\mathscr{P}f(x) = \int_{H/H \cap P} f(xh)d\dot{h}$ $(x \in G, f \in \mathscr{E}(G/H))$. Here $d\dot{h}$ is a H-invariant Radon-measure on $H/H \cap P$. The integral converges at least for those $\nu \in (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*$ satisfying $\mathrm{Re}\langle \nu - \rho, \alpha\rangle > 0$ for all $\alpha \in \Sigma(\mathfrak{n}, \mathfrak{a})$($\Sigma(\mathfrak{n}, \mathfrak{a})$ is the set of roots of $\mathfrak{a}$ in $\mathfrak{n}$ and $\rho \in \mathfrak{a}^*$ is given by $\rho(X) = \frac{1}{2}\mathrm{Trace}(\mathrm{ad}(X) \mid \mathfrak{n})$ for $X \in \mathfrak{a}$). The restriction of functions from G to K defines an isomorphism from $I_P^\infty(\nu)$ onto the space $\mathscr{D}(K/M^a)$ of smooth functions on K/M^a, with $M^a = M \cap K \cap H$. For $f \in \mathscr{D}(K/M^a)$ let $f_\nu \in I_P^\infty(\nu)$ be the function with $f_\nu \mid K = f$ and put $\mathscr{P}_\nu f = \mathscr{P}f_{-\nu}$. In

Received March 4, 1994; in revised form December 16, 1994.

case $H = K$ this is just the Poisson integral for the non-compact Riemannian space G/K. In this case every joint eigenfunction of the space $\mathsf{D}(G/K)$ of invariant differential operators on G/K in $\mathscr{E}(G/K)$ can be represented by a Poisson integral of a hyperfunction on K/M. The problem to extend this statement to other symmetric spaces was the motivation of Oshima and Sekiguchi to introduce and study the Poisson transform for certain symmetric spaces G/K_ε in [12].

Now let (ξ, V_ξ) be an $(M \cap H)$-spherical irreducible smooth representation of M. Since it is $(M \cap H)$-spherical, there is some non-trivial $\eta \in V_\xi'^{M \cap H}$ (the space of $(M \cap H)$-invariant continuous linear functionals on V_ξ), determining an intertwining operator $\Phi_\eta : V_\xi \to \mathscr{E}(M/M \cap H)$ by $\Phi_\eta(v)(m) = \langle \xi(m^{-1})v, \eta \rangle$ $(v \in V_\xi, m \in M)$. Φ_η maps V_ξ onto a submodule of $\mathscr{E}(M/M \cap H)$ (note that $M/M \cap H$ is compact). The parabolically induced representation $I_P^\infty(\xi : \nu)$ can be identified with a subrepresentation of $I_P^\infty(\nu)$ and we get a Poisson transform $\mathscr{P} : I_P^\infty(\xi : \nu) \to \mathscr{E}(G/\mathrm{H})$. $\mathscr{P}_\nu$ will also be used to demote the Poisson transform in the compact picture of $I_P^\infty(\xi : -\nu)$. In this generality the Poisson transform was defined in [11] (under slight restrictions on G/H) and [2] (for more general G/H) and proved to have a meromorphic continuation to $(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*$. This was generalized by Brylinski and Delorme in [5] to general $\sigma\theta$-stable P and ξ in the discrete series for $M/M \cap H$, and recently by the author to ξ of moderate growth.

The motivation for studying the image of the Poisson transform does not only come from non-compact Riemannian spaces. For a moment let G_1 be a reductive Lie group with Cartan involution θ_1 and a maximal compact subgroup $K_1 = G_1^{\theta_1}$. Put $G = G_1 \times G_1$, let σ and θ be the involutions on G given by $\sigma(x, y) = (y, x)$ and $\theta(x, y) = (\theta_1(x), \theta_1(y))$ $(x, y \in G_1)$, and put $H = G^\sigma$. The map $G \to G_1$, $(x, y) \mapsto xy^{-1}$ induces an isomorphism of G/H to G_1. A $\sigma\theta$-stable parabolic subgroup of G has the form $P = P_1 \times \bar{P}_1$, P_1 being a parabolic subgroup of G_1 and $\bar{P}_1 = \theta_1(P_1)$. Let $P_1 = M_1 A_1 N_1$ be the Langlands decomposition of P_1. A (generalized) principal series representation $I_P(\xi : \nu)$ (with unitary ξ) is equivalent to a $(\mathfrak{g}, K)$-module of the form $I_{P_1}(\xi_1, \nu_1) \otimes I_{\bar{P}_1}(\xi_1' : -\nu_1)$, with an irreducible (unitary) representation (ξ_1, V_{ξ_1}) of M_1 and $\nu_1 \in \mathfrak{a}_1^*$. Let $J_{\bar{P}_1|P_1}(\xi_1 : \nu_1) : I_{P_1}(\xi_1 : \nu_1) \to I_{\bar{P}_1}(\xi_1 : \nu_1)$ be the standard intertwining operator. The Poisson transform $\mathscr{P} : I_P(\xi : \nu) \to \mathscr{E}(G/H)$ is given by $\mathscr{P}(\phi \otimes \psi)(x) = \langle \pi_{\bar{P}_1, \xi_1, \nu_1}(x) J_{\bar{P}_1|P_1}(\xi_1 : \nu_1)\phi, \psi \rangle$ for $\phi \in I_{P_1}(\xi_1, \nu_1)$' $\psi \in I_{\bar{P}_1}(\xi_1' : -\nu_1)$. Thus the Poisson transform for G/H is given by standard intertwining operators on G_1 (All this is explained or follows from the explanations given in [5] §4). Recall that by the Langlands classification every irreducible $(\mathfrak{g}, K)$-module V is equivalent to the image of certain standard intertwining operators $J_{\bar{P}_1|P_1}(\xi_1 : \nu_1)$. This means that the space of K-finite coefficientfunctions for V is equal to the K-finite image $\mathscr{P} I_P(\xi : \nu)$. (This can be strengthened to functions of moderate growth.) Note that P_1, ξ_1 and ν_1 can be determined by the asymptotic expansions of coefficientfunctions for V. This leads to a general problem of generalizing the Langlands classification to symmetric spaces using the Poisson transform.

Let r be the number of open H-orbits in G/P, and let $P_1, \ldots, P_r$ be minimal $\sigma\theta$-stable parabolic subgroups of G such that $HP_1, \ldots, HP_r$ are disjoint and open in G. To be able to represent joint eigenfunctions of $\mathbf{D}(G/H)$ in $\mathscr{E}(G/H)$ by a Poisson integral it is known, that one has to use sums of Poisson integrals for $P_1, \ldots, P_r$. This was first done by Oshima and Sekiguchi in [12] for G/K_ε-spaces. The Poisson transform is best known for rank one spaces G/H. It can be normalised to be holomorphic on $(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*$. In [10] W. A. Kosters handles the space $\mathrm{SL}(n, \mathbf{R})/S(\mathrm{GL}(1, \mathbf{R}) \times \mathrm{GL}(n-1, \mathbf{R}))$(for this space $r = 1$) and proves that $\mathscr{P}_\lambda$ is an isomorphism for "generic" λ (see [10] Theorem 5.4 for a more precise statement with hyperfunctions). H. Schlichtkrull has given a complete description of the image of the (normalized) Poisson transform for the hyperbolic spaces ([15] Theorem 7.1) and gives an interesting solution to the above described problem of Oshima and Sekiguchi for these spaces ([15] Theorem 7.4). Much research has been done in establishing the Plancherelformula for G/H (using the H-invariant linear functionals mapping $f \in I_P^\infty(\xi : \nu)$ to $\mathscr{P}f(e)$). The interested reader should consult [4] for a survey on the Plancherelformula (and other questions).

This paper is organized as follows. In §2 some standard notations are introduced. In §3 I give the construction of a manifold B and a representation $\sigma_{P,\nu}$ an $\mathscr{D}(B)$, equivalent to the direct sum of $I_P^\infty(\nu_i)$ (with ν_i "conjugate" to ν). In §4 the construction of standard intertwining operators is extended to an operator $\mathscr{D}(B) \to \mathscr{D}(B)$ intertwining $\sigma_{P,\nu}$ and $\sigma_{\bar{P},\nu}$. In §5 the Poisson transform $\mathscr{P}_\nu : \mathscr{D}(B) \to \mathscr{E}(G/H)$ is defined and the asymptotics of $\mathscr{P}_\nu F$ ($F \in \mathscr{D}(B)$) is related to standard intertwining operators. As a consequence a weak injectivity result is given (Corollary 5.8). In §6 only split symmetric spaces G/H are handled. Then $\mathfrak{a}^a = \mathfrak{a} \cap \mathfrak{q}$ is maximal abelian in $\mathfrak{q} \cap \mathfrak{p}$ and $\mathfrak{q}$. Let $\mathscr{E}_\lambda(G/H)$ be the joint eigenspaces of $\mathbf{D}(G/H)$ in $\mathscr{E}(G/H)$ for $\lambda \in (\mathfrak{a}^a)^*$ (see §2). It is proved that the multiplicity of K-types in $\mathscr{D}(B)$ and $\mathscr{E}_\lambda(G/H)$ are always the same. To proof this it is assumed that H is essentially connected. This means $H = M^a H_e$ (H_e being the connected component of the unit element in H). If G is connected and semisimple and H is also connected this follows from a result given by T. Oshima in his lecture at the conference on harmonic analysis on Lie groups in "The Danish Lie group seminar" in august 1991. In §7 I look at the image of the Poisson transform. Again only for split spaces G/H with H essentially connected. A set $A(G/H)$ of minimal K-types of some principal series $I_P(\xi : \nu)$ is introduced. They are shown to have the same λ-norm (Lemma 7.3). For $\lambda \in (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*$, satisfying $\mathrm{Re}\langle \lambda + \rho, \alpha \rangle \leqq 0$ for all $\alpha \in \Sigma(\mathfrak{n}, \mathfrak{a})$, it is proved that $\mathscr{P}_\lambda$ maps the δ-isotypic component $\mathscr{D}(B)_\delta$ of $\mathscr{D}(B)$ isomorphically onto the δ-isotypic component $\mathscr{E}_\lambda(G/H)_\delta$ of $\mathscr{E}_\lambda(G/H)$ (see §2 for notations) for all $\delta \in A(G/H)$ (Proposition 7.5).

Then a construction of the finite dimensional space $V(\xi)$ of [2] is given, parametrizing the operators from $\mathscr{D}(K:\xi)$ to $\mathscr{D}(B)$ intertwining $\pi_{P,\xi,\nu}$ and $\sigma_{P,\nu}$, giving an (extended) Poisson transform $\mathscr{P}_\xi : I_P(\xi:\nu) \otimes V(\xi) \to \mathscr{E}(G/H)$. The image of $\mathscr{P}_\xi$ is described for ν satisfying $\mathrm{Re}\langle \nu - \rho, \alpha\rangle \geqq 0$ for all $\alpha \in \Sigma(\mathfrak{n}, \alpha)$ (Corollary 7.6 and Theorem 7.8).

Now let V be an irreducible $(\mathfrak{g}, K)$-submodule of the space $\mathscr{E}_\lambda(G/H)_K$ of K-finite functions in $\mathscr{E}_\lambda(G/H)$. By Théorème 2 and 3 of [6] V is equivalent to an irreducible quotient of a (generalized) principal series representation $I_P(\xi:\nu)$ with a $\sigma\theta$-stable parabolic subgroup $P = MAN$, ξ in the discrete series for $M/M \cap H$ and a $\Sigma(\mathfrak{n}, \mathfrak{a})$-dominant $\nu \in (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*$. If P is minimal $\sigma\theta$-stable and ν in not just $\Sigma(\mathfrak{n}, \mathfrak{a})$-dominant but satisfies the stronger condition $\mathrm{Re}\langle \nu - \rho, \alpha\rangle \geqq 0$ for all $\alpha \in \Sigma(\mathfrak{n}, \mathfrak{a})$, then V is equivalent to the unique irreducible quotient $J_P(\xi:\nu)$ of $I_P(\xi:\nu)$ and by Theorem 7.8 there is a unique $T \in V(\xi)$ such that $V = \mathscr{P}_\xi(I_P(\xi:\nu) \otimes \mathbf{C}T)$. This also follows from [2] Theorem 5.10 (proved using different methods but not restricted to split spaces G/H) for "generic" ν (see [2] for a precise formulation). For the minimal $\sigma\theta$-stable parabolic subgroups P it remains to determine when $\mathscr{P}_\xi(I_P(\xi:\nu) \otimes \mathbf{C}T)$ is irreducible and also to handle the case when ν only satisfies the weaker condition of being $\Sigma(\mathfrak{n}, \mathfrak{a})$-dominant.

2. Some notations and preliminary results.

The standard notations $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ will be used to denote the ring of integers, and the field of real and complex numbers, $\mathbf{N}$ for the positive integers and $\mathbf{N}_0$ for the nonnegative integers. Let $\mathfrak{U}(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$, $\mathfrak{Z}(\mathfrak{g})$ the center of $\mathfrak{U}(\mathfrak{g})$ and $\mathfrak{S}(\mathfrak{g})$ the symmetric algebra for $\mathfrak{g}$. Further let $\mathrm{D}(G/H)$ be the space of invariant differential operators on G/H and let

$$\Gamma : \mathfrak{U}(\mathfrak{g})^H \to \mathrm{D}(G/H) \tag{1}$$

be the canonical homomorphism. (If H operates on a vector space V, then V^H denotes the subspace of H-fixed vectors.) Then Γ is an epimorphism with kernel $\mathfrak{U}(\mathfrak{g})^H \cap \mathfrak{U}(\mathfrak{g})\mathfrak{h}$ ([8] Ch. II. Theorem 4.6).

Let $\hat{K}$ be the set of equivalence classes of (continuous) irreducible representations of K. If V is a K-module and $\delta \in \hat{K}$ let V_δ be the δ-isotypic component of V and V_K the submodule of vectors $v \in V$ contained in some finite dimensional submodule of V.

Let $P = MAN$ be the Langlands decomposition of a fixed minimal $\sigma\theta$-stable parabolic subgroup of G, and $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ the corresponding decomposition of the complexified Lie algebra. Put $\mathfrak{a}^a = \mathfrak{a} \cap \mathfrak{q}$. Then $\mathfrak{a}^a$ is a maximal abelian subspace in $\mathfrak{q} \cap \mathfrak{p}$. Extend $\mathfrak{a}^a$ to a maximal abelian subspace $\mathfrak{a}^d$ of $\mathfrak{q}$. By [14] Theorem 10 the set $\Sigma(\mathfrak{g}, \mathfrak{a}^a)$ of roots of $\mathfrak{a}^a$ in $\mathfrak{g}$ is a (restricted) root system. Since P is $\sigma\theta$-stable $\alpha \mid \mathfrak{a}^a \neq 0$ for all $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ and $\Sigma(\mathfrak{n}, \mathfrak{a}^a)$ is a positive system of roots for $\Sigma(\mathfrak{g}, \mathfrak{a}^a)$. Let

$\Sigma^+(\mathfrak{g}, \mathfrak{a}^d)$ be a positive system of roots for $\Sigma(\mathfrak{g}, \mathfrak{a}^d)$, such that $\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a}^d)$ and $\alpha | \mathfrak{a}^a \neq 0$ implies $\alpha | \mathfrak{a}^a \in \Sigma(\mathfrak{n}, \mathfrak{a}^a)$, and let $\mathfrak{n}^d$ be the sum of the root spaces $\mathfrak{g}_\alpha$, $\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a}^d)$. Then $\mathfrak{n} \subset \mathfrak{n}^d$. Let $W^d = W(\mathfrak{g}, \mathfrak{a}^d)$ be the Weyl group for $\Sigma(\mathfrak{g}, \mathfrak{a}^d)$.

Recall the construction of the Harish-Chandra isomorphism. Let $\tilde{\gamma}: \mathfrak{U}(\mathfrak{g})^H \to \mathfrak{U}(\mathfrak{a}^d)^{W^d}$ be the projection with respect to the decomposition $\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{a}^d) \oplus (\mathfrak{n}^d \mathfrak{U}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g})\mathfrak{h})$ and let $T_{\rho^d}: \mathfrak{U}(\mathfrak{a}^d) \to \mathfrak{U}(\mathfrak{a}^d)$ be righttranslation by $\rho^d \in (\mathfrak{a}^d)^*$ ($\rho^d(X) = \frac{1}{2}\mathrm{Trace}(\mathrm{ad}(X) | \mathfrak{n}^d)$, $X \in \mathfrak{a}^d$). Then

$$(2) \qquad \gamma: \mathfrak{U}(\mathfrak{g})^H \to \mathfrak{U}(\mathfrak{a}^d)^{W^d}$$

is given by $\gamma(u) = T_{\rho^d}(\tilde{\gamma}(u))$, $u \in \mathfrak{U}(\mathfrak{g})^H$. γ factorizes (through Γ) to an isomorphism (also denoted by γ) between $\mathrm{D}(G/H)$ and $\mathfrak{U}(\mathfrak{a}^d)^{W^d}$. For $\lambda \in (\mathfrak{a}^d)^*$ let $\chi_\lambda: \mathrm{D}(G/H) \to \mathbf{C}$ (or χ_λ^G) be defined by $\chi_\lambda(D) = \langle \gamma(D), \lambda \rangle$, $D \in \mathrm{D}(G/H)$. ($\chi_\lambda(\Gamma(u)) = \langle \tilde{\gamma}(u), \lambda + \rho^d \rangle$, $u \in \mathfrak{U}(\mathfrak{g})^H$. Here $\langle , \rangle$ is used for evaluation of an element in $\mathfrak{U}(\mathfrak{a}^d)$ at $\lambda \in (\mathfrak{a}^d)^*$.) As usual let $\mathscr{E}_\lambda(G/H)$ be the space of functions $f \in \mathscr{E}(G/H)$, satisfying $Df = \chi_\lambda(D)f$, for all $D \in \mathrm{D}(G/H)$. Now put $L = MA (L = P \cap \theta(P))$, and let $\mathfrak{l}$ be the corresponding complexified Lie algebra. There is a canonical isomorphism $\mathfrak{U}(\mathfrak{l}) \cong \mathfrak{U}(\mathfrak{m}) \otimes \mathfrak{U}(\mathfrak{a})$. Let $\mathfrak{r}$ be the orthogonal complement of $\mathfrak{l} \cap \mathfrak{h}$ in $\mathfrak{h}$ (with respect to $\langle , \rangle$). The projection to the first component in the decomposition $\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{l}) \oplus (\mathfrak{n}\mathfrak{U}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g})\mathfrak{r})$ gives a homomorphism

$$(3) \qquad \tilde{\gamma}^P: \mathfrak{U}(\mathfrak{g})^H \to \mathfrak{U}(\mathfrak{l})^{L \cap H}.$$

For $\nu \in \mathfrak{a}^*$ let

$$(4) \qquad \gamma_\nu^P: \mathfrak{U}(\mathfrak{g})^H \to \mathfrak{U}(\mathfrak{m})^{M \cap H},$$

be defined by $\gamma_\nu^P(u) = \langle \tilde{\gamma}^P(u), \nu + \rho \rangle$. Here $\rho(X) = \frac{1}{2}\mathrm{Trace}(\mathrm{ad}(X) | \mathfrak{n})$, $(X \in \mathfrak{a})$, and $\langle \cdot, \nu + \rho \rangle$ denotes the evaluation at $\nu + \rho$ interpreted as a homomorphism from $\mathfrak{U}(\mathfrak{m}) \otimes \mathfrak{U}(\mathfrak{a})$ to $\mathfrak{U}(\mathfrak{m})$. For $\lambda \in (\mathfrak{a}^d)^*$ put $\mu = \lambda | \mathfrak{a}^d \cap \mathfrak{k}$ and $\nu = \lambda | \mathfrak{a}^d \cap \mathfrak{p}$. Then one gets $\chi_\lambda(u) = \chi_\mu^M(\gamma_\nu^P(u))$, $u \in \mathfrak{U}(\mathfrak{g})^H$ (here $\chi_\mu^M: \mathfrak{U}(\mathfrak{m})^{M \cap H} \to \mathbf{C}$).

3. The principal series.

In this section I look at the principal series for G/H, give a definition of the manifold B and the representations $\sigma_{P,\nu}$ on $\mathscr{D}(B)$ (adding together the principal series representations for all H-conjugacy classes of minimal $\sigma\theta$-parabolic subgroups of G). These representations suit well for handling the Poisson transform.

For a representation (ξ, V_ξ) of M on the Fréchet space V_ξ and $\nu \in \mathfrak{a}^*$ let $\pi_{P,\xi,\nu}$ be the smoothly induced representation of G given by lefttranslation on the space $I_P^\infty(\xi : \nu)$ of smooth functions $f: G \to V_\xi$ satisfying

$$(5) \qquad f(gman) = a^{-(\nu+\rho)}\xi(m)^{-1}f(g) \quad (g \in G, m \in M, a \in A, n \in N)$$

($I_P^\infty(\xi : \nu)$ is endowed with the usual Fréchet space topology.) Let $I_P(\xi : \nu)$ be the

$(\mathfrak{g}, K)$-module of K-finite vectors in $I_P^\infty(\xi : \nu)$. Further let $\mathscr{D}(K; V_\xi)$ be the space of smooth V_ξ-valued functions on K, and $\mathscr{D}(K : \xi)$ the subspace of $f \in \mathscr{D}(K; V_\xi)$ satisfying $f(km) = \xi(m)^{-1} f(k)$, for all $k \in K$, $m \in K \cap M$.

As noted in the introduction an $\eta \in V_\xi'^{M \cap H}$ determines an intertwining operator from V_ξ to $\mathscr{D}(M/M \cap H)$. I will handle the Poisson transform for subrepresentations of (the leftregular representation on) $\mathscr{D}(M/M \cap H)$. Recall that since P is minimal $\sigma\theta$-stable $\mathfrak{m} \cap \mathfrak{p} \subset \mathfrak{m} \cap \mathfrak{h}$ and thus $M/M \cap H$ is compact. Let $\delta_e \in \mathscr{D}'(M/M \cap H)$ be the Dirac distribution at $e \in M$.

Lemma 3.1. *Let (ξ, V_ξ) be a subrepresentation of the leftregular representation of $(M \cap K)$ on $\mathscr{D}(M \cap K/M^a)$. Then $T : \mathscr{D}(K : \xi) \to \mathscr{D}(K/M^a)$, given by $Tf(k) = \langle f(k), \delta_e \rangle$, $(f \in \mathscr{D}(K : \xi)$, $k \in K)$ is a K-isomorphism to a submodule of $\mathscr{D}(K/M^a)$. If ξ is equal to the leftregular representation on $\mathscr{D}(M/M \cap H)$ then T is an isomorphism.*

Proof. The evaluation by $\delta_e \in \mathscr{D}'(M/M \cap H)$ induces a continuous map $\mathscr{D}(K; V_\xi) \to \mathscr{D}(\mathrm{K})$. One easily sees that Tf is right-M^a-invariant, for all $f \in \mathscr{D}(K : \xi)$. Thus T is a continuous intertwining operator.

Suppose $Tf = 0$. Then $\langle f(k), \delta_e \rangle = 0$, for all $k \in K$. But then $f(k)(m) = \langle \xi(m^{-1}) f(k), \delta_e \rangle = \langle f(km), \delta_e \rangle = 0$ $(k \in K,\ m \in M \cap K)$. Since $M = (M \cap K)(M \cap H)$ we get $f(k)(m) = 0$ for all $k \in K$ and $m \in M$. Thus $f = 0$, proving the injectivity of T.

Now suppose $\xi = l$ is equal to the leftregular representation on $\mathscr{D}(M/M \cap H)$ and let $g \in \mathscr{D}(K/M^a)$. Then the map $f : K \to \mathscr{D}(M \cap K/M^a)$, $k \mapsto (l(k^{-1})g)|_{M \cap K}$ is smooth, and satisfies $Tf = g$, proving the surjectivity of T. By the open mapping theorem T is open and thus an isomorphism.

If ξ is any subrepresentation of the leftregular representation l on $\mathscr{D}(M/M \cap H)$, then $\mathscr{D}(K : \xi)$ can be identified with the submodule of $f \in \mathscr{D}(K : l)$ satisfying $f(k) \in V_\xi$ for all $k \in K$. Thus $T : \mathscr{D}(K : \xi) \to \mathscr{D}(K/M^a)$ is a K-isomorphism to a submodule of $\mathscr{D}(K/M^a)$.

Let $\pi_{P,\nu}$ be the representation of G given by lefttranslation on the space $I_P^\infty(\nu)$ of smooth functions $f : G \to \mathbf{C}$ satisfying

$$f(gman) = a^{-(\nu+\rho)} f(g) \quad (g \in G,\ m \in M \cap H,\ a \in A,\ n \in N) \tag{6}$$

$(\pi_{P,\nu}, I_P^\infty(\nu))$ is just the representation of G induced from the one-dimensional character $man \mapsto a^\nu$ of $(M \cap H)AN$. Let $I_P(\nu)$ be the $(\mathfrak{g}, K)$-module of K-finite vectors in $I_P^\infty(\nu)$. Let (ξ, V_ξ) be a subrepresentation of $\mathscr{D}(M/M \cap H)$ and let

$$T_P : \mathrm{I}_P^\infty(\xi : \nu) \to I_P^\infty(\nu), \tag{7}$$

be given by $T_P f(g) = \langle f(g), \delta_e \rangle (f \in I_P^\infty(\xi : \nu),\ g \in G)$. Then T_P is a G-monomorphism. It will be more convenient to look at $\pi_{P,\nu}$ than the representations $\pi_{P,\xi,\nu}$.

Let $P_m = {}^{\circ}M_m A_m N_m$ be the Langlands decomposition of a minimal parabolic subgroup of G contained in P. Let $\kappa: G \to K$ be the projection onto the K-component in the Iwasawa-decomposition $G = KA_m N_m$, and let $H_P: G \to \mathfrak{a}_0$ and $m_P: G \to M$ be the analytic maps determined by $x = \kappa(x) m_P(x) \exp(H_P(x)) n$, for some $n \in N$ $(x \in G)$. Define $\kappa_P: G \to K/M^a$ by $\kappa_P(x) = \pi_1(\kappa(x)$, $\pi_1: K \to K/M^a$ being the canonical projection, and let $\bar{H}_P: G \to \mathfrak{a}_0 \cap \mathfrak{q}_0$ be given by $\bar{H}_P(x) = \pi_2(H_P(x))$ $(x \in G)$, $\pi_2: \mathfrak{a}_0 \to \mathfrak{a}_0 \cap \mathfrak{q}_0$ being the projection along $\mathfrak{a}_0 \cap \mathfrak{h}_0$. Let $L = MA$ be the Levy component of P. The map $G \to K/M^a \times A/A \cap H$, $x \mapsto (\kappa_P(x), \exp(\bar{H}_P(x))$ induces an isomorphism $G/(L \cap H)N \cong K/M^a \times A/A \cap H$. $\kappa_P(x)$ and $\bar{H}_P(x)$ $(x \in G)$ are uniquely determined by $x \in \kappa_P(x)(L \cap H)\exp(\bar{H}_P(x))N$.

For $\nu \in (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*$ and $f \in \mathscr{D}(K/M^a)$ put

$$(\pi_{P,\nu}(g)f)(k) = e^{-\langle \nu+\rho, H_P(g^{-1}k)\rangle} f(\kappa_P(g^{-1}k)), \quad g \in G,\ k \in K$$

Since H_P is right-M-invariant and κ_P is right-M^a-invariant, $\pi_{P,\nu}(g)f \in \mathscr{D}(K/M^a)$ $(g \in G)$. Note that $\langle \nu + \rho, H_P(x)\rangle = \langle \nu + \rho, \bar{H}_P(x)\rangle$ $(x \in G)$. Again use T_P for the composition of the isomorphism in (7) and restriction to K. The map is given by $T_P f(k) = \langle f(k), \delta_e\rangle$, $k \in K$ $(f \in I_P^\infty(\xi : \nu))$. We now immediately get

LEMMA 3.2. *$\pi_{P,\nu}$ is a (continuous) representation of G on $\mathscr{D}(K/M^a)$. The restriction map $I_P^\infty(\nu) \to \mathscr{D}(K/M^a)$, $f \mapsto f \mid K$, is an isomorphism intertwining the leftregular representation on $I_P^\infty(\nu)$ and $\pi_{P,\nu}$ on $\mathscr{D}(K/M^a)$.*

This justifies the use of $\pi_{P,\nu}$ for the representation of G on $I_P^\infty(\nu)$ and $\mathscr{D}(K/M^a)$. For $f \in \mathscr{D}(K/M^a)$ let f_ν denote the element of $I_P^\infty(\nu)$ satisfying $f_\nu \mid K = f$.

I now want to construct the representation $\sigma_{P,\nu}$ on $\mathscr{D}(B)$. For this I need some preparations. Let $W = W(\mathfrak{g}, \mathfrak{a}^a)$ be the Weyl group of the (restricted) root system $\Sigma(\mathfrak{g}, \mathfrak{a}^a)$. Let $N_K(\mathfrak{a}^a)$ (resp. $Z_K(\mathfrak{a}^a)$) be the normalizer (resp. centralizer) of $\mathfrak{a}^a$ in K. The adjoint map $\mathrm{Ad}: N_K(\mathfrak{a}^a) \to \mathrm{End}(\mathfrak{a}^s)$, $k \mapsto \mathrm{Ad}(k) \mid \mathfrak{a}^a$ (End(.) being the space of endomorphisms) maps $N_K(\mathfrak{a}^a)/Z_K(\mathfrak{a}^a)$ isomorphically onto W (see [2] Lemma 1.2). Let $W_{K \cap H}$ be the image of $N_{K\cap H}(\mathfrak{a}^a)$ in W. Recall that $M^a = M \cap K \cap H$. By [2] Lemma 2.2

$$M^a = Z_{K \cap H}(\mathfrak{a}^a) \tag{8}$$

Thus $N_{K \cap H}(\mathfrak{a}^a)$ normalizes M^a and $W_{K\cap H}$ operates on K/M^a by multiplication from the right. We can thus define

$$B = K/M^a \times_{W_{K\cap H}} W. \tag{9}$$

For $w \in W$ let P^w be the parabolic $\bar{w} P \bar{w}^{-1}$, with $\bar{w} \in N_K(\mathfrak{a}^a)$ a representative of w. By [2] Lemma 2.2 again $L = MA$ is the centralizer of $\mathfrak{a}^a$ in G. Therefore $N_K(\mathfrak{a}^a)$ normalizes L, M and A (in particular $N_K(\mathfrak{a}^a) \subset N_K(\mathfrak{a})$). Let $\kappa_w: G \to K/M^a$ and $H_w: G \to \mathfrak{a}_0 \cap \mathfrak{q}_0$ be defined by

(10) $$\kappa_w = \kappa_{P^w},$$

(11) $$H_w = \bar{H}_{P^w}.$$

LEMMA 3.3. *Let $u \in N_{K\cap H}(\mathfrak{a}^a)$ and $w \in W$. Then*

a) $\kappa_{uw}(x) = \kappa_w(xu)u^{-1}$, $x \in G$,

b) $H_{uw}(x) = \mathrm{Ad}(u)H_w(xu)$, $x \in G$.

PROOF. For $x \in G$ we have

$$x \in \kappa_w(xu)u^{-1}(u(L \cap H)u^{-1})\exp(\mathrm{Ad}(u)H_w(xu))(uN^w u^{-1}).$$

Since $u \in N_{K\cap H}(\mathfrak{a}^a)$, u normalizes $L \cap H$, and we get

$$x \in \kappa_w(xu)u^{-1}(L \cap H)\exp(\mathrm{Ad}(u)H_w(xu))N^{uw}.$$

Since $\kappa_w(xu)u^{-1} \in K/M^a$ and $\mathrm{Ad}(u)H_w(xu) \in \mathfrak{a}_0 \cap \mathfrak{q}_0$ the lemma follows.

Let R denote the representation of $W_{K\cap H}$ on $\mathscr{D}(K/M^a)$ given by $(R(u)f)(k) = f(ku)(f \in \mathscr{D}(K/M^a), k \in K/M^a, u \in W_{K\cap H})$. $\mathscr{D}(B)$ can be identified with the space of smooth functions $F: K \times W \to \mathbf{C}$ satisfying

(12) $$F(km, w) = F(k, w),\ k \in K,\ m \in M^a,\ w \in W,$$

(13) $$F(k, uw) = (R(u) \otimes 1)F(k, w),\ k \in K,\ u \in W_{K\cap H},\ w \in W.$$

For $F \in \mathscr{D}(B)$ and $w \in W$ let $F_w \in \mathscr{D}(K/M^a)$ be defined by $F_w(k) = F(k, w)$, $k \in K$. For $g \in G$ and $F \in \mathscr{D}(B)$ let $\sigma_{P,\nu}(g)F : K \times W \to \mathbf{C}$ be defined by

(14) $$(\sigma_{P,\nu}(g)F)(k, w) = (\pi_{P^w, w\nu}(g)F_w)(k)$$

(15) $$= e^{-\langle w(\nu+\rho), H_w(g^{-1}k)\rangle}F(\kappa_w(g^{-1}k), w),$$

for $k \in K$, $w \in W$.

LEMMA 3.4. *Let $\nu \in (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*$ and put $r = |W_{K\cap H}\backslash W|$. Then*

a) *$\sigma_{P,\nu}(g)F \in \mathscr{D}(B)$, for all $F \in \mathscr{D}(B)$ and $g \in G$. $\sigma_{P,\nu}$ is a (continuous) representation of G.*

b) *Let $w_1, \ldots, w_r \in W$ be representatives of the equivalence classes $W_{K\cap H}\backslash W$. Then the map $\mathscr{D}(B) \to \bigoplus_{i=1} I^{\infty}_{P^{w_i}}(w_i\nu)$, $F \mapsto (F^{P^{w_i}}_{w_i, w_1\nu}, i = 1, \ldots, r)$ is a G-isomorphism.*

PROOF. (a) Note that κ_w and H_w are right-M^a-invariant. But then (15) implies that $\sigma_{P,\nu}(g)F$ satiesfies (12). Let $u \in N_{K\cap \mathrm{H}}(\mathfrak{a}^a)$ and $w \in W$. Then (using Lemma 3.3).

$$\begin{aligned}(\sigma_{P,\nu}(g)F)(k, uw) &= e^{-\langle uw(\nu+\rho), H_{uw}(g^{-1}k)\rangle}F(\kappa_{uw}(g^{-1}k), uw)\\ &= e^{-\langle uw(\nu+\rho), \mathrm{Ad}(u)H_w(g^{-1}ku)\rangle}F(\kappa_w(g^{-1}ku)u^{-1}, uw)\\ &= e^{-\langle w(\nu+\rho), H_w(g^{-1}ku)\rangle}F(\kappa_w(g^{-1}ku), w)\\ &= (\sigma_{P,\nu}(g)F)(ku, w)\end{aligned}$$

Thus (13) is satisfied and $\sigma_{P,\nu}(g)F \in \mathscr{D}(B)$. That $\sigma_{P,\nu}$ is a continuous representation of G follows from part (b) wich is easily verified.

4. Intertwining operators.

In this section I recall the definition of standard intertwining operators for induced representations and their relationship with asymptotics of coefficient function for the these representations. The statements are formulated with applications to the Poisson transform in mind.

If (ξ, V_ξ) is a representation of M, then let $\mathscr{C}(P:\xi:\nu)$ denote the space of continuous functions $f: G \to V_\xi$ satisfying (5). Let $d\bar{n}$ be a Haarmeasure on $\bar{N}$ normalized by $\int_{\bar{N}} e^{-\langle 2\rho, H_P(\bar{n})\rangle} d\bar{n} = 1$. Lemma 4.2 below is formulated to be used for the leftregular representation of M on $\mathscr{D}(M/M \cap H)$. Thus we need to know that $\xi(M)$ is equicontinuous. This is proved in the following Lemma

LEMMA 4.1. *Let (ξ, V_ξ) be a continuous representation of M. Suppose there is some $\phi \in V_\xi'^{M \cap H}$ such that $\{v \in V_\xi \mid \phi(\xi(m)v) = 0$ for all $m \in M\} = \{0\}$. Then $\xi(M)$ is equicontinuous.*

PROOF. Let $\mathfrak{m}_n$ be the subalgebra of $\mathfrak{m}$ generated by $\mathfrak{m} \cap \mathfrak{p}$. Since M is reductive, $\mathfrak{m}_n$ is an $\mathrm{Ad}(M)$-stable ideal in $\mathfrak{m}$. Let M_n be the corresponding analytic subgroup of M. M_n is normal in M and, since $\mathfrak{m} \cap \mathfrak{p} \subset \mathfrak{m} \cap \mathfrak{h}$, M_n is contained in $M \cap H$ and $M = (M \cap K)M_n$. Since ϕ is M_n-invariant we get $\phi(\xi(m)(\xi(m_n)v - v)) = 0$, for all $m \in M$, $m_n \in M_n$, $v \in V_\xi$. Thus $\xi \mid M_n$ is trivial and therefore $\xi(M) = \xi(M \cap K)$. But this set is equicontinuous, since ξ is continuous and $M \cap K$ is compact.

Let $\mathfrak{a}_+^*$ be the set of $\nu \in \mathfrak{a}^*$ satisfying $\mathrm{Re}\langle \nu, \alpha\rangle \geqq 0$ for all $\alpha \in \Sigma(\mathfrak{n}, \mathfrak{a})$.

LEMMA 4.2. *Let (ξ, V_ξ) be a (continuous) representation of M, such that $\xi(M)$ is equicontinuous, and let $\nu \in \rho + \mathfrak{a}_+^*$. For $f \in \mathscr{C}(P:\xi:\nu)$ the integral*

$$(J_{\bar{P}|P}(\xi:\nu)f)(x) = \int_{\bar{N}} f(x\bar{n})\, d\bar{n}, \quad x \in G \tag{16}$$

converges absolutely and uniformly (for x) on compact subsets of G. It defines a nontrivial (continuous) intertwining operator $J_{\bar{P}|P}(\xi:\nu): \mathscr{C}(P:\xi:\nu) \to \mathscr{C}(\bar{P}:\xi:\nu)$. Let ξ' be the operation of M on V_ξ' and let $g: G \to V_\xi'$ satisfy (5). Suppose there is some $U \subset G$, such that $e \in U$ and such that $g \mid U$ is continuous and bounded and $g \mid G \backslash U = 0$. Then

$$\lim_{a_P \to \infty} a^{\rho-\nu}\langle \pi_{P,\xi,\nu}(a)f, g\rangle = \langle (J_{\bar{P}|P}(\xi:\nu)f)(e), g(e)\rangle \tag{17}$$

($a_P \to \infty$ meaning $a^\alpha \to \infty$ for all $\alpha \in \Sigma(\mathfrak{n}, \mathfrak{a})$).

PROOF. Let $f \in \mathscr{C}(P:\xi:\nu)$ and let γ be a continuous seminorm on V_ξ. Then

$$\|f(x\bar{n})\|_\gamma = e^{-\mathrm{Re}\langle \nu+\rho, H_P(\bar{n})\rangle} \|\xi(m_P(\bar{n})^{-1})f(x\kappa_P(\bar{n}))\|_\gamma \tag{18}$$

Since $\xi(M)$ is equicontinuous there is some constant $C > 0$ such that $\|f(x\bar{n})\|_\gamma \leqq Ce^{-\mathrm{Re}\langle \nu+\rho, H_P(\bar{n})\rangle}$. C depends on γ, f and x, but is bounded for x in a compact subset of G. Now $\nu \in \rho + \mathfrak{a}_+^*$ implies $e^{-\mathrm{Re}\langle \nu-\rho, H_P(\bar{n})\rangle} \leqq 1$ (see [8] Corollary IV.6.6) and thus $\|f(x\bar{n})\|_\gamma \leqq Ce^{-\langle 2\rho, H_P(\bar{n})\rangle}$. Thus $\int_{\bar{N}} \|f(x\bar{n})\|_\gamma d\bar{n} \leqq C\int_{\bar{N}} e^{-\langle 2\rho, H_P(\bar{n})\rangle} d\bar{n} = C$, proving the absolute convergence (uniformly for x in any compact subset of G).

The rest of the proof is standard (compare [18] Lemma 1.2 and Lemma 5.1 or [3] Lemma 15.6).

I now want to extend the standard intertwining operators to $\mathscr{D}(B)$. For $F \in \mathscr{D}(B)$ let $J^W_{\bar{P}|P}(\nu)F : \mathrm{K} \times \mathrm{W} \to \mathrm{C}$ be defined by $(J^W_{\bar{P}|P}(\nu)F)(k, w) = J_{\bar{P}^w|P^w}(w\nu)F_w)(k)$, for $k \in K$, $w \in W$. Let $(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*_+$ be the set of $\nu \in (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*$ satisfying $\mathrm{Re}\langle \nu, \alpha\rangle \geqq 0$ for all $\alpha \in \Sigma(\mathfrak{n}, \mathfrak{a})$.

LEMMA 4.3. *Let $\nu \in \rho + (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*_+$ and $F \in \mathscr{D}(B)$, Then $J^W_{\bar{P}|P}(\nu)F \in \mathscr{D}(B)$ and $J^W_{\bar{P}|P}(\nu) : \mathscr{D}(B) \to \mathscr{D}(B)$ intertwines $\sigma_{P,\nu}$ and $\sigma_{\bar{P},\nu}$.*

PROOF. For $u \in N_{K\cap H}(\mathfrak{a}^a)$ and $w \in W$ we have $F_{uw} - R(u)F_w$. Extend F_w to $F^{P^w}_{w,w\nu} \in I_{P^w}(w\nu)$. Let $m \in M \cap H$, $a \in A$ and $n \in N^{uw}$. Then

$$\begin{aligned}(R(u)F^{P^w}_{w,w\nu})(kman) &= F^{P^w}_{w,w\nu}(kmanu)\\ &= a^{-uw(\nu+\rho)}F^{P^w}_{w,w\nu}(ku)\\ &= a^{-uw(\nu+\rho)}F^{P^{uw}}_{uw,uw\nu}(k)\\ &= F^{P^{uw}}_{uw,uw\nu}(kman), \quad k \in K\end{aligned}$$

i.e. $R(u)F^{P^w}_{w,w\nu} = F^{P^{uw}}_{uw,uw\nu}$. For simplicity now write F_w instead of $F^{P^w}_{w,w\nu}$. We now become (letting $d\bar{n}$ denote the normalized Haarmeasure on the various $\bar{N}^w$'s)

$$\begin{aligned}(J^W_{\bar{P}|P}(\nu)F)(k, uw) &= (J_{\bar{P}^{uw}|P^{uw}}(uw\nu)F_{uw})(k) = \int_{\bar{N}^{uw}} F_{uw}(k\bar{n})\,d\bar{n}\\ &= \int_{\bar{N}^w} F_{uw}(ku\bar{n}u^{-1})\,d\bar{n} \quad = \int_{\bar{N}^w} F_w(ku\bar{n})\,d\bar{n}\\ &= (J_{\bar{P}^w|P^w}(w\nu)F_w)(ku) \quad = (J^W_{\bar{P}|P}(\nu)F)(ku, w).\end{aligned}$$

Thus $J^W_{\bar{P}|P}(\nu)F \in \mathscr{D}(B)$. The intertwining property is clear from the definition of $\sigma_{P,\nu}$ and the intertwining property of $J_{\bar{P}^w|P^w}(w\nu)$.

5. The Poisson transform.

In this section I give a definition of the Poisson kernel associated to P and give some properties of the associated Poisson transform. Then the Poisson transform $\mathcal{P}_\nu : \mathcal{D}(B) \to \mathcal{E}(G/H)$ will be defined and a weak injectivity result proved.

For $\nu \in (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*$ let $p_\nu : G \to \mathcal{D}'(M/M \cap H)$ (the space of distributions on $M/M \cap H$) be the *Poisson kernel*, defined by

$$(19) \qquad p_\nu(x) = \begin{cases} a^{-(\nu+\rho)}\delta_{m^{-1}} & \text{for } x = hman,\ h \in H,\ m \in M,\ a \in A,\ n \in N \\ 0 & \text{otherwise} \end{cases}$$

(with $\langle \delta_m, f \rangle = f(m)$, for $f \in \mathcal{D}(M/M \cap H)$, $m \in M$.)

LEMMA 5.1. *p_ν is well defined and satisfies $p_\nu(hman) = a^{-(\nu+\rho)}\xi'(m)^{-1}p_\nu(x)$, for all $x \in G$, $m \in M$, $a \in A$, $n \in N$. (ξ' denoting the leftregular representation of M on $\mathcal{D}'(M/M \cap H)$). If $\nu \in -(\rho + (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*_+)$, then p_ν is (strongly) bounded.*

PROOF. p_ν is well defined and smooth on HP. The boundedness statement follows from the following lemma.

LEMMA 5.2. *Let (ξ, V_ξ) be a (continuous) representation of M and $\nu \in -(\rho + (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*_+)$. Suppose $p : G \to V_\xi$ is a function satisfying*

a) $p(hxman) = a^{-(\nu+\rho)}\xi(m)^{-1}p(x)$, for $x \in G$, $h \in H$, $m \in M$, $a \in A$, $n \in N$.

b) $p|_{G \setminus HP} = 0$.

Then p is bounded.

This follows from the proof in [2] Proposition 5.6 and the remark after that proof. Under more restrictive hypothesis it is proved in [11] Lemma 4.1. The decomposition $HP = H(M \cap K)\exp(\mathfrak{a}_0 \cap \mathfrak{q}_0)N$ gives smooth maps

$$\mu_P : HP \to M^a \backslash M \cap K,$$

$$a_P : HP \to \exp(\mathfrak{a}_0 \cap \mathfrak{q}_0).$$

For $x \in HP$, $\mu_P(x)$ and $a_P(x)$ are uniquely determined by $x \in H\mu_P(x)a_P(x)N$. Let $d\dot{h}$ be an H-invariant measure on $H/H \cap P$ ($H \cap P$ is equal to $H \cap L$) normalized such that

$$(20) \qquad \int_{K/K \cap P} f(k)\,dk = \int_{H/H \cap P} f(\kappa(h))e^{-\langle 2\rho, H_P(h)\rangle}\,d\dot{h}$$

for every $f \in \mathscr{C}(G/P)$ (the space of continuous functions on G/P) with support in HP/P (see [11] Lemma 1.3). The following proposition gives a version of the Poisson transform needed here. It is a modified version of the one given in [11] §6 (we cannot use [11] directly except when $\mathcal{D}(M/M \cap H)$ is finite dimensional).

PROPOSITION 5.3. *Let $\nu \in \rho + (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*_+$ and $f \in I_P^\infty(\nu)$. Then the integral*

$$\mathscr{P}f(x) = \int_{H/H\cap P} f(xh)\,d\dot{h} \tag{21}$$

converges absolutely and uniformly (for x) on compact subsets of G and defines a continuous intertwining operator $\mathscr{P}: I_P^\infty(\nu) \to \mathscr{E}(G/H)$. *Let* ξ *denote the representation M on* $\mathscr{D}(M/M \cap H)$ *from the left. Let* $g \in I_P^\infty(\xi : \nu)$ *satisfy* $T_P g = f$. *Then*

$$\mathscr{P}f(x) = \int_K \langle g(k), p_{-\nu}(x^{-1}k)\rangle\, dk \tag{22}$$

$$= \int_{K\cap xHP} f(k\mu_P(x^{-1}k)^{-1})a_P(x^{-1}k)^{\nu-\rho}\, dk \tag{23}$$

for all $x \in G$.

PROOF. For $f \in I_P(\nu)$ we have $f(xh) = f(x\kappa(h))e^{-\langle \nu+\rho, H_P(h)\rangle}$, $h \in H$. For $h \in H$ and x in some compact subset of $G\,|f(x\kappa(h))|$ remaines bounded. By (20) it is enough to prove that $e^{-\mathrm{Re}\langle \nu-\rho, H_P(h)\rangle}$ is bounded on H. By [1] Theorem 1.1 $H_P(h) = \Sigma x_\alpha H_\alpha$, with $x_\alpha \geqq 0$, the sum being taken over simple roots for $\Sigma(\mathfrak{n}^-, \mathfrak{a}^a)$ (with $\mathfrak{n}^- = \{X \in \mathfrak{n} \mid \sigma\theta(X) = -X\}$) and $H_\alpha \in \mathfrak{a}^a$ defined by $\alpha(X) = \langle X, H_\alpha\rangle$, $X \in \mathfrak{a}^n$. But then $e^{-\mathrm{Re}\langle \nu-\rho, H_P(h)\rangle}$ is bounded since $\nu \in \rho + (\mathfrak{a}/\mathfrak{a}\cap\mathfrak{h})^*_+$. The intertwining property is now obvious. The equation (22) follows from (20) and (23) from the definition of p_ν.

Recall the definition of γ_ν^P from (4). The following theorem is proved with a similar argument as [11] Theorem 4.3.

THEOREM 5.4. *Let* $\nu \in -(\rho + (\mathfrak{a}/\mathfrak{a}\cap\mathfrak{h})^*_+)$. *Then*

$$D\mathscr{P}_\nu f = \mathscr{P}_\nu(R(\gamma_\nu^P(D))f),$$

for all $D \in \mathrm{D}(G/H)$ *and* $f \in \mathscr{D}(K/M^a)$.

PROOF. Choose $u \in \mathfrak{U}(\mathfrak{g})^H$ satisfying $\Gamma(u) = D$ (see (1)). Then $u = \tilde{\gamma}^P(u) + v$, with $v \in (\mathfrak{n}\mathfrak{U}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g})\mathfrak{r})$ (see (3)). Let $\mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$, $u \mapsto u^t$ be the anti-automorphism, with $Xt = -X$ for $X \in \mathfrak{g}$, and ξ' denote the leftregular representation of M on $\mathscr{D}'(M/M\cap H)$. Then (since p_ν is smooth on HP)

$$\begin{aligned}(L(u)p_\nu)(hman) &= \xi'(m^{-1})a^{-(\nu+\rho)}(R(u^t)p_\nu)(e)\\ &= \xi'(m^{-1})a^{-(\nu+\rho)}\xi'(\gamma_\nu^P(u))p_\nu(e)\\ &= \xi'(\mathrm{Ad}(m^{-1})\gamma_\nu^P(u))p_\nu(hman),\end{aligned}$$

for $h \in H, m \in M, a \in A, n \in N$. For $f \in \mathscr{D}(K/M^a)$ let $\tilde{f} \in \mathscr{D}(K:\xi)$ satisfy $T\tilde{f} = f$. We then get

$$(R(u)\mathscr{P}_\nu f)(x) = \int_{K\cap xHP} \langle \tilde{f}(k), (L(u)p_\nu)(x^{-1}k)\rangle \, dk$$

$$= \int_{K\cap xHP} \langle \xi(\mathrm{Ad}(\mu_P(x^{-1}k)^{-1})\gamma_\nu^P(u)^t)\tilde{f}(k), p_\nu(x^{-1}k)\rangle \, dk$$

$$= \int_{K\cap xHP} (\xi(\mathrm{Ad}(\mu_P(x^{-1}k)^{-1})\gamma_\nu^P(u)^t)f(k))(\mu_P(x^{-1}k)^{-1})a_P(x^{-1}k)^{-(\nu+\rho)} \, dk$$

$$= \int_{K\cap xHP} \langle (R(\gamma_\nu^P(u))f)^{\sim}(k), p_\nu(x^{-1}k)\rangle \, dk$$

$$= \mathscr{P}_\nu(R(\gamma_\nu^P(u)f)(x), \quad x \in G/H$$

proving the theorem.

COROLLARY 5.5. *Suppose G/H is split (i.e. $\mathfrak{a}^a = \mathfrak{a}^d$) and let $\lambda \in -(\rho + (\mathfrak{a}/\mathfrak{a}\cap\mathfrak{h})^*_+)$. Then $\mathscr{P}_\lambda\mathscr{D}(K/M^a) \subset \mathscr{E}_\lambda(G/H)$.*

We now turn to the definition of the Poisson transform for $\mathscr{D}(B)$.

LEMMA 5.6. *Let $u \in W_{K\cap H}$ and $\nu \in -(\rho + (\mathfrak{a}/\mathfrak{a}\cap\mathfrak{h})^*_+)$. Then*

$$\mathscr{P}_{u\nu}^{P^w} f(x) = \mathscr{P}_\nu^P(R(u^{-1})f)(x), \quad \text{for} \quad x \in G/H,$$

for all $f \in \mathscr{D}(K/M^a)$.

PROOF. For $u \in N_{K\cap H}(\mathfrak{a}^a)$ and $x \in HP^u$ we get

$$x \in Hu^{-1}(u\mu_P(xu)u^{-1})(ua_P(xu)u^{-1})uNu^{-1}.$$

Since u normalizes M^a and $\mathfrak{a}_0 \cap \mathfrak{q}_0$ this implies $\mu_{P^u}(x) = u\mu_P(xu)u^{-1}$ and $a_{P^u}(x) = ua_P(xu)u^{-1}$. Thus we get (using (23))

$$\mathscr{P}_{u\nu}^{P^w} f(x) = \int_{K\cap xHuPu^{-1}} f(k\mu_{P^u}(x^{-1}k)^{-1})a_{P^u}(x^{-1}k)^{-u(\nu+\rho)} \, dk$$

$$= \int_{K\cap xHPu^{-1}} f(ku\mu_P(x^{-1}ku)^{-1}u^{-1})a_P(x^{-1}ku)^{-(\nu+\rho)} \, dk$$

$$= \int_{K\cap xHP} f(k\mu_P(x^{-1}k)^{-1}u^{-1})a_P(x^{-1}k)^{-(\nu+\rho)} \, dk$$

$$= \mathscr{P}_\nu^P(R(u^{-1})f)(x).$$

$W_{K\cap H}$ operates on $\mathbf{C}[W]$ (the group algebra of $W = W(\mathfrak{g}, \mathfrak{a}^a)$) from the left. Thus $\mathscr{D}(K/M^a)\otimes_{W_{K\cap H}}\mathbf{C}[W]$ is a well defined K-module (K operating on $\mathscr{D}K/M^a$) from the left) that can be identified with the $\mathscr{D}(B)$.

Let $\nu\in -(\rho + (\mathfrak{a}/\mathfrak{a}\cap\mathfrak{h})^*_+)$. For $f\in\mathscr{D}(K/M^a)$ and $T\in\mathbf{C}[W]$ put $\mathscr{P}_\nu(f\otimes T) = \Sigma_{w\in W}\, T(w)\mathscr{P}^{P^w}_{w\nu}f$. Using Lemma 5.6 we easily get $\mathscr{P}_\nu(f\otimes l(u)T) = \mathscr{P}_\nu(R(u)f\otimes T)$, for $u\in W_{K\cap H}$ ($l(u)$ denoting the leftregular representation of W on $\mathbf{C}[W]$). Thus we get a well defined map from $\mathscr{D}(K/M^a)\otimes_{W_{K\cap H}}\mathbf{C}[W]$ to $\mathscr{E}(G/H)$ giving the *Poisson transform*:

$$\mathscr{P}_\nu:\mathscr{D}(B)\to\mathscr{E}(G/H). \tag{24}$$

Using the description of $\mathscr{D}(B)$ by (12) and (13) and putting $t = |W_{K\cap H}|$ we get $\mathscr{P}_\nu F = t^{-1}\Sigma_{w\in W}\mathscr{P}^{P^w}_{w\nu}F_w$, for $F\in\mathscr{D}(B)$.

I now want to prove a weak injectivity result for the Poisson transform. This will be done by relating boundary values with standard intertwining operators.

THEOREM 5.7. *Let $F\in\mathscr{D}(B)$ and $\nu\in\rho + (\mathfrak{a}/\alpha\cap\mathfrak{h})^*_+$. Then*

$$\lim_{a_{P^w}\to\infty} a^{w(\rho-\nu)}\mathscr{P}_{-\nu}F(ka^{-1}) = (J^W_{\bar{P}|P}(\nu)F)(k, w),$$

for $k\in K$, $w\in W$.

PROOF. Let $f\in I^\infty_P(\nu)$. Let ξ denote the leftregular representation of M on $\mathscr{D}(M/M\cap H)$. Choose $g\in I^\infty_P(\xi:\nu)$ with $f = T_P g$. Then by Lemma 4.2

$$\begin{aligned}
\lim_{a_{P^w}\to\infty} a^{w(\rho-\nu)}\mathscr{P}^P_{-\nu}F(ka^{-1}) &= \lim_{a_P\to\infty}(waw^{-1})^{w(\rho-\nu)}\mathscr{P}^P_{-\nu}f(kwa^{-1}w^{-1})\\
&= \lim_{a_P\to\infty} a^{\rho-\nu}\langle g, \pi_{P,\xi',-\nu}(kwa^{-1}w^{-1})p_{-\nu}\rangle\\
&= \lim_{a_P\to\infty} a^{\rho-\nu}\langle \pi_{P,\xi,\nu}(a)\pi_{P,\xi',\nu}(w^{-1}k^{-1})g, \pi_{P,\xi,-\nu}(w^{-1})p_{-\nu}\rangle\\
&= \langle (J_{\bar{P}|P}(\xi:\nu)g)(kw), p_{-\nu}(w)\rangle.
\end{aligned}$$

But the last term is 0 if $w\notin W_{K\cap H}$. If $w\in W_{K\cap H}$ it is equal to (using Lemma 4.2)

$$\begin{aligned}
\langle (J_{\bar{P}|P}(\xi:\nu)g)(kw), \delta_e\rangle &= (T_{\bar{P}}J_{\bar{P}|P}(\xi:\nu)g)(kw)\\
&= (R(w)J_{\bar{P}|P}(\nu)f)(k).
\end{aligned}$$

Now let $F\in\mathscr{D}(B)$. With $t = |W_{K\cap H}|$ we become

$$\begin{aligned}
\lim_{a_{P^w}\to\infty} a^{w(\rho-\nu)}\mathscr{P}^P_{-\nu}F(ka^{-1}) &= \lim_{a_{P^w}\to\infty} t^{-1}\sum_{u\in W} a^{w(\rho-\nu)}\mathscr{P}^{P_{uw}}_{-uw\nu}F_{uw}(ka^{-1})\\
&= t^{-1}\sum_{u\in W_{K\cap H}}\lim_{a_{P^w}\to\infty} a^{w(\rho-\nu)}\mathscr{P}^{P_{uw}}_{-uw\nu}F_{uw}(ka^{-1})\\
&= t^{-1}\sum_{u\in W_{K\cap H}}\lim_{a_{P^w}\to\infty} a^{w(\rho-\nu)}\mathscr{P}^{P^w}_{-w\nu}(R(u^{-1})F_{uw})(ka^{-1})\\
&= \lim_{a_{P^w}\to\infty} a^{w(\rho-\nu)}\mathscr{P}^{P^w}_{-u\nu}F_w(ka^{-1})\\
&= (J_{\bar P^w|P^w}(w\nu)F_w)(k).
\end{aligned}$$

By definition this is $(J^W_{\bar P|P}(\nu)F)(k, w)$.

COROLLARY 5.8. *Let $\delta\in\hat K$. Then there exists a non-zero holomorphic function q_δ on $\rho + (\mathfrak{a}/\mathfrak{a}\cap\mathfrak{h})^*_+$ such that if $\nu\in\rho + (\mathfrak{a}/\mathfrak{a}\cap\mathfrak{h})^*_+$ and $q_\delta(\nu)\neq 0$, then $\mathscr{P}_\nu\,|\,\mathscr{D}(B)_\delta$ is injective.*

PROOF. Let $F\in\mathscr{D}(B)_\delta$ be in the kernel of $\mathscr{P}_\nu$. Then $J_{\bar P^w|P^w}(w\nu)F_w$ is zero for all $w\in W$. It is therefore enough to construct holomorphic functions q^w_δ on $w\rho + w\mathfrak{a}^*_+$ such that $J_{\bar P^w|P^w}(w\nu)\,|\,I_{P^w}(w\nu)_\delta$ is injective if $q^w_\delta(w\nu)\neq 0$. We may assume $w = e$. By Frobenius reciprocity δ has finite multiplicity in $I_P(\nu)$. Thus $I_P(\nu)_\delta$ is contained in a finite sum of $I_P(\xi:\nu)$ (ξ some irreducible $(M\cap H)$-spherical representations of M). Therefore it is enough to prove the analogous statement for $I_P(\xi:\nu)_\delta)$. On the submodule $I_P(\xi:\nu)$ of $I_P(\nu)$. $J_{\bar P|P}(\nu)$ is given by $\mathrm{J}_{\bar P|P}(\xi:\nu)$. Let E_δ be the projection onto the δ-isotypic component $I_P(\xi:\nu)_\delta$ of $I_P(\xi:\nu)$. The restriction of $\mathrm{J}_{\bar P|P}(\xi:\nu)$ to $I_P(\xi:\nu)_\delta$ is injective if the determinant $q_{\xi,\delta}$ of $E_\delta J_{\bar P|P}(\xi:\nu)E_\delta\in\mathrm{End}(I_P(\xi:\nu)_\delta)$ is non-zero. This function is holomorphic on $\rho + (\mathfrak{a}/\mathfrak{a}\cap\mathfrak{h})^*_+$ (compare [18] Lemma 5..3). But then [2] Proposition 3.7 shows, using the meromorphic continuation of $J_{\bar P|P}(\xi:\nu)$, that $q_{\xi,\delta}$ is non-zero. This finishes the proof of the corollary.

6. The multiplicity formula.

I now turn to the proof of the multiplicity formula, starting with a lower bound using the meromorphicity of the Poisson transform. The proof that this also is an upper bound is done by looking at the Taylor series of K-spherical functions on G/H. This will only be done for G/H split. Recall that $r = |W_{K\cap H}\backslash W|$ is equal to the number of open H-orbits in G/P.

LEMMA 6.1. *Suppose G/H is split. Let $\lambda\in(\mathfrak{a}^a)^*$ and (δ, E) an irreducible representation of K. Then $m(\delta,\mathscr{E}_\lambda(G/H))\geqq|W_{K\cap H}\backslash W|\dim E'^{M^a}$.*

PROOF. $\mathfrak{a}^a = \mathfrak{a}\cap\mathfrak{q}$ can be identified with $\mathfrak{a}/\mathfrak{a}\cap\mathfrak{h}$. By Frobenius reciprocity δ has finite multiplicity in $\mathscr{D}(B)$. The Poisson transform, restricted to $\mathscr{D}(B)_\delta$ can be continued meromorphically to all of $(\mathfrak{a}^a)^*$. Since $\mathscr{D}(B)_\delta$ is contained in a finite sum of some $I_{P^w}(\xi, wv)$'s, this follows from the existence of a meromorphic continuation of $\mathscr{P}^{P^w}_{\xi, wv}$ to all of $(\mathfrak{a}^a)^*$ (proved in [11] Theorem 5.1 under certain restrictions on (G, H) and more generally in [2] Theorem 5.10). Put $F = \mathscr{D}(B)_\delta$. We now use the following lemma that can be proved in the same way as [12] Proposition 2.21

LEMMA 6.2. *Let V be a Fréchet space and F a finite dimensional vector space. Further let $U \subset \mathbf{C}$ be open and connected and let $f: U \to \mathrm{Hom}_{\mathbf{C}}(F, V)$ be meromorphic. Suppose there is some $v \in U$ such that f is holomorphic at v and $f(v)$ is injective. Let $\lambda \in U$. Then there is a meromorphic map $A: U \to \mathrm{End}(F)$ such that*

$$fA: U \to \mathrm{Hom}_{\mathbf{C}}(F, V),\ \mu \mapsto f(\mu) \circ A(\mu)$$

is holomorphic at λ and $(fA)(\lambda)$ is injective.

By Corollary 5.8 $\mathscr{P}_v \,|\, \mathscr{D}(B)_\delta$ is injective for "most" $v \in \rho + (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*_+$. By the previous lemma we immediately get that $m(\delta, \mathscr{E}_\lambda(G/H))$ is bounded by $m(\delta, \mathscr{D}(B))$ from below. As a K-module $\mathscr{D}(K/M^a)^r$. The statement of the lemma now follows from Frobenius reciprocity.

We now turn to the upper bound on the multiplicities. Let (δ, E) be an irreducible representation of K and let $(\delta^\vee, E')$ denote the dual representation. Further let $\mathscr{E}_\lambda(G/H)_\delta$ be the δ-isotypic subspace of $\mathscr{E}_\lambda(G/H)$ $(\lambda \in (\mathfrak{a}^d)^*)$ and $\mathscr{E}_\delta(G/H; E)$ the space of smooth maps $F: G/H \to E$ satisfying $F(kx) = \delta(k)F(x)$, for $x \in G/H, k \in K$. $\mathrm{D}(G/H)$ operates on $\mathscr{E}_\delta(G/H; E)$ from the right. Let $\mathscr{E}_{\lambda,\delta}(G/H; E)$ be the corresponding eigenspace $(\lambda \in (\mathfrak{a}^d)^*)$. Then $E \otimes \mathscr{E}_{\lambda,\delta^\vee}(G/H; E') \ni (v, F) \mapsto F_v \in \mathscr{E}_\lambda(G/H)_\delta$, with $F_v(x) = \langle v, F(x) \rangle$, $x \in G/H$, is an isomorphism. Thus the multiplicity of δ in $\mathscr{E}_\lambda(G/H))$ is equal to the dimension of $\mathscr{E}_{\lambda,\delta^\vee}(G/H; E')$.

We will get an upper bound on the dimension of this space by looking at certain terms in the Taylor expansions of $F \in \mathscr{E}_{\lambda,\delta^\vee}(G/H; E')$. Let

$$\pi: \mathfrak{S}(\mathfrak{q}) \to \mathfrak{S}(\mathfrak{p} \cap \mathfrak{q})$$

be the projection onto the second compenent in the decomposition $\mathfrak{S}(\mathfrak{q}) = (\mathfrak{k} \cap \mathfrak{q})\mathfrak{S}(\mathfrak{q}) \oplus \mathfrak{S}(\mathfrak{p} \cap \mathfrak{q})$. If $p \in \mathfrak{S}(\mathfrak{q})^H$ then $\pi(\mathrm{p}) \in \mathfrak{S}(\mathfrak{p} \cap \mathfrak{q})^{H \cap K}$.

Let $\mathscr{H}(\mathfrak{p} \cap \mathfrak{q})$ denote the space of harmonic elements in $\mathfrak{S}(\mathfrak{p} \cap \mathfrak{q})$ (with respect to $K \cap H$). The following lemma shows that in case H is *essentially connected*, i.e. $H = H_e Z_{K \cap H}(\mathfrak{a}^a)$, then $\mathscr{H}(\mathfrak{p} \cap \mathfrak{q})$ is just the space of harmonic elements with respect to $(K \cap H)_e$, allowing us to use the results of [9].

LEMMA 6.3. *Let H be essentially connected. Then $\mathfrak{S}(\mathfrak{p} \cap \mathfrak{q})^{K \cap H} = \mathfrak{S}(\mathfrak{p} \cap \mathfrak{q})^{\mathfrak{k} \cap \mathfrak{h}}$.*

PROOF. Since H is essentially connected $W_{K \cap H} = W_{(K \cap H)_e}$. By Chevalley the orthogonal projection from $\mathfrak{p} \cap \mathfrak{q}$ onto $\mathfrak{a}^a$ determines an isomorphism π_1 between $\mathfrak{S}(\mathfrak{p} \cap \mathfrak{q})^{\mathfrak{k} \cap \mathfrak{h}}$ and $\mathfrak{S}(\mathfrak{a}^a)^{W_{K \cap H}}$ (see [8] Corollary II.5.12). Let $w \in Z_{K \cap H}(\mathfrak{a}^a)$ and

$p \in \mathfrak{S}(\mathfrak{p} \cap \mathfrak{q})^{\mathfrak{k} \cap \mathfrak{h}}$. Then $\pi_1(\mathrm{Ad}(w)p) = \mathrm{Ad}(w)\pi_1(p) = \pi_1(p)$. Since π_1 is injective $\mathrm{Ad}(w)p = p$ for all $w \in Z_{K \cap H}(\mathfrak{a}^a)$. Thus $p \in \mathfrak{S}(\mathfrak{p} \cap \mathfrak{q})^{K \cap H}$ proving the lemma.

The following lemma will only be used for G/H split. But for future references I will state it here more generally. As before let $\mathfrak{a}^d$ be a maximally abelian subspce of $\mathfrak{q}$ containing $\mathfrak{a}^a$, $W^d = W(\mathfrak{g}, \mathfrak{a}^d)$ and $W = W(\mathfrak{g}, \mathfrak{a}^a)$. The Cartan involution θ defines an automorphism on W^d by $w^\theta(X) = \theta w(\theta(X))$, for $X \in \mathfrak{a}^d$. Put $W^{d\theta} = \{w \in W^d \mid w^\theta = w\}$ (this equal to $\{w \in W^d \mid w(\mathfrak{a}^d \cap \mathfrak{p}) = (\mathfrak{a}^d \cap \mathfrak{p})\}$) and let W_θ^d be the subgroup of $W^{d\theta}$ generated by reflections s_α with $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}^d)$ vanishing on $\mathfrak{a}^d \cap \mathfrak{p}$. Restriction to $\mathfrak{a}^d \cap \mathfrak{p}$ defines a map $W^d \to W$ giving an exact sequence (see [13] Lemma (7.2) (ii))

$$1 \to W_\theta^d \to W^{d\theta} \to W \to 1$$

In particular if $p \in \mathfrak{S}(\mathfrak{a}^d)^{W^d}$ then $\pi(p) \in \mathfrak{S}(\mathfrak{a}^a)^W$. Let

$$\tilde{\pi}: \mathfrak{S}(\mathfrak{a}^d)^{W^d} \to \mathfrak{S}(\mathfrak{a}^a)^W$$

denote the restriction of π to $\mathfrak{S}(\mathfrak{a}^d)^{W^d}$.

ASSUMPTION 6.4. *This assumption is said to be valid if $\tilde{\pi}$ is surjective.*

If G/H is split then $\mathfrak{a}^d = \mathfrak{a}^a$, so Assumption 6.4 is valid.

LEMMA 6.5. *Suppose Assumption 6.4 is valid. Put $r = |W_{K \cap H} \backslash W|$. There exist homogeneous elements $v_1 = 1, v_2, \ldots, v_r \in \mathfrak{S}(\mathfrak{p} \cap \mathfrak{q})^{K \cap H}$ such that*

$$\mathfrak{S}(\mathfrak{p} \cap \mathfrak{q}) = \sum_{i=1}^{r} v_i \pi(\mathfrak{S}(\mathfrak{q})^H) \mathscr{H}(\mathfrak{q} \cap \mathfrak{q}).$$

Thus every $p \in \mathfrak{S}(\mathfrak{p} \cap \mathfrak{q})$ has a decomposition into linear combination of elements of the form $v_i \pi(q) p$, with $q \in \mathfrak{S}(\mathfrak{q})^H$ and $p \in \mathscr{H}(\mathfrak{p} \cap \mathfrak{q})$.

PROOF. By [8] Theorem III.1.1 $\mathfrak{S}(\mathfrak{p} \cap \mathfrak{q}) = \mathfrak{S}(\mathfrak{p} \cap \mathfrak{q})^{K \cap H} \mathscr{H}(\mathfrak{p} \cap \mathfrak{q})$. By Chevalley the orthogonal projection of $\mathfrak{p} \cap \mathfrak{q}$ onto $\mathfrak{a}^a$ defines an isomorphism between $\mathfrak{S}(\mathfrak{p} \cap \mathfrak{q})^{K \cap H}$ and $\mathfrak{S}(\mathfrak{a}^a)^{W_{K \cap H}}$ (using $K \cap H = (K \cap H)_e N_{K \cap H}(\mathfrak{a}^a)$). In this proof let G_c be the adjoint group of $\mathfrak{g}$. σ (as an involution on $\mathfrak{g}$) defines an involution on G_c also denoted by σ. Using [9] Proposition 1 and Chevalley we get $\mathfrak{S}(q)^{G_c^\sigma} = \mathfrak{G}(\mathfrak{q})^H \cong \mathfrak{S}(\mathfrak{a}^d)^{W^d}$ (using the orthogonal projection of $\mathfrak{q}$ onto $\mathfrak{a}^d$).

By Assumption 6.4 $\pi(\mathfrak{S}(\mathfrak{q})^H)$ is the preimage of $\mathfrak{S}(\mathfrak{a}^a)^W$ under the isomorphism $\mathfrak{S}(\mathfrak{p} \cap \mathfrak{q})^{K \cap H} \cong \mathfrak{S}(\mathfrak{a}^a)^{W_{K \cap H}}$. By the first part of [7] Lemma 8 there exist r homogeneous elements $\bar{v}_1 = 1, \bar{v}_2, \ldots, \bar{v}_r \in \mathfrak{S}(\mathfrak{a}^a)^{W_{K \cap H}}$ such that $\mathfrak{S}(\mathfrak{a}^a)^{W_{K \cap H}} = \oplus_{i=1}^r \mathfrak{S}(\mathfrak{a}^a)^W$ (in [7] Lemma 8 the subgroup $W_{K \cap H}$ is also required to be generated by reflections. But this is not used to proof of the first part of that lemma). Now the statement follows letting $v_i \in \mathfrak{S}(\mathfrak{p} \cap \mathfrak{q})^{K \cap H}$ be the canonical elements mapping to $\bar{v}_i$ under the isomorphism $\mathfrak{S}(\mathfrak{p} \cap \mathfrak{q})^{K \cap H} \cong \mathfrak{S}(\mathfrak{a}^a)^{W_{K \cap H}}$.

Let $\mathfrak{S}_m(\mathfrak{g})$ (resp. $\mathfrak{S}^m(\mathfrak{g})$) and $\mathfrak{U}_m(\mathfrak{g})$ ($m \in \mathbf{Z}$) be the standard filtration (resp. gradation) of $\mathfrak{S}(\mathfrak{g})$ and $\mathfrak{U}(\mathfrak{g})$ by degree. Further let $\Lambda : \mathfrak{S}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$ be the symmetrization map.

THEOREM 6.6. *Suppose G/H is split and H is essentially connected. Let $\lambda \in (\mathfrak{a}^a)^*$ and (δ, E) an irreducible representation of K. Then*

$$m(\delta, \mathscr{E}_\lambda(G/H)) = |W_{K \cap H} \backslash W| \dim E'^{M^a}.$$

PROOF. Let $v_1, \ldots, v_r \in \mathfrak{S}(\mathfrak{p} \cap \mathfrak{q})^{K \cap H}$ be as in Lemma 6.5. For $F \in \mathscr{E}_{\lambda, \delta^\vee}(G/H; E')$ let $T_F^i : \mathscr{H}(\mathfrak{p} \cap \mathfrak{q}) \to E^*$, be defined by $T_F^i(p) = (R(\Lambda(pv_i)F)(e)$, $i = 1, \ldots, r$, $p \in \mathscr{H}(\mathfrak{p} \cap \mathfrak{q})$

(1) $T_F^i \in \operatorname{Hom}_{K \cap H}(\mathscr{H}(\mathfrak{p} \cap \mathfrak{q}), E')$, $i = 1, \ldots, r$.

The symmetrization map Λ intertwins the adjoint representation on $\mathfrak{S}(\mathfrak{g})$ and $\mathfrak{U}(\mathfrak{g})$. Let $k \in K \cap H$ and $u = \Lambda(pv_i)$. Since v_i is $\operatorname{Ad}(K \cap H)$-stable we have $T_F^i(\operatorname{Ad}(k)p) = (R(\operatorname{Ad}(k)u)F)(e) = \delta^\vee(k)(R(u)F)(e) = \delta^\vee(k)T_F^i(p)$, proving (1).

(2) $F \in \mathscr{E}_{\lambda, \delta^\vee}(G/H; E')$, $T_F^i = 0$, $i = 1, \ldots, r$, implies $F = 0$.

Suppose $T_F^i = 0$, i.e. $(R(\Lambda(pv_i))F)(e) = 0$ $(i = 1, \ldots, r)$. I will prove $(R(u)F)(e) = 0$, for all $u \in \mathfrak{U}(\mathfrak{g})$ using induction on the degree of u. Thus let $u \in \mathfrak{U}_m(\mathfrak{g})$, $m \in \mathbf{Z}$. If $m < 0$ then $u = 0$ and there is nothing to be proved. Now let $m \geqq 0$ and assume $(R(v)F)(e) = 0$ for all $v \in \mathfrak{U}_{m-1}(\mathfrak{g})$. By Poincare-Birkoff-Witt $u \in (\mathfrak{k}\mathfrak{U}_{m-1}(\mathfrak{g}) + \mathfrak{U}_{m-1}(\mathfrak{g})\mathfrak{h}) \oplus \Lambda(\mathfrak{S}_m(\mathfrak{p} \cap \mathfrak{q}))$. Let $X \in \mathfrak{k}$, $v \in \mathfrak{U}_{m-1}(\mathfrak{g})$. Then $(R(Xv)F)(e) = \delta^\vee(X)(R(v)F)(e)$. But this is 0 since $v \in \mathfrak{U}_{m-1}(\mathfrak{g})$. The same proof also gives $(R(u)F)(e) = 0$ for $u \in \mathfrak{U}_{m-1}(\mathfrak{g})\mathfrak{h}$. We may therefore assume $u \in \Lambda(\mathfrak{S}_m(\mathfrak{p} \cap \mathfrak{q}))$. But then u is a linear combination of elements of the form $\Lambda(v_i pq)$ with $p \in \mathscr{H}(\mathfrak{p} \cap \mathfrak{q})$ and $q \in \pi(\mathfrak{S}(\mathfrak{q})^H)$ (and $v_i pq \in \mathfrak{S}_m(\mathfrak{p} \cap \mathfrak{q})$). Let $\tilde{q} \in \mathfrak{S}_m(\mathfrak{q})^H$ be such that $q - \tilde{q} \in (\mathfrak{k} \cap \mathfrak{q})\mathfrak{S}_{m-1}(\mathfrak{p} \cap \mathfrak{q})$. Let $r, s \in \mathfrak{S}(\mathfrak{g})$ be such that $rs \in \mathfrak{S}_m(\mathfrak{g})$. The symmetrization map Λ has the property $\Lambda(rs) = \Lambda(r)\Lambda(s) + \mathfrak{U}_{m-1}(\mathfrak{g})$. Since $(R(\mathfrak{U}_{m-1}(\mathfrak{g}))F)(e) = \{0\}$ by induction hypothesis, we get $(R(\Lambda(rs))F)(e) = (R(\Lambda(r)\Lambda(s))F(e)$. Now using this we get $(R(\Lambda(v_i pq))F)(e) = (R(\Lambda(v_i p\tilde{q}))F)(e) = (R(\Lambda(v_i p))R(\Lambda(\tilde{q}))F)(e)$. But this is equal to $\chi_\lambda(\Lambda(\tilde{q}))(R(\Lambda(v_i p))F)(e) = 0$. Since F is analytic it must vanish on the connected component of eH in G/H and since F is spherical it must vanish on all G/H.

By (1) and (2) $m(\delta, \mathscr{E}_\lambda(G/H))$ is bounded from above by r times the dimension of $\operatorname{Hom}_{K \cap H}(\mathscr{H}(\mathfrak{p} \cap \mathfrak{q}), E')$. But this space is isomorphic to $\operatorname{Hom}_{K \cap H}(E', \mathscr{H}(\mathfrak{p} \cap \mathfrak{q})^c)$ ($\mathscr{H}(\mathfrak{p} \cap \mathfrak{q})^c$ the contragredient module of $\mathscr{H}(\mathfrak{p} \cap \mathfrak{q})$). $\mathscr{H}(\mathfrak{p} \cap \mathfrak{q})^c$ can be identified with the space $\mathscr{H}'$ of harmonic polynomials on $\mathfrak{p} \cap \mathfrak{q}$. Let $X \in \mathfrak{a}^a$ be regular, H_c^a the adjoint group of $\mathfrak{h}^a$ and $H_c^{a\theta} = \{g \in H_c^a \mid \theta(g) = g\}$ (the involution θ on $\mathfrak{h}^a$ defines an involution θ on H_c^a). By [9] Proposition 1, $H_c^{a\theta}X = (H_c^{a\theta})_c X$. Let O_X denote this orbit. By [9] Theorem 17 the map $\mathscr{H}'$ $p \mapsto p \mid O_X$ is an isomorphism to the space of rational functions on (the variety) O_X. In particular $p \mid O_X$ is holomorphic. Since $\operatorname{Ad}((H \cap K)_e)$ is a real form of $(H_c^{a\theta})_e$, the map $\mathscr{H}' \to \mathscr{E}(H \cap K/M^a)$, $p \mapsto p|_{\operatorname{Ad}(H \cap K)X}$

is injective. Therefore $\dim \mathrm{Hom}_{H\cap K}(\mathscr{H}(\mathfrak{p}\cap\mathfrak{q}), E') \leqq \dim \mathrm{Hom}_{H\cap K}(E, \mathscr{E}(H\cap H/M^a)) = E'^{M^a}$. This finishes the proof.

7. The image of the Poisson transform.

I now want to give a description of the image of the Poisson transform $\mathscr{P}_{-\lambda}$ for $\lambda \in \rho + (\mathfrak{a}/\mathfrak{a}\cap\mathfrak{h})^*_+$ in case G/H is split. First recall the definition of a lowest K-type.

DEFINITION 7.1. *Let $\mathfrak{t}_0$ be a Cartan subalgebra of $\mathfrak{k}_0$ and $\mu \in \mathfrak{t}^*$ the highest weight of an irreducible representation δ of K, with respect to some positive system $\Sigma^+(\mathfrak{k}, \mathfrak{t})$. The norm of δ is defined to be $\|\delta\| = \langle \mu + 2\rho_c, \mu + 2\rho_c\rangle$ (ρ_c is half the sum of roots in $\Sigma^+(\mathfrak{k}, \mathfrak{t})$. Let V be any K-module. δ is a lowest K-type of V if*

a) *δ occurs in V.*

b) *$\|\delta\| = \min\{\|\delta\| \mid \xi$ a K-type occuring in $V\}$*

If L is any subgroup of K and ξ a representation of L let $A(\xi)$ denote the set of lowest K-types of the induced representation $\mathscr{D}(K:\xi)$.

Let $\hat{M}^{M\cap H}$ be the set of equivalence classes of irreducible $(M\cap H)$-spherical representations of M. For $\xi \in \hat{M}^{M\cap H}$ let ξ also denote a representation in this equivalence class. For $\xi \in \hat{M}^{M\cap H}$ let $A(\xi)$ denote the set $A(\xi|_{M\cap K})$. *Then* $A(\xi)$ is the set of minimal K-types of $I_P(\xi : \nu)$ ($\nu \in \mathfrak{a}^*$). Let $A(G/H)$ be the union of the sets $A(\xi)$, the union taken over all $\xi \in \hat{M}^{M\cap H}$.

THEOREM 7.2. a) *Let $\nu \in \mathfrak{a}^*$ and $\xi \in \hat{M}^{M\cap H}$. Then every K-type in $A(\xi)$ has multiplicity one in $I_P(\xi : \nu)$.*

b) *If $\nu \in \rho + (\mathfrak{a}/\mathfrak{a}\cap\mathfrak{h})^*_+$ then $I_P(\xi : \nu)$ has a unique irreducible quotient. Let $J_P(\xi : \nu)$ denote this unique quotient. Every K-type in $A(\xi)$ is contained in $J_P(\zeta : \nu)$.*

c) *Let ξ_1 and ξ_2 be unitary and finite dimensional representations of M and $\nu \in \rho + (\mathfrak{a}/\mathfrak{a}\cap\mathfrak{h})^*_+$. If $J_P(\xi_1 : \nu)$ and $J_P(\xi_2 : \nu)$ are equivalent, then ξ_1 and ξ_2 are equivalent.*

PROOF. First note that the multiplicity of K-types in $I_P(\xi : \nu)$ is independent of ν. Let $P_m = M_m A_m N_m$ be the Langlands decomposition of a minimal parabolic subgroup of G contained in P. Then $P_m \cap M = M_m(A_m \cap M)(N_m \cap M)$ is the Langlands decomposition of a minimal parabolic subgroup of M. Let $\rho_M \in (\mathfrak{a}_m \cap \mathfrak{m})^*$ be half the sum of the roots of $\mathfrak{a}_m \cap \mathfrak{m}$ in $\mathfrak{n}_m \cap \mathfrak{m}$ (counted with multiplicity). By [2] Lemma 4.4 $\xi \mid M_m$ is irreducible and ξ is equivalent to a quotient of $I^M_{P_m\cap M}(\xi \mid M_m : \rho_M)$.

We now choose $\nu \in \mathfrak{a}^*$ such that $\langle \rho_M + \nu, \alpha\rangle > 0$, for all $\alpha \in \Sigma(\mathfrak{n}_m, \mathfrak{a}_m)$ By induction in stages $I_P(\xi : \nu)$ is a quotient of $I_{P_m}(\xi \mid M_m : \rho_M + \nu)$. Since $\rho_M + \nu$ is strictly positive $I_{P_m}(\xi \mid M : \rho_M + \nu)$ has a unique irreducible quotient, the Langlands quotient $J_{P_m}(\xi \mid M_m : \rho_M + \nu)$.

Since $I_{P_m}(\xi \mid M_m : \rho_M + \nu)$ and $J_{P_m}(\xi \mid \rho_M + \nu)$ have the same lowest K-types, $A(\xi)$ is the set of lowest K-types of $I_{P_m}(\xi \mid M_m : \rho_M + \nu)$. These K-types all have multiplicity one in $I_{P_m}(\xi \mid M_m : \rho_M + \nu)$ ([17] Theorem 6.5.9) and therefore also in $I_P(\xi : \nu)$ proving part a) and b).

If $J_{P_m}(\xi_1|_{M_m} : \nu + \rho_M)$ and $J_{P_m}(\xi_2|_{M_m} : \nu + \rho_M)$ are equivalent, then there is some $w \in W(\mathfrak{g}, \mathfrak{a}_m)$ satisfying $(\xi_1|_{M_m})^w \cong \xi_{M_m}$ and $w(\nu + \rho_M) = \nu + \rho$. But since $\nu + \rho_M$ is strictly dominant $w = \mathrm{Id}$ and therefore also $\xi_1 \cong \xi_2$.

Now recall the definition of λ-norm of a K-type and the definition of λ-lowest K-type of a $(\mathfrak{g}, K)$-module from [17] Definition 5.4.1 (called lambda-lowest there).

LEMMA 7.3. *Let G/H be split. Then all the K-types in $A(G/H)$ have the same λ-norm.*

PROOF. Let $\xi \in \hat{M}^{M \cap H}$. Let $P_m = M_m A_m N_m$ be the Langlands decomposition of a minimal parabolic subgroup of G contained in P. Then $M_m \subset M$ and $A \subset A_m$. We look at $\mathfrak{a}_m^*$ as a subspace of $\mathfrak{a}_m^*$. Let $\lambda \in \rho + (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})_+^*$. Then $\nu = \lambda + \rho_M \in \mathfrak{a}_m^*$ is strictly dominant (ρ_M is given by the minimal parabolic subgroup $M \cap P_m$ of M). Put $\chi = \xi|_{M_m}$. Then χ is irreducible. Since G/H is split, $M_e \subset H$. Thus $(M_m)_e \subset H$ and χ is $(M_m)_e$-spherical. Since $M_m/(M_m)_e$ is an abelian group, χ must be one-dimensional and $\chi|_{(M_m)e}$ trivial. The λ-norm of a minimal K-type of $I_{P_m}(\chi : \nu)$ can be calculated from $\chi|_{(M_m)e} = \xi|_{(M_m)e}$ (use [17] Theorem 6.6.15, Lemma 6.6.12, Theorem 6.5.9a) and b) and Lemma 6.5.6), giving a value independent of ξ. In the proof of Theorem 7.2 we saw that $I_P(\xi : \lambda)$ and $I_{P_m}(\delta : \nu)$ have the same minimal K-types. This finishes the proof.

This Lemma implies that if $\xi \in \hat{M}^{M \cap H}$ and δ is any K-type occuring in $I_P(\xi : \lambda)$, then δ is a minimal K-type of $I_P(\zeta : \lambda)$ if and only if $\delta \in A(G/H)$. Using Theorem 7.2 we immediatly get

COROLLARY 7.4. *Suppose G/H is split and let $\lambda \in \rho + (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})_+^*$. Then $J_{\bar{P}|P}(\lambda) : I_P(\lambda)_\delta \to I_{\bar{P}}(\lambda)_\delta$ is bijective for all $\delta \in A(G/H)$.*

Using the connection between the asymptotics of $\mathscr{P}_\nu F$ ($F \in \mathscr{D}(B)$) and standard intertwining operators, given in §5 we are now in the position to prove the following proposition. It is the main result of this section.

PROPOSITION 7.5. *Suppose G/H is split and H is essentially connected. Let $\lambda \in -(\rho + (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})_+^*)$. Then $\mathscr{P}_\lambda : \mathscr{D}(B)_\delta \to \mathscr{E}_\lambda(G/H)_\delta$ is bijective for all $\delta \in A(G/H)$. Thus if $f \in \mathscr{E}_\lambda(G/H)_\delta$ then there exists a unique $F \in \mathscr{D}(B)_\delta$, with $\mathscr{P}_\lambda F = f$.*

PROOF. By Theorem 6.6 the dimension of the spaces $\mathscr{D}(B)_\delta$ and $\mathscr{E}_\lambda(G/H)_\delta$ are equal. We thus only have to show the injectivity of $\mathscr{P}_\lambda$ when restricted to $\mathscr{D}(B)_\delta$. But this follows immediately from Theorem 5.7 and Corollary 7.4.

COROLLARY 7.6. *Suppose G/H is split and H is essentially connected. Let $\lambda \in -(\rho + (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*_+)$. The ($K$-finite) image of $\mathscr{P}_\lambda$ is equal to the $\mathfrak{U}(\mathfrak{g})$-submodule of $\mathscr{E}_\lambda(G/H)_K$ generated by $\mathscr{E}_\lambda(G/H)_{A(G/H)}$ (the linear subspace of $\mathscr{E}_\lambda(G/H)_K$ generated by $\mathscr{E}_\lambda(G/H)_\delta$, with $\delta \in A(G/H)$).*

COROLLARY 7.7. *Suppose G/H is split and H is essentially connected. Let $\xi \in M^{M \cap H}$ and $\lambda \in -(\rho + (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*_+)$. Then the multiplicity of $J_P(\xi : -\lambda)$ in the composition series of $\mathscr{E}_\lambda(G/H)$ is equal to*

$$d_\xi = \|W_{K \cap H}\|^{-1} \sum_{w \in W} \dim V_\xi'^{w^{-1}(M \cap H)w} \tag{25}$$

PROOF. Put $V = J_P(\xi : -\lambda)$. By Corollary 7.6 the multiplicity of V in $\mathscr{E}_\lambda(G/H)$ is equal to the multiplicity in the image of $\mathscr{P}_\lambda$. By Theorem 7.2 b) the multiplicity of V in the image of $\mathscr{P}_\lambda$ is equal to the multiplicity of V in $\sigma_{P, -\lambda}$. By Lemma 3.4 this multiplicity is equal to $\|W_{K \cap H}\|^{-1}$ times the sum (over $w \in W$) of the multiplicities of V in $I_P(-w\nu)$. For $w \in W$ and $\chi \in \hat{M}^{M \cap H} I_{P^w}(\chi : -w\lambda)$ is equivalent to $I_P(\chi^{w^{-1}} : -\lambda)$. By Theorem 7.2 c) V occurs in $I_{P^w}(\chi : -w\lambda)$ if and only if χ is equivalent to ξ^w. Therefore the multiplicity of V in $I_P(-w\nu)$ is equal to $\dim V_{\xi^w}'^{M \cap H} = \dim V_\xi'^{w^{-1}(M \cap H)w}$, proving the corollary.

We saw in the proof just given that for $\nu \in \rho + (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*_+$ $I_P(\xi : \nu)$ embeds d_ξ-times into $(\sigma_{P,\nu}, \mathscr{D}(B))$. We can thus construct a space $V(\xi)$ and a monomorphism

$$S_\xi : \mathscr{D}(K : \xi) \otimes V(\xi) \to \mathscr{D}(B). \tag{26}$$

intertwining $\pi_{P,\xi,\nu} \otimes I$ and $\sigma_{P,\nu}$ (I denoting the trivial representation on $V(\xi)$). I will now give a construction of $V(\xi)$ and S_ξ. This space is easily seen to be isomorphic to the space also called $V(\xi)$ in (eq. (5) of) [3] (used there to represent the intertwining operators from $I_P(\xi : \nu)$ to $\mathscr{E}(G/H)$ for "generic" ν). Let $\bar{W} = N_K(\mathfrak{a}^a)$ and let $V(\xi)$ be the space of functions $T : \bar{W} \to V_\xi'$ satisfying

i) $T(\bar{w}) \in V_\xi'^{\bar{w}^{-1}(M \cap H)\bar{w}}$, $\bar{w} \in \bar{W}$.
ii) $T(\bar{w}m) = \xi'(m)^{-1} T(\bar{w})$, $\bar{w} \in \bar{W}$, $m \in Z_K(\mathfrak{a}^a)$.
iii) $T(\bar{u}\bar{w}) = T(\bar{w})$, $\bar{w} \in \bar{W}$, $\bar{u} \in N_{K \cap H}(\mathfrak{a}^a)$.

For $f \in I_P(\xi : \lambda)$ and $T \in V(\xi)$ let $S_\xi(f \otimes T) : K \times W \to \mathbf{C}$ be defined by

$$S_\xi(f \otimes T)(k, w) = \langle T(\bar{w}), f(k\bar{w}) \rangle, \quad k \in K, w \in W.$$

Here $\bar{w} \in \bar{W}$ is any representative of w. ii) above guarantees that $S_\xi(f \otimes T)$ is well defined. One easily sees that $S_\xi(f \otimes T) \in \mathscr{D}(B)$ and that S_ξ intertwines $\pi_{P,\xi,\nu} \otimes I$ and $\sigma_{P,\nu}$. (This follows from the definition of $\sigma_{P,\nu}$ and the well known fact that $R(\bar{w}) : I_P(\xi : \nu) \to I_{P^w}(\xi^{\bar{w}} : w\nu)$ is an intertwining operator, $R(\bar{w})$ denoting righttranslation by $\bar{w}$.) For $\nu \in -(\rho + (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*_+)$ let

$$\mathscr{P}_{\xi,\nu}: \mathscr{D}(K:\xi) \otimes V(\xi) \to \mathscr{E}(G/H)$$

be defined by $\mathscr{P}_{\xi,\nu}(f \otimes T) = \mathscr{P}_\nu(S_\xi(f \otimes T))$. Let $\mathscr{P}_\xi : I_P(\xi : -\nu) \otimes V(\xi) \to \mathscr{E}(G/H)$ denote the map $\mathscr{P}_\xi(f \otimes T) = \mathscr{P}_{\xi,\nu}(f \mid K \otimes T)(f \in I_P(\xi : -\nu)$ and $T \in V(\xi))$. Then $\mathscr{P}_\xi$ is an inertwining operator. The space $V(\xi)$ has dimension d_ξ (given by (25). Corollary 7.7 and the discussion above now immediatly gives.

THEOREM 7.8. *Suppose G/H is split and H is essentially connected. Let $\xi \in \hat{M}^{M \cap H}$ and $\lambda \in -(\rho + (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*_+)$ and suppose V is a submodule of $\mathscr{E}_\lambda(G/H)_K$. Then the following are equivalent*

a) *$J_P(\xi : -\lambda)$ is a unique irreducible quotient of V.*

b) *There exists a $T \in V(\xi)$ such that $V = \mathscr{P}_\xi(I_P(\xi : -\lambda) \otimes \mathbf{C}T)$.*

$\mathbf{C}T$ in b) *is uniquely determined.*

The image of $\mathscr{P}_{\xi,\lambda}(\lambda \in -(\rho + (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*_+))$ can now be characterized as the maximal submodule of $\mathscr{E}_\lambda(G/H)$ having $J_P(\xi, -\lambda)$ as its only irreducible quotient.

COROLLARY 7.9. *Suppose G/H is split and H is essentially connected. Let $\lambda \in -(\rho + (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*_+)$. Let V be an irreducible submodule of $\mathscr{E}_\lambda(G/H)$ and assume there is some $f \in V$ and $w \in W$ such that $\lim_{a^{P^w} \to \infty} a^{w(\rho-\lambda)} f(xa^{-1})$ does not vanish identically for $x \in G$. Then there exist a $T \in V(\xi)$ unique up to a constant such that $V = \mathscr{P}(I_P(\xi : -\lambda) \otimes \mathbf{C}T)$.*

PROOF. By the subrepresentation theorem for G/H (due to T. Oshima, see [6] Théorème 1 or [16] Theorem 4.6) V is isomorphic to a submodule of $I_{\bar{P}}(\xi : -\lambda)$ for some $\xi \in \hat{M}^{M \cap H}$. Since $\lambda \in -(\rho + (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h})^*_+)$ $J_P(\xi : -\lambda)$ is the unique irreducible submodule of $I_{\bar{P}}(\xi : -\lambda)$. The corollary now follows from Theorem 7.8.

REFERENCES

1. E. P. van den Ban, *A convexity theorem for semisimple symmetric spaces, Pacific J. Math.* 124 (1986), 21–55.
2. *E. P. van den Ban, The principal series for a reductive symmetric space* I. *H-fixed distribution vectors,* Ann. Sci. Ec. Norm. Sup., 21 (1988), 359–412.
3. E. P. van den Ban, *The principal series for a reductive symmetric space,* II. *Eisenstein Inteegrals,* J. Funct. Anal. 109 (1992), 331–441.
4. E. P. van den Ban, M. Flensted-Jensen and H. Schlichtkrull, *Basic harmonic analysis on pseudo-Riemannian symmetric spaces,* J. Funct. Anal. (1994), 69–101.
5. J.-L. Brylinski and P. Delorme, *Vecteurs distributions H-invariants pour les séries principales géneralisées d'espaces symétriques réductifs et prolongement méromorphe d'integrales d'Eisenstein,* Invent. Math. (1992), 619–664.
6. P. Delorme, *Injection de modules spheriques pour les espaces symetrique reductifs dans certains representations induits.* In "Proceedings, Luminy, 1985", Lecture Notes in Math. (1987) 1243, 108–114.
7. Harish-Chandra, *Spherical functions on a semisimple Lie group,* I, Amer. J. Math. 80 (1958), 241–310.

8. Sigurdur Helgason, *Groups and Geometric Analysis*, Academic Press, 1984.
9. B. Kostant and S. Rallis, *Orbits and representations associayed with symmetric spaces*, Amer. J. Math. 93 (1971), 753–809.
10. W. A. Kosters, *Eigenspaces of the Laplace-Beltrami-operator on* $\mathrm{SL}(n, \mathbb{R})/S(\mathrm{GL}(1) \times \mathrm{GL}(n-1))$. I. and II. Ind. Math., 47 (1985), 99–145.
11. G. Ólafsson, *Fourier and Poisson transformation associated to a semisimple symmetric space*, Invent. Math. 90 (1987), 605–629.
12. T. Oshima and J. Sekiguchi, *Eigenspaces of invariant differential operators on an affine symmetric space*, Invent. Math. 57 (1980), 1–81.
13. T. Oshima and J. Sekiguchi, *The restricted root system of a semisimple symmetric pair*, *Advanced Studies in Pure Mathematics* 4 (1984), 433–497.
14. W. Rossmann, *The structure of semisimple symmetric spaces*, Canad. J. Math. 31 (1979), 156–180.
15. H. Schlichtkrull, *Eigenspaces of the Laplacian on Hyperbolic Spaces: Composition Series and Integral Transsforms*, J. Funct. Anal. 70 (1987), 194–219.
16. H. Thorleifsson, *On the Langlands classification for H-spherical representations for some semisimple spaces G/H*. Mathem. Goett., 2 (1992).
17. D. Vogan, *Representations of real reductive Lie groups*, Progr. Math. 15, 1981.
18. D. Vogan and N. R. Wallach, *Intertwining Operators for Real Reductive Groups*, Adv. Math. 82 (1990), 203–243.

MATHEMATISCHES INSTITUT
UNIVERSITÄT GÖTTINGEN
BUNSENSTRASSE 3–5
37073 GÖTTINGEN
GERMANY