MATH. SCAND. 80 (1997), 5–24

ON THE SELF-INTERSECTION SET AND THE IMAGE OF A GENERIC MAP

CARLOS BIASI AND OSAMU SAEKI*

Abstract.

Let $f : M \to N$ be a continuous map of a closed m-dimensional manifold into an n-dimensional manifold with $k = n - m > 0$. We define a primary obstruction to the existence of a homotopy between f and a smooth embedding which is related to the self-intersection set of a generic map homotopic to f. When f is a smooth generic map in the sense of [9], we show that f is a smooth embedding if and only if the primary obstruction vanishes and the $(m - k + 1)$th Betti numbers of M and the image $f(M)$ coincide, generalizing the authors' previous result [4] for immersions with normal crossings. As a corollary we obtain a converse of the Jordan-Brouwer theorem for codimension-1 generic maps, which is a generalization of the results in [3, 1, 2, 20] for immersions with normal crossings. Using generic maps, we show the vanishing of the primary obstruction for injective maps. Furthermore, for non-generic smooth maps, we find a homology class in the closure of the self-intersection set which corresponds to the primary obstruction.

1. Introduction.

Let $f : M \to N$ be a smooth map of a closed m-dimensional manifold into an n-dimensional manifold with $k = n - m > 0$. In this paper, we consider the following problems: *Is f homotopic to an embedding? If this is the case, is f itself an embedding?*

For the first problem, we define a primary obstruction $\theta_1(f)$ to the existence of a homotopy between f and a smooth embedding as a homology class in $H_{m-k}(M; \mathbf{Z}_2)$. This homology class is represented by the closure of the self-intersection set of a generic map [19] homotopic to f and it is shown that it is a homotopy invariant. Thus, if f is homotopic to an embedding, $\theta_1(f)$ necessarily vanishes. Nevertheless, we warn the reader that the vanishing of this primary obstruction does not necessarily imply the existence of a homotopy between f and a smooth embedding.

For the second problem, we assume that $\theta_1(f)$ vanishes and want to find a condition which guarantees that f is an embedding. This recognition problem is difficult to solve in general. Thus, in this paper, we assume that f is generic in the sense of [9] and study the topology of the image $f(M)$ of f. One of the main results of this paper is Theorem 4.1 which states that such a

* The second author is partly supported by CNPq, Brazil.
Received March 27, 1995.

generic map f is an embedding if and only if the $(m-k+1)$th Betti numbers of M and $f(M)$ coincide. This is a direct generalization of a result in [4] for immersions with normal crossings. As a corollary to this result, we obtain a converse of the Jordan-Brouwer theorem for codimension-1 generic maps (Corollary 4.6), which generalizes the results in [3, 1, 2, 20] for codimension-1 immersions with normal crossings.

The paper is organized as follows. In §2, we define two cohomology classes $w_k(f), v(f) \in H^k(M; \mathbf{Z}_2)$ and give various equivalent definitions for $v(f)$. Using these cohomology classes we define the primary obstruction $\theta_1(f) \in H_{m-k}(M; \mathbf{Z}_2)$. We also give a sufficient condition for the vanishing of this class. In §3, we recall the class of generic maps in the sense of [9] and compare it with the class of generic maps treated in [19]. In fact, a generic map in the sense of [9] is necessarily generic in the sense of [19] and is much stronger. In particular, it is shown that the closure of the self-intersection set and the image of a generic map in the sense of [9] are triangulable. In §4, we prove our main theorem (Theorem 4.1) and give its applications. In §5, we show that the primary obstruction class $\theta_1(f)$ vanishes for maps which are topological embeddings, using generic maps. The idea of the proof is to approximate the map by a generic map which is close enough to the original map so that the self-intersection set is small and is null-homologous. In §6, we consider non-generic maps and, assuming that the closure A of the self-intersection set is an ANR (absolute neighborhood retract), we find a homology class u in $H_{m-k}(A; \mathbf{Z}_2)$ such that $i_*(u) = \theta_1(f)$, where $i : A \to M$ is the inclusion map. Using this result, we give an alternative proof of a result obtained in [20] concerning the number of connected components of the complement of a codimension-1 map.

Throughout the paper, all homology and cohomology groups have $\mathbf{Z}_2$ coefficients unless otherwise indicated.

The authors would like to thank du Plessis for invaluable comments about the stratification of generic maps. The second author would like to thank the people in ICMSC-USP, Instituto de Ciências Matemáticas de São Carlos, Universidade de São Paulo, Brazil, where this work has been done.

2. Primary obstruction.

Let $f : M \to N$ be a continuous map of an m-dimensional manifold M into an n-dimensional manifold N. We suppose that $k = n - m > 0$ and that the map f is proper. For the moment, we assume no differentiability of M, N or f. In this section, we define a homotopy invariant of f which is a primary obstruction to the existence of a homotopy between f and an embedding.

Let the stable normal bundle $f^*TN \oplus \nu_M$ of f be denoted by ν_f, where ν_M

is the stable normal bundle of the manifold M. Then we denote by $w_k(f)(\in H^k(M))$ the kth Stiefel-Whitney class of the stable vector bundle ν_f. Note that this is a homotopy invariant; i.e., if f and $g : M \to N$ are homotopic, then $w_k(f) = w_k(g)$. This is easily seen, since $w_k(f)$ is the degree k term of $f^*w(N) \cup \bar{w}(M)$, where $w(N)$ is the total Stiefel-Whitney class of N and $\bar{w}(M)$ is the total dual Stiefel-Whitney class of M.

For the proper continuous map $f : M \to N$, we define $v(f) \in H^k(M)$ to be the image of the fundamental class $[M] \in H^c_m(M)$ by the composite

$$H^c_m(M) \xrightarrow{f_*} H^c_m(N) \xrightarrow{D_N^{-1}} H^k(N) \xrightarrow{f^*} H^k(M),$$

where H^c_* denotes the (singular) homology of the compatible family with respect to the compact subsets ([22, Chapter 6, Section 3]), and D_N denotes the Poincaré duality isomorphism. By the definition, it is easy to see that $v(f)$ is a homotopy invariant (when M is not compact, the homotopy should be through proper maps).

We note that when M and N are smooth and f is an immersion, the above definitions of $w_k(f)$ and $v(f)$ coincide with those of $w_k(\nu_f)$ and $v_k(f)$ respectively given in [4]. See also [16] and [12, Proposition 4.1].

REMARK 2.1. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
H_m(M) & \xrightarrow{f_*} & H_m(N) & \xrightarrow{D_N^{-1}} & H^k_c(N) & \xrightarrow{f^*} & H^k_c(M) \\
\downarrow{\scriptstyle \alpha_M} & & \downarrow{\scriptstyle \alpha_N} & & \downarrow{\scriptstyle \beta_N} & & \downarrow{\scriptstyle \beta_M} \\
H^c_m(M) & \xrightarrow{f_*} & H^c_m(N) & \xrightarrow{D_N^{-1}} & H^k(N) & \xrightarrow{f^*} & H^k(M),
\end{array}$$

where H^*_c denotes the (singular) cohomology with compact supports ([22, p.323]) and $\alpha_M, \alpha_N, \beta_M$ and β_N are the natural homomorphisms. When M is compact, α_M and β_M are isomorphisms and we see that $v(f)$ is equal to the image of the fundamental class $[M] \in H_m(M)$ by the composite $\beta_M \circ f^* \circ D_N^{-1} \circ f_*$.

REMARK 2.2. Consider the following commutative diagram:

$$\begin{array}{ccccc}
H^c_m(N, N-f(M)) & \xrightarrow{\gamma^{-1}} & \check{H}^k(f(M)) & \xrightarrow{f^*} & \check{H}^k(M) \\
\uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle (i')^*} & & \downarrow{\scriptstyle \delta} \\
H^c_m(N) & \xrightarrow{D_N^{-1}} & H^k(N) & \xrightarrow{f^*} & H^k(M),
\end{array}$$

where $i : (N, \emptyset) \to (N, N-f(M))$ and $i' : f(M) \to N$ are the inclusion maps, $\check{H}^*$ denotes the Čech cohomology group (or the Alexander-Čech cohomology group) ([22]), δ denotes the canonical isomorphism ([22, Chapter 6]), and γ denotes the duality isomorphism ([22, p.342]). Thus $v(f) \in H^k(M)$ is equal to the image of $[M] \in H^c_m(M)$ by the composite $\delta \circ f^* \circ \gamma^{-1} \circ i_* \circ f_* : H^c_m(M) \to H^k(M)$. Since this composite factors through $H^c_m(N, N - f(M))$, we see that $v(f)$ depends only on the map $f : M \to V$, where V is an arbitrary neighborhood of $f(M)$ in N. Note that this is also true for $w_k(f)$.

As a corollary to this observation, we have the following: *given a continuous map $f : M \to N$, if $v(f) \neq 0$, then one cannot perform surgery operations in $N - f(M)$ so that $f_*[M] = 0$ in the resulting manifold.* For example, the inclusion $\mathbf{R}P^1 \to \mathbf{R}P^2$ is such an example.

REMARK 2.3. When $f(M)$ is an ANR, we have the following commutative diagram:

$$\begin{array}{ccccc} H^c_m(M) & \xrightarrow{f_*} & H^c_m(N) & \xrightarrow{D_N^{-1}} & H^k(N) \\ f_* \downarrow & & i'_* \uparrow & & i^* \uparrow \\ H^c_m(f(M)) & \xrightarrow{=} & H^c_m(f(M)) & \xrightarrow{\gamma'^{-1}} & H^k(N, N - f(M)), \end{array}$$

where γ' denotes the duality isomorphism (see [11, p.179] and [22]). Thus, in this case, we see that $v(f) \in H^k(M)$ is equal to the image of $[M] \in H^c_m(M)$ by the composite $f^* \circ i^* \circ \gamma'^{-1} \circ f_* : H^c_m(M) \to H^k(M)$.

PROBLEM 2.4. Find a geometric interpretation of $w_k(f)$ and $v(f)$ for a general proper continuous map f (see [4, Remark 2.1]).

As has been seen in [4] (see also [12] and [17, Corollary 11.4]), we know that if M and N are smooth manifolds and f is a smooth embedding, then $w_k(f) = v(f)$.

DEFINITION 2.5. For a proper continuous map $f : M \to N$, we define the homology class $\theta_1(f) \in H^c_{m-k}(M)$ by $\theta_1(f) = D_M(v(f) - w_k(f))$, where $D_M : H^k(M) \to H^c_{m-k}(M)$ is the Poincaré duality isomorphism. Note that this is a homotopy invariant of f. Furthermore, if f is a proper continuous map between smooth manifolds homotopic to a smooth embedding, then $\theta_1(f)$ vanishes.

We will see in a later section that if f is homotopic to a *topological* embedding of a closed smooth manifold into a smooth manifold, then $\theta_1(f)$ vanishes (see Theorem 5.1).

The reason why we use the homology class $\theta_1(f)$ instead of its Poincaré dual is that, if f is a generic smooth map, then the homology class $\theta_1(f)$ is exactly the one represented by the closure of the self-intersection set of f, which will be seen in the next section (see Remark 3.9).

EXAMPLE 2.6. Let $f : K \to S^3$ be a continuous map, where K is the Klein bottle. Then it is easy to see that $\theta_1(f)$ does not vanish, since $w_1(f)$ does not vanish while $\nu(f)$ vanishes. Hence f is not homotopic to a smooth embedding.

In some cases we have the vanishing of the obstruction $\theta_1(f)$. The following proposition will be proved in §4.

PROPOSITION 2.7. *Let $f : M \to N$ be a proper continuous map of an m-dimensional manifold into an n-dimensional manifold with $k = n - m > 0$. If $f_* : H^c_{m-k}(M) \to H^c_{m-k}(N)$ is injective, then $\theta_1(f) = 0$ in $H^c_{m-k}(M)$.*

We note that, since $\theta_1(f)$ depends only on the neighborhood of $f(M)$ (see Remark 2.2), the above proposition is valid also when f_* on the $(m-k)$-th homology is injective after a sequence of surgeries performed in $N - f(M)$.

EXAMPLE 2.8. Consider a continuous map $f : T^2 \to T^3$ of the 2-dimensional torus into the 3-dimensional one. Since they are orientable, $w_1(f)$ always vanishes. If $f_*[T^2] = 0$, then we see that $\nu(f) = 0$ by the definition. If $f_*[T^2] \neq 0$, then it is not difficult to see that $f_* : H_1(T^2) \to H_1(T^3)$ is injective. Hence by Proposition 2.7, we have $\theta_1(f) = 0$. Thus, for a continuous map $f : T^2 \to T^3$, we always have $w_1(f) = \nu(f) = 0$ and $\theta_1(f) = 0$. The same is true for every continuous map $f : S^p \times S^p \to S^p \times S^p \times S^p$ with p odd.

3. Generic maps.

The purpose of this section is to define a certain class of smooth maps between smooth manifolds which are generic in the sense that every map can be approximated by this class of maps and which at the same time have a good behavior with respect to the self-intersection set.

Let M and N be smooth manifolds of dimensions m and n respectively. For the moment, we do not assume that M is compact.

DEFINITION 3.1. Define $l(m,n)$ to be the minimum integer l such that $l > n$ and that $\operatorname{codim} W^l(m,n) > n$, where $W^l(m,n)$ is the real algebraic variety in $J^l(m,n)$ defined in [9, p.120]. Note that this positive integer is well-defined by virtue of [9, Theorem (7.2) (p.121)]. Note that, for every integer l' with $l' \geq l(m,n)$, $\operatorname{codim} W^{l'}(m,n) > n$ (see the proof of [9, Theorem (7.2)]).

DEFINITION 3.2. Define $\Omega(= \Omega(M,N))$ to be $\Omega^{l(m,n)}(M,N)$, which is the space of the proper smooth maps $f : M \to N$ which are multi-transverse with respect to the stratification $\mathscr{A}^{l(m,n)}(M,N)$ with $J^{l(m,n)}f(M) \cap W^{l(m,n)}(M,N) = \emptyset$. (For details, see [9, Proposition (4.1) (p.146)].) We call each element of Ω a *generic map* by virtue of the following theorem.

THEOREM 3.3. ([9]) *The subspace Ω is open and dense in the space*

$C^\infty_{\mathrm{pr}}(M,N)$ *of all proper smooth maps of M into N with respect to the Whitney* C^∞ *topology.*

DEFINITION 3.4. Let $f : M \to N$ be a proper smooth map with $m < n$. We say that f is *generic for the double points*, if it is so in the sense of Ronga [19, Définition (p.228)]; in other words, if the 1-jet extension $J^1 f : M \to J^1(M,N)$ of f is transverse to the submanifolds Σ^i for all i and if the r-fold product map $f^r : M^r \to N^r$ of f is transverse to the diagonal δ of N^r on $M^r - \Delta$ for every $r = 2,3,4,\cdots$, where Σ^i is the space of the 1-jets whose kernel dimension is equal to i, $\delta = \{(y,\cdots,y) \in N^r\}$ and $\Delta = \{(x_1,\cdots,x_r) \in M^r | x_i = x_j \text{ for some } i \neq j\}$.

LEMMA 3.5. *If* $f : M \to N$ *is generic with* $m < n$, *then it is also generic for the double points.*

PROOF. In the following, we set $l = l(m,n)(\geq n+1)$. It is not difficult to show successively that the following conditions are equivalent for fixed points $y \in N$ and $(x_1,\cdots,x_r) \in M^r - \Delta$ with $f(x_i) = y$.

(a) $d(f^r)_{(x_1,\cdots,x_r)}(T(M^r)_{(x_1,\cdots,x_r)}) \oplus T\delta_{(y,\cdots,y)} = T(N^r)_{(y,\cdots,y)}$.

(b) $\{(v_1,\cdots,v_r)|v_i \in df_{x_i}(TM_{x_i})\} \oplus \{(v,\cdots,v)|v \in TN_y\} = TN_y \oplus \cdots \oplus TN_y$.

(c) $\dim\{(v_1,\cdots,v_r)|v_i \in df_{x_i}(TM_{x_i})\} + \dim\{(v,\cdots,v)|v \in TN_y\} - \dim\{(v,\cdots,v)|v \in df_{x_i}(TM_{x_i}) \text{ for all } i\} = r \cdot \dim TN_y$.

(d) $\sum_{i=1}^r \dim(df_{x_i}(TM_{x_i})) + n - \dim(df_{x_1}(TM_{x_1}) \cap \cdots \cap df_{x_r}(TM_{x_r})) = rn$.

(e) $\mathrm{codim}(df_{x_1}(TM_{x_1}) \cap \cdots \cap df_{x_r}(TM_{x_r})) = \sum_{i=1}^r \mathrm{codim}(df_{x_i}(TM_{x_i}))$.

Now consider the natural projection $\pi : TN_y \to \oplus_{i=1}^r TN_y/df_{x_i}(TM_{x_i})$. Then we see that this is surjective if and only if $\dim(TN_y) - \dim(\ker \pi) = \sum_{i=1}^r \mathrm{codim}(df_{x_i}(TM_{x_i}))$, which is equivalent to the condition (e). On the other hand, since f is multi-transverse with respect to $\mathscr{A}^l(M,N)$, the stratified set $(M,(J^l f)^{-1}(\mathscr{A}^l(M,N)))$ has regular intersections with respect to f. Hence for all $y \in N$ and all $(x_1,\cdots,x_r) \in M^r - \Delta$ with $f(x_i) = y$ the projection $TN_y \to \oplus_{i=1}^r TN_y/df_{x_i}(T(X_{x_i})_{x_i})$ is surjective, where X_{x_i} is the stratum of $(J^l f)^{-1}(A^l(M,N))$ which contains x_i. Since the natural projection $TN_y/df_{x_i}(T(X_{x_i})_{x_i}) \to TN_y/df_{x_i}(TM_{x_i})$ is surjective, we see that the projection π is surjective. Hence the condition (a) holds for all $y \in N$ and all $(x_1,\cdots,x_r) \in M^r - \Delta$ with $f(x_i) = y$, and f^r is transverse to δ on $M^r - \Delta$.

Next we show that $J^1 f : M \to J^1(M,N)$ is transverse to Σ^i for all i. Consider the following commutative diagram:

$$\begin{array}{ccl} & & J^l(M,N) = (S_0 \cup S_1 \cup \cdots \cup S_m) \cup W^l(M,N) \\ & J^l f \nearrow & \quad \downarrow \pi^{l,1} \\ M & \xrightarrow{J^1 f} & J^1(M,N) = \Sigma^0 \cup \Sigma^1 \cup \cdots \cup \Sigma^m, \end{array}$$

where $\pi^{l,1}$ is the natural projection and $S_0, S_1, \cdots, S_m$ are as defined in [9, p.135]. Note that $\mathrm{codim} S_i = i$ ([9, Proposition (2.3) (d) (p.136)]). Then we see that $(\pi^{l,1})^{-1}(\Sigma^i) - W^l(M,N) = \cup_{j \geq i(k+i)}(\text{some components of } S_j)$, where $k = n - m$. Since $J^l f(M) \cap W^l(M,N) = \emptyset$ and $J^l f$ is transverse to S_j, we see that $J^1 f$ is transverse to Σ^i. This completes the proof.

We note that the converse of Lemma 3.5 does not hold in general. For details, see [5, §4].

THEOREM 3.6. ([9, Proposition (3.3) (p.142)]) *If f is generic, then $(\mathscr{A}, \mathscr{A}')$ Thom stratifies f, where $\mathscr{A} = ((J^l f)^{-1} \mathscr{A}^l(M,N))_f$, $\mathscr{A}' = \{f(X) | X \in \mathscr{A}\} \cup \{N - f(M)\}$ and $l = l(m,n)$. In particular, $f(M)$ admits a Whitney stratification $\{f(X) | X \in \mathscr{A}\}$.*

REMARK 3.7. Let $(A, \mathscr{A})$ be a locally closed Whitney stratified subset of a smooth manifold M. Then the partition $\mathscr{A}^c$ of A into the connected components of all strata in $\mathscr{A}$ is a Whitney stratification of A. Furthermore, $\mathscr{A}^c$ satisfies the frontier condition (see [9, Theorem (5.6) and Corollary (5.7) (p.61)]).

For a generic map $f: M \to N$ $(m < n)$, we set $M_2(f) = \{x \in M | f^{-1}(f(x)) = \{x, y\}$ with $x \neq y$, and df_x and df_y are non-singular$\}$. Note that $M_2(f)$ is a union of strata of $\mathscr{A}^c$, where $\mathscr{A}$ is as in Theorem 3.6. Note that, by [19] and Lemma 3.5, $\overline{M_2(f)} = M(f) \cup \Sigma(f)$, where $M(f) = \{x \in M | f^{-1}(f(x)) \neq \{x\}\}$ and $\Sigma(f) = \{x \in M | \dim(\ker df_x) \geq 1\}$.

If f is generic, then it is not difficult to see that $(\mathscr{A}^c | \overline{M_2(f)}, (\mathscr{A}')^c | f(\overline{M_2(f)}))$ Thom stratifies $f | \overline{M_2(f)} : \overline{M_2(f)} \to f(\overline{M_2(f)})$, where $(\mathscr{A}')^c | f(\overline{M_2(f)}) = \{f(X) | X \in \mathscr{A}^c | \overline{M_2(f)}\}$. Thus we have the following.

LEMMA 3.8. *If $f \in \Omega$, then $\overline{M_2(f)} (= M(f) \cup \Sigma(f))$ and $f(\overline{M_2(f)})$ admit Whitney stratifications. In particular, they are triangulable.*

The latter result follows from [10]. Note that the image $f(M)$ is also triangulable.

We note that, for maps which are generic for the double points, a result corresponding to Lemma 3.8 does not hold. In fact, in [5, §4], an example of a smooth map which is generic for the double points and whose image is not even an ANR is given. The main reason that we are using generic maps instead of maps which are generic for the double points is that the closure of the self-intersection set and the image of a generic map are triangulable while they are not necessarily triangulable for maps which are generic for the double points.

REMARK 3.9. Let $f: M \to N$ be a proper generic map. Then by [19, Théorème 2.6] and our Lemma 3.5, we see that $\overline{M_2(f)} = M(f) \cup \Sigma(f)$ carries a non-trivial fundamental class $\mu(f) = [\overline{M_2(f)}] \in H^c_{m-k}(\overline{M_2(f)})$. Furthermore, we have $j_*(\mu(f)) = \theta_1(f)$, where $j : \overline{M_2(f)} \to M$ is the inclusion map. In other words, $j_*(\mu(f))$ is the primary obstruction to the existence of a homotopy between f and a smooth embedding.

We also note that the above result does not hold for non-generic maps. For example, consider the smooth map $f : T^2 \to S^3$ as in Figure 1, where T^2 is the 2-dimensional torus. Then we see easily that $\theta_1(f)$ vanishes. Furthermore, (the closure of) the self-intersection set $A = \overline{M(f)}(\subset T^2)$ of f carries a fundamental class $\mu(f) \in H_1(A)$. However $j_*(\mu(f))$ does not vanish and does not coincide with $\theta_1(f)$. Of course f is not generic.

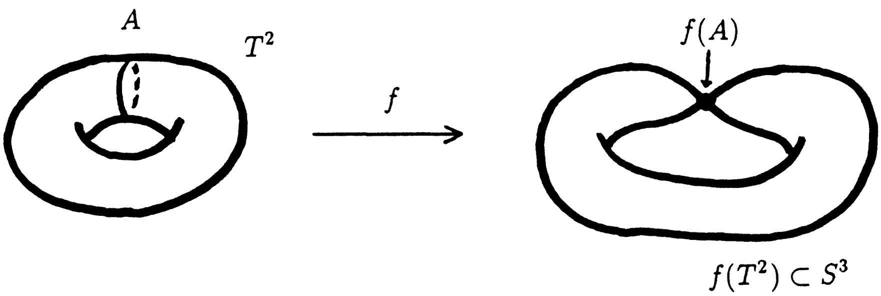

Figure 1

4. A characterization of embeddings among generic maps.

THEOREM 4.1. *Let $f : M \to N$ be a generic map of a closed m-dimensional manifold into an n-dimensional manifold with $k = n - m > 0$. Then f is a smooth embedding if and only if $\theta_1(f) = 0$ in $H_{m-k}(M)$ and $\beta_{m-k+1}(f(M)) = \beta_{m-k+1}(M)$, where β_j denotes the j-th Betti number with $\mathbf{Z}_2$-coefficient.*

PROOF. Let A be the closure of $M(f) = \{x \in M | f^{-1}(f(x)) \neq \{x\}\}$ and set $B = f(A)$. Note that by Lemma 3.8 A and B are compact polyhedrons. Note also that $\dim A = \dim B = m - k$. We suppose that f is not a smooth embedding. This implies either $\Sigma(f) \neq \emptyset$ or $M(f) \neq \emptyset$. Since $A = M(f) \cup \Sigma(f)$ [19], we have $A \neq \emptyset$. Then by the same argument as in [4, §2], we obtain the exact sequence,

$$0 \to H_{m-k+1}(M) \to H_{m-k+1}(f(M))$$
$$\to H_{m-k}(A) \overset{\alpha}{\to} H_{m-k}(B) \oplus H_{m-k}(M) \to \cdots,$$

where $\alpha = (f|A)_* \oplus j_*$ and $j : A \to M$ is the inclusion map. Then we have

$$\beta_{m-k+1}(f(M)) = \beta_{m-k+1}(M) + \dim\ker\alpha.$$

Now consider the fundamental class $\mu(f) = [A] \in H_{m-k}(A)$, which is non-zero.

LEMMA 4.2. $(f|A)_*[A] = 0$ in $H_{m-k}(B)$.

PROOF. First note that $f|A$ is a double cover when restricted to $M_2(f)$. Furthermore, it is not difficult to show that $A - M_2(f)$ is a subcomplex of A with respect to a suitable triangulation and that its dimension is strictly smaller than that of A. Hence we have the conclusion.

On the other hand, by Remark 3.9, we have $j_*[A] = \theta_1(f)$. Now suppose that $\theta_1(f) = 0$. Then we have $j_*[A] = 0$, which implies that

$$\begin{aligned}\beta_{m-k+1}(f(M)) &= \beta_{m-k+1}(M) + \dim\ker\alpha \\ &> \beta_{m-k+1}(M),\end{aligned}$$

since $[A] \in H_{m-k}(A)$ is a non-zero element of $\ker\alpha$. Thus, if f is not a smooth embedding, we have $\beta_{m-k+1}(f(M)) > \beta_{m-k+1}(M)$ or $\theta_1(f) \neq 0$.

On the other hand, if f is a smooth embedding, we clearly have $\beta_{m-k+1}(f(M)) = \beta_{m-k+1}(M)$. Furthermore, we have $\theta_1(f) = 0$ (see Definition 2.5). This completes the proof of Theorem 4.1.

We note that the conclusion of Theorem 4.1 (with $H_{m-k}(M)$ replaced by $H^c_{m-k}(M)$ and β_{m-k+1} by β^c_{m-k+1}) holds also when M is not compact and f is proper, provided that $\beta^c_{m-k+1}(M)$ is finite, where $\beta^c_{m-k+1}(X)$ denotes the dimension of the homology group $H^c_{m-k+1}(X)$ for a space X.

In [5], the same result as Theorem 4.1 has been obtained for maps which are generic for the double points (in the sense of Ronga [19]), where the Betti numbers should be replaced by the Betti numbers with respect to the Čech homology. When the map is generic (in the sense of [9]), the image $f(M)$ is triangulable and these Betti numbers coincide. In other words, our Theorem 4.1 is an easy consequence of the result in [5]. However, we have included a complete proof here, since it is much simpler than that given in [5] and it clarifies the idea. Furthermore, we do not know if the corresponding result for proper maps holds for maps which are generic for the double points (see [5, ☆3]).

By the same argument as in [4] (or by the definition of $v(f)$), we obtain the following corollaries.

COROLLARY 4.3. *Let $f : M \to N$ be a generic map of a closed m-dimensional manifold into an n-dimensional manifold with $k = n - m > 0$. Suppose that either $f^* : H^k(N) \to H^k(M)$ or $f_* : H_m(M) \to H_m(N)$ is the zero map.*

Then f is a smooth embedding if and only if $w_k(f) = 0$ in $H^k(M)$ and $\beta_{m-k+1}(f(M)) = \beta_{m-k+1}(M)$.

COROLLARY 4.4. *Let $f : M \to N$ be a generic map of a closed m-dimensional manifold into an n-dimensional manifold with $k = n - m > 0$. Suppose that $\beta_k(N) = \beta_{2k-1}(N) = \tilde{\beta}_{2k-2}(N) = 0$ and $w_k(f) = 0$. Here $\tilde{\beta}_j$ denotes the dimension of the reduced j-th homology group with $\mathbf{Z}_2$-coefficient. Then f is a smooth embedding if and only if $\tilde{\beta}_{2k-2}(N - f(M)) = \beta_{k-1}(M)$.*

Compare the above corollary with the results of [13].
Here we give a proof of Proposition 2.7.

PROOF OF PROPOSITION 2.7. By a homotopy we may assume that f is generic. Then by Lemma 4.2, we see that $f_*(j_*[A]) = j'_*((f|A)_*[A]) = 0$ in $H^c_{m-k}(N)$, where $j' : B \to N$ is the inclusion map. Since f_* is injective, we see that $\theta_1(f) = j_*[A] = 0$. This completes the proof.

Using Proposition 2.7, we have the following.

COROLLARY 4.5. *Let $f : M \to N$ be a generic map of a closed m-dimensional manifold into an n-dimensional manifold with $k = n - m > 0$. Suppose that $f_* : H_{m-k}(M) \to H_{m-k}(N)$ is injective. Then f is a smooth embedding if and only if $\beta_{m-k+1}(f(M)) = \beta_{m-k+1}(M)$.*

The following corollary to Corollary 4.4 is a converse of the Jordan-Brouwer Theorem for generic maps, which generalizes the results of [3, 1, 2] (see also [20]). Note that, when $k = 1$ and $H_1(N) = 0$, we always have $v(f) = 0$ (Remark 2.1) and $w_1(f) = 0$ if and only if M is orientable.

COROLLARY 4.6. *Let $f : M \to N$ be a generic map of a closed orientable m-dimensional manifold into a connected $(m + 1)$-dimensional manifold with $H_1(N) = 0$. Then f is a smooth embedding if and only if $\beta_0(N - f(M)) = \beta_0(M) + 1$.*

As a corollary to Corollary 4.5, we also have the following.

COROLLARY 4.7. *Let $f : M \to N$ be a generic map of a closed m-dimensional manifold into a connected $(m + 1)$-dimensional manifold with $H_1(N) = 0$. Suppose that $f_* : H_{m-1}(M) \to H_{m-1}(N)$ is injective. Then f is a smooth embedding if and only if $\beta_0(N - f(M)) = \beta_0(M) + 1$.*

PROOF. By the same argument as in the proof of Corollary 1.3 of [4], we see that $\beta_0(N - f(M)) = \beta_m(f(M)) + 1$. Then the result follows from Corollary 4.5.

Note that a converse of the Jordan-Brouwer Theorem has been obtained also for quasi-regular immersions ([21]) and that a quasi-regular immersion is not necessarily generic.

Theorem 4.1 suggests the following definition.

DEFINITION 4.8. Let $f : M \to N$ be a continuous map of a closed m-dimensional manifold into an n-dimensional manifold with $k = n - m > 0$. We define $\theta_2(f) \in \{0, 1, 2, \cdots\}$ to be the minimum of $\beta_{m-k+1}(g(M)) - \beta_{m-k+1}(M)$ over all generic maps $g : M \to N$ homotopic to f. Note that this is well-defined by Theorem 3.3 and that it is a homotopy invariant of f.

Theorem 4.1 can now be reformulated as follows: *a continuous map $f : M \to N$ is homotopic to a smooth embedding if and only if $\theta_1(f) = 0$ in $H_{m-k}(M)$ and $\theta_2(f) = 0$.* We do not know if there exists an effective method to determine $\theta_2(f)$. If fact, we even do not know if $\theta_2(f)$ can be arbitrarily large for a given pair of manifolds (M, N).

Here we note the following.

LEMMA 4.9. *Let $f : M \to N$ be a generic map as in Theorem 4.1. Then $\beta_{m-k+1}(f(M)) = \beta_{m-k+1}(M)$ if $\tilde{H}_{m-k+1}(C_{\tilde{f}}) = 0$, where $C_{\tilde{f}}$ is the mapping cone of the map $\tilde{f} = f : M \to f(M)$ ($C_{\tilde{f}}$ is the space obtained from $(M \times [0, 1]/M \times \{0\}) \cup f(M)$ with $(x, 1)$ identified with $f(x)$ for all $x \in M$).*

PROOF. Let $Z_{\tilde{f}}$ be the mapping cylinder of $\tilde{f}$; i.e., $Z_{\tilde{f}}$ is the space obtained from $(M \times [0, 1]) \cup f(M)$ with $(x, 1)$ identified with $f(x)$ for all $x \in M$. Now consider the exact sequence

$$H_{m-k+1}(M) \xrightarrow{i_*} H_{m-k+1}(Z_{\tilde{f}}) \to H_{m-k+1}(Z_{\tilde{f}}, M),$$

where $i : M \to Z_{\tilde{f}}$ is the inclusion onto $M \times \{0\}$. Then the map i_* is equivalent to the map $\tilde{f}_* : H_{m-k+1}(M) \to H_{m-k+1}(f(M))$, which is injective by the proof of Theorem 4.1. Furthermore we see that $H_{m-k+1}(Z_{\tilde{f}}, M)$ is isomorphic to $\tilde{H}_{m-k+1}(C_{\tilde{f}})$. Hence if $\tilde{H}_{m-k+1}(C_{\tilde{f}}) = 0$, then $\beta_{m-k+1}(f(M)) = \beta_{m-k+1}(M)$. This completes the proof.

As a corollary to Theorem 4.1 and the above lemma, we have the following.

COROLLARY 4.10. *Let $f : M \to N$ be a generic map of a closed m-dimensional manifold into an n-dimensional manifold with $k = n - m > 0$. Then f is a smooth embedding if and only if $\theta_1(f) = 0$ in $H_{m-k}(M)$ and $\tilde{H}_{m-k+1}(C_{\tilde{f}}) = 0$.*

In view of Corollary 4.10, for a continuous map $f : M \to N$ as in Definition 4.8, we can define $\theta_2'(f)$ as the minimum of $\dim \tilde{H}_{m-k+1}(C_{\tilde{g}})$ over all generic maps g homotopic to f and we have that f is homotopic to a smooth embedding if and only if $\theta_1(f) = 0$ in $H_{m-k}(M)$ and $\theta_2'(f) = 0$.

In some cases, we have geometric formulas for $\beta_{m-k+1}(f(M)) - \beta_{m-k+1}(M)$ and $\dim\tilde{H}_{m-k+1}(C_{\tilde{f}})$ for generic f.

PROPOSITION 4.11. *Let $f : M \to N$ be a generic map of a closed m-dimensional manifold into an $(2m-1)$-dimensional manifold with $m \geq 3$. Then if $H_1(M) = 0$, we have $\beta_2(f(M)) - \beta_2(M) = \dim\tilde{H}_2(C_{\tilde{f}}) = \beta_0(B)$, where $B = f(A) = f(\overline{M(f)})$.*

PROOF. By the proof of Theorem 4.1, we have

$$\beta_2(f(M)) - \beta_2(M) = \dim \ker((f|A)_* : H_1(A) \to H_1(B)).$$

On the other hand, since f is a generic map of an m-dimensional manifold into an $(2m-1)$-dimensional manifold, we know that f is an immersion with normal crossings except at a finite number of points, where f has Whitney Umbrellas (see [24, 25]). Hence A is a closed 1-dimensional submanifold of M and B is a compact 1-dimensional submanifold of N with boundaries corresponding to the image of the Whitney Umbrellas. Then for each connected component C of B, $f|f^{-1}(C) : f^{-1}(C) \to C$ is equivalent to the trivial double cover $S^1 \cup S^1 \to S^1$, the non-trivial double cover $S^1 \to S^1$, or the projection $\pi : S^1 \to [-1, 1]$, where $\pi(z) = \mathrm{Re}(z)$ $(z \in \mathbf{C}, |z| = 1)$. In each case, we have $\dim \ker((f|f^{-1}(C))_* : H_1(f^{-1}(C)) \to H_1(C)) = 1$. Hence we have $\dim \ker(f|A)_* = \beta_0(B)$. The equality $\beta_2(f(M)) - \beta_2(M) = \dim\tilde{H}_2(C_{\tilde{f}})$ follows from the exact sequence

$$0 \to H_2(M) \to H_2(Z_{\tilde{f}}) \to H_2(C_{\tilde{f}}) \to H_1(M)$$

together with the isomorphisms $H_2(Z_{\tilde{f}}) \cong H_2(f(M))$ and $H_1(M) \cong 0$. This completes the proof.

Compare Proposition 4.11 with [14, Theorem 1.2 (2)] and [2, Theorem 1.3 and Remark 2.10].

5. Vanishing of the primary obstruction for topological embeddings.

THEOREM 5.1. *Let $f : M \to N$ be a topological embedding of a closed m-dimensional smooth manifold into an n-dimensional smooth manifold with $k = n - m > 0$. Then $\theta_1(f) \in H_{m-k}(M)$ vanishes.*

PROOF. We have a Riemannian metric on M, which we fix here. We denote by $d(x, y)$ the distance between x and y in M with respect to the fixed metric. Then there exists a positive constant ε which satisfies the following: putting

$$X = \{(x, y, t) \in M \times M \times \mathsf{R} | d(x, y) < \varepsilon, |t| \leq d(x, y)/2\},$$

we have a continuous map $\varphi : X \to M$ such that, for every $(x, y) \in M \times M$ with $d(x, y) < \varepsilon$, $\varphi(x, y, -d(x, y)/2) = x, \varphi(x, y, d(x, y)/2) = y$ and $\alpha(t) = \varphi(x, y, t)$ is a geodesic curve connecting x and y parametrized by arc length (for example, see [23, Theorem 14 (p.454)]). Note that $\varphi(x, y, t) = \varphi(y, x, -t)$ for all $(x, y, t) \in X$.

Since M is compact and f is a topological embedding, there exists a generic map $g : M \to N$ homotopic to f which satisfies that $g(x) = g(y)$ implies that $d(x, y) < \varepsilon$. Thus, for the proof of Theorem 5.1, it suffices to prove the following.

LEMMA 5.2. *Let $f : M \to N$ be a generic map of a closed m-dimensional Riemannian manifold into an n-dimensional smooth manifold with $k = n - m > 0$. Suppose that $f(x) = f(y)$ implies that $d(x, y) < \varepsilon$. Then $\theta_1(f) = 0$ in $H_{m-k}(M)$.*

PROOF. Let $F_2(M)$ be the "fat square" of M as defined in [19]; i.e., $F_2(M) = (M \times M - \Delta_M) \cup P(TM)$, where Δ_M is the diagonal set and $P(TM)$ is the total space of the projective tangent bundle of M. We denote by $\sigma : F_2(M) \to M \times M$ the natural projection and by $p_1 : M \times M \to M$ the projection to the first factor. Put $\pi = p_1 \circ \sigma$. Recall that $\overline{M_2(f)} = \pi(\tilde{M}_2(f))$, where $\tilde{M}_2(f) = \{z \in \Omega | S(f)_{(z)} = 0\}$, Ω is a neighborhood of $P(TM)$ in $F_2(M)$, γ is a line bundle over $F_2(M)$ which extends the canonical line bundle over $P(TM)$, and $S(f)$ is a section over Ω of $Hom(\gamma, \pi^* f^*(TN))$ defined by $S(f)_{(x,y)} = f(y) - f(x)$ for $x \neq y$ and $S(f)_{(x,x)} = df_x$ by means of natural identifications. For details, see [19]. Note that $\tilde{M}_2(f)$ is a regular submanifold of $F_2(M)$ of dimension $m - k$.

Define the involution $\tau : M \times M \to M \times M$ by $\tau(x, y) = (y, x)$. The involution τ can be lifted to an involution of $F_2(M)$, which we denote by $\tilde{\tau} : F_2(M) \to F_2(M)$. Note that $\tilde{\tau}|(M \times M - \Delta_M) = \tau|(M \times M - \Delta_M), \tilde{\tau}|P(TM) = \mathrm{id}_{P(TM)}$ and that the diagram

$$\begin{array}{ccc} F_2(M) & \xrightarrow{\tilde{\tau}} & F_2(M) \\ \sigma \downarrow & & \downarrow \sigma \\ M \times M & \xrightarrow{\tau} & M \times M \end{array}$$

is commutative. Note that $\tilde{M}_2(f)$ is invariant under τ; i.e., $\tilde{\tau}(\tilde{M}_2(f)) = \tilde{M}_2(f)$.

By our hypothesis, we have

$$\sigma(\tilde{M}_2(f)) \subset \{(x, y) \in M \times M | d(x, y) < \varepsilon\}.$$

Define the continuous map $\beta : X_2(f) \to M$ by $X_2(f) = \{(p, t) \in \tilde{M}_2(f) \times \mathbf{R} | \, |t| \leq d(x, y)/2, (x, y) = \sigma(p)\}$ and $\beta(p, t) = \varphi(\sigma(p), t)$. Note that $X_2(f)$ is a compact polyhedron. We also note that β is invariant under the involution $\kappa : X_2(f) \to X_2(f)$ defined by $\kappa(p,t) = (\tilde{\tau}(p), -t)$, since

$$\begin{aligned}\beta \circ \kappa(p,t) &= \beta(\tilde{\tau}(p), -t)\\ &= \varphi(\sigma(\tilde{\tau}(p)), -t)\\ &= \varphi(\tau(\sigma(p)), -t)\\ &= \varphi(\sigma(p), t)\\ &= \beta(p,t).\end{aligned}$$

Thus β induces a continuous map $\bar{\beta} : \Delta \to M$, where $\Delta = X_2(f)/\kappa$ is a compact polyhedron of dimension $m-k+1$ (in fact, Δ is a compact smooth manifold with corners. See the remark below). In other words, $\bar{\beta}$ can be regarded as an $(m-k+1)$-chain in M. Set $Y = \{(p,t) \in X_2(f) \mid |t| = d(x,y)/2, (x,y) = \sigma(p)\}$. Then for $(p,t) \in Y$, denoting the equivalence class of (p,t) in $\Delta = X_2(f)/\kappa$ by $[p,t]$, we have

$$\begin{aligned}\bar{\beta}([p,t]) &= \beta(p,t)\\ &= \varphi(\sigma(p), t)\\ &= x = p_1 \circ \sigma(p) = \pi(p)\end{aligned}$$

if $t = -d(x,y)/2$ and

$$\varphi(\sigma(p), t) = y = p_1(\tau(\sigma(p))) = p_1(\sigma(\tilde{\tau}(p))) = \pi(\tilde{\tau}(p))$$

if $t = d(x,y)/2$. Thus we have $\bar{\beta}(\partial\Delta) = \bar{\beta}(Y/\kappa) = \pi(\tilde{M}_2(f)) = \overline{M_2(f)}$ and $\partial\bar{\beta} = \overline{M_2(f)}$ as $\mathbf{Z}_2$-cycles of dimension $m-k$. Hence we have

$$\theta_1(f) = D_M(w_k(f) - v(f)) = [\overline{M_2(f)}] = 0.$$

This completes the proof of Lemma 5.2 and hence of Theorem 5.1.

REMARK 5.3. Set $\hat{X}_2(f) = \{(p,t) \in X_2(f) | \sigma(p) \in \Delta_M, t = 0\}$. Then the pair $(X_2(f), \hat{X}_2(f))$ is locally homeomorphic to $(\{(x,y) \in \mathbf{R}^2 | \, |y| \leq |x|\}, (0,0)) \times \mathbf{R}^{m-k-1}$. Furthermore, the involution $\kappa : X_2(f) \to X_2(f)$ is of the form $((x,y),a) \mapsto ((-x,-y),a)$ under the local identification. Thus $\Delta = X_2(f)/\kappa$ is a smooth manifold with corner $\hat{X}_2(f)/\kappa$. We can smooth the corner easily and then Δ is a compact smooth manifold with boundary. Furthermore, the map $\partial\Delta \to \tilde{M}_2(f)$ defined by $[p,t] \mapsto p$ is a diffeomorphism. Hence $\pi|\tilde{M}_2(f) : \tilde{M}_2(f) \to M$ is null-bordant; i.e., $[\pi|\tilde{M}_2(f)] = 0$ in $\eta_{m-k}(M)$ (see [7]).

As a corollary to Lemma 5.2, we have the following.

COROLLARY 5.4. *Let $f : M \to N$ be a continuous map of a closed m-dimensional Riemannian manifold into an n-dimensional smooth manifold with*

$k = n - m > 0$. Suppose that $f(x) = f(y)$ implies that $d(x, y) < \varepsilon$, where ε is as in the proof of Theorem 5.1. Then $\theta_1(f) = 0$ in $H_{m-k}(M)$.

REMARK 5.5. Note that there do exist topological embeddings which are not homotopic to smooth embeddings. For example, consider a continuous map $f : S^2 \to S^2 \times S^2$ such that $f_*[S^2] = 2[S^2 \times \{*\}] + 3[\{*\} \times S^2]$ in $H_2(S^2 \times S^2; \mathbf{Z})(\cong \pi_2(S^2 \times S^2))$. Then, by [8], f is homotopic to a topological (locally flat) embedding. However, by [15], f is not homotopic to a smooth embedding.

6. The primary obstruction for non-generic maps.

Let $f : M \to N$ be a continuous map and $A \subset M$ the closure of the self-intersection set of f; i.e., A is the closure of $M(f) = \{x \in M | f^{-1}(f(x)) \neq \{x\}\}$. In this section, we will find a class in the homology of A which represents the obstruction $\theta_1(f)$.

THEOREM 6.1. *Let $f : M \to N$ be a continuous map of a closed m-dimensional smooth manifold into an n-dimensional smooth manifold with $k = n - m > 0$. Suppose that A is an* ANR. *Then there exists a class $u \in H_{m-k}(A)$ such that $j_*(u) = \theta_1(f)$, where $j : A \to M$ denotes the inclusion. Furthermore, if $B = f(A)$ is also an* ANR, *then we can choose u as above so that $(f|A)_*(u) = 0$ in $H_{m-k}(B)$.*

REMARK 6.2. When A is not an ANR, we can find an appropriate class u as above in the Čech homology of A. Alternatively, we can find a closed subset A' of M which contains A and which is an ANR. Then we can find an appropriate class u in the homology of A'. However, the hypothesis on B seems to be difficult to remove.

PROOF OF THEOREM 6.1. Since A is an ANR, there exists an open neighborhood V of A with a retraction $r : V \to A$. Then by [11, p.175], there exists another open neighborhood V' of A such that $\overline{V'}$ is contained in V and that the diagram

$$\begin{array}{ccc} V' & \xrightarrow{i_1} & V \\ & {\scriptstyle r|V'} \searrow \quad \nearrow {\scriptstyle i_2} & \\ & A & \end{array}$$

is homotopy commutative, where i_1 and i_2 are the inclusions. Take a smaller open neighborhood V'' of A such that $\overline{V''} \subset V'$. Then we take a compact codimension-0 submanifold W with boundary in M such that $\overline{V''} \subset \operatorname{Int} W \subset W \subset V'$. Furthermore, take a smaller codimension-0 submanifold W' in $\operatorname{Int} W$ whose interior contains $\overline{V''}$. We introduce a Riemannian metric on M such that both ∂W and $\partial W'$ are totally geodesic. Then there exists an $\varepsilon > 0$ such that $x \in W - \operatorname{Int} W'$ and $d(x,y) < \varepsilon$ imply that a geodesic connecting x and y with total length $d(x,y)$ is entirely contained in $V' - \overline{V''}$ and that the property for M mentioned in the proof of Theorem 5.1 is satisfied. Then there exists a generic map $g : M \to N$ homotopic to f such that $g(x) = g(y)$ with $x \notin V''$ or $y \notin V''$ implies that $d(x,y) < \varepsilon$.

Now consider the map $\pi = p_1 \circ \sigma : F_2(M) \to M$ and its restriction to $\tilde{M}_2(g)$ as in the proof of Theorem 5.1. Then there exists a compact codimension-0 submanifold with boundary T of $\tilde{M}_2(g)$ such that $\tilde{M}_2(g) \cap \pi^{-1}(M - \operatorname{Int} W) \subset \operatorname{Int} T \subset T \subset \tilde{M}_2(g) \cap \pi^{-1}(M - W')$. Then $\pi(\partial T)$ is contained in $\operatorname{Int} W - W'$. Hence, by the choice of ε and g, we see that $\pi(\tilde{\tau}(\partial T))$ is contained in $V' - \overline{V''}$, where $\tilde{\tau} : F_2(M) \to F_2(M)$ is the involution defined in the proof of Theorem 5.1. Furthermore, the geodesic connecting $\pi(x)$ and $\pi(\tilde{\tau}(x))$ is entirely contained in $V' - \overline{V''}$ for every $x \in \partial T$. Set $\tilde{T} = T \cup \tilde{\tau}(T)$ and set $X_T = \{(p,t) \in \tilde{T} \times \mathbf{R} \mid |t| \leq d(x,y)/2, (x,y) = \sigma(p)\} (\subset X_2(g))$ (for notation, see the proof of Theorem 5.1). Then we see that X_T is invariant under the involution $\kappa : X_2(g) \to X_2(g)$ and that we can define a continuous map $\bar{\beta}' : X_T/\kappa \to M$ as in the proof of Theorem 5.1. Furthermore, we may assume that the involution $\tilde{\tau}$ is piecewise linear with respect to a triangulation of $F_2(M)$ and that $\tilde{M}_2(g)$ and T are subcomplexes of $F_2(M)$. Then $\tilde{T}$ is also a subcomplex of $F_2(M)$ and $\bar{\beta}'$ defines an $(m-k+1)$-chain in M. The boundary of this chain consists of $\pi(\tilde{T})$ and a part entirely contained in $V' - \overline{V''}$. Thus we have shown that $(j')_* \circ \pi_*[\tilde{M}_2(g)] = (j')_*[\overline{M_2(g)}]$ lies in the image of

$$(i')_* : H_{m-k}(V') \to H_{m-k}(M),$$

where $j' : \overline{M_2(g)} \to M$ and $i' : V' \to M$ are the inclusions. In other words, there exists a class $u' \in H_{m-k}(V')$ such that $(i')_*(u') = (j')_*[\overline{M_2(g)}]$. Now, by the choice of the neighborhoods V and V' of A, we have the following commutative diagram:

$$\begin{array}{ccc} H_{m-k}(V') & \xrightarrow{(i_1)_*} & H_{m-k}(V) \\ {\scriptstyle (r|V')_*} \searrow & & \nearrow {\scriptstyle (i_2)_*} \\ & H_{m-k}(A). & \end{array}$$

Then, setting $u = (r|V')_*(u') \in H_{m-k}(A)$, we see that $j_*(u) = (j')_*[\overline{M_2(g)}] = \theta_1(g) = \theta_1(f)$. This completes the proof of the first half of the theorem.

Now we suppose that $B = f(A)$ is an ANR. Then there exists a neighborhood U of B in N such that there exists a retraction $r' : U \to B$. We may assume that $V \subset f^{-1}(U)$, that $g(V) \subset U$ and that $f|V$ and $g|V$ are homotopic as maps of V into U. Let c be the $(m-k)$-cycle in M given by $c = \pi(\tilde{M}_2(g)) - \partial\bar{\beta}'$. Note that c is contained in V' and that $c = c_1 + c_2$, where $c_1 = \pi(\overline{\tilde{M}_2(g) - \tilde{T}})$, $c_2 = \bar{\beta}'(X')$, $X' = \{(p,t) \in \partial\tilde{T} \times \mathbf{R} \mid |t| \leq d(x,y)/2, (x,y) = \sigma(p)\}/\kappa$ and $\partial\tilde{T} = (\partial T - \tilde{\tau}(\text{Int}T)) \cup (\tilde{\tau}(\partial T) - \text{Int}T)$. We first show that $(g|V')_*[c] = 0$ in $H_{m-k}(U)$. Since $g(c_1)$ and $g(c_2)$ are $(m-k)$-cycles in U, we have only to show that each of them is null homologous in U. For $g(c_1)$, this is proved by an argument similar to the proof of Lemma 4.2. As to $g(c_2)$, we can construct an $(m-k+1)$-chain in U whose boundary is $g(c_2)$ as follows. We may assume that U is the interior of a compact codimension-0 submanifold U' of N. Then we introduce a Riemannian metric on N such that $\partial U'$ is totally geodesic. Then, choosing $\varepsilon > 0$ sufficiently small, we may assume that, for every point $a \in \partial\tilde{T}$, the image $g(\gamma_a)$ of a geodesic γ_a connecting $\pi(a)$ and $\pi(\tilde{\tau}(a))$ in M is contained in an open set U_a in U such that any two points of U_a are connected by a unique geodesic contained in a larger open set U'_a such that $U_a \subset U'_a \subset U$. Note that $g(\gamma_a)$ is a closed curve in U_a. Let P_a be the union of the geodesic segments in U'_a connecting $g(\pi(a))$ and the points in $g(\gamma_a)$. Note that P_a is the image of a continuous map of a 2-disk into U. Then let P be the $(m-k+1)$-chain in U defined by the union of all P_a over all $a \in \partial\tilde{T}$; in other words, P is the image of a continuous map of a 2-disk bundle over $\partial\tilde{T}$. Then it is easy to see that the boundary of P coincides with $g(c_2)$. Hence we have shown that $g(c) = g(c_1) + g(c_2)$ is null homologous in U.

Using the fact that $f|V$ and $g|V$ are homotopic, we have the following commutative diagram:

$$\begin{array}{ccc}
H_{m-k}(\overline{M_2(g)}) & \xrightarrow{(g|\overline{M_2(g)})_*} & H_{m-k}(g(\overline{M_2(g)})) \\
{\scriptstyle (i_3)_*}\downarrow & & {\scriptstyle (i_4)_*}\downarrow \\
H_{m-k}(V) & \xrightarrow{(f|V)_*} & H_{m-k}(U) \\
{\scriptstyle (i_2)_*}\uparrow & & {\scriptstyle (i_5)_*}\uparrow \\
H_{m-k}(A) & \xrightarrow{(f|A)_*} & H_{m-k}(B),
\end{array}$$

where i_3, i_4 and i_5 are the inclusions. Then by Lemma 4.2, we see that $(g|\overline{M_2(g)})_*[\overline{M_2(g)}] = 0$ in $H_{m-k}(g(\overline{M_2(g)}))$. Furthermore, by the proof of the first half, we see that $(i_2)_*(u) = (i_3)_*[\overline{M_2(g)}]$ holds for u constructed as above. Thus we see that $(i_5)_* \circ (f|A)_*(u) = (f|V)_* \circ (i_2)_*(u) = 0$. Note that $(i_5)_*$ is injective, since B is a retract of U. Hence we see that $(f|A)_*(u) = 0$, completing the proof.

LEMMA 6.3. *In Theorem 6.1, suppose that B is an* ANR *and that*

$u \in H_{m-k}(A)$ as in the second half of the theorem is not zero. Then if $\theta_1(f)$ vanishes, then $f_* : H_{m-k+1}(M) \to H_{m-k+1}(f(M))$ is not surjective.

PROOF. Using the hypothesis that A and B are ANR's, as in the proof of Theorem 4.1, we have the exact sequence

$$H_{m-k+1}(A) \to H_{m-k+1}(B) \oplus H_{m-k+1}(M) \xrightarrow{(j'')_* \oplus f_*} H_{m-k+1}(f(M))$$
$$\to H_{m-k}(A) \xrightarrow{\alpha} H_{m-k}(B) \oplus H_{m-k}(M),$$

where α is defined as before and $j'' : B \to f(M)$ is the inclusion map. Then by Theorem 6.1, $\ker \alpha$ is not zero, and hence $f_* : H_{m-k+1}(M) \to H_{m-k+1}(f(M))$ cannot be surjective. This completes the proof.

Note that in the above lemma, if $H_{m-k+1}(A) = 0$, then $f_* : H_{m-k+1}(M) \to H_{m-k+1}(f(M))$ is injective.

As an application of Theorem 6.1, we prove the following.

PROPOSITION 6.4. *Let $f : M \to N$ be a continuous map of a closed orientable connected m-dimensional smooth manifold into a connected $(m+1)$-dimensional smooth manifold with $H_1(N) = 0$. Suppose that A and B are ANR's. Then if f has a normal crossing point of multiplicity 2, then $\beta_0(N - f(M)) \geq 3$.*

For the definition of a normal crossing point of multiplicity 2, see [20]. Note that the fact that $\beta_0(N - f(M)) \geq 2$ has been obtained in [18] without the assumptions on A, B, the orientability of M, and the existence of a normal crossing point. They only need the assumption that A is not the whole manifold M. We also note that the above result has been obtained in [20] without the assumptions on A and B by using a method completely different from ours. In fact, in [20], it is shown that, if f has a normal crossing point of multiplicity m, then we have $\beta_0(N - f(M)) \geq m + 1$. We do not know if we can prove this result using our method.

PROOF OF PROPOSITION 6.4. First note that $A \neq M$, since f has a normal crossing point of multiplicity 2. Then we see that $H^m(A) = 0$ (see [22, p.342]) and hence that $H_m(A) = 0$ by the universal coefficient theorem.

Consider the following exact sequence of the pair $(N, N - f(M))$:

$$H_1(N) \to H_1(N, N - f(M)) \to \tilde{H}_0(N - f(M)) \to \tilde{H}_0(N).$$

By our hypothesis, we see that $H_1(N, N - f(M))$ is isomorphic to $\tilde{H}_0(N - f(M))$. On the other hand $H_1(N, N - f(M))$ is isomorphic to $\check{H}^m(f(M))$ by [22, Theorem 10 (p.342)]. Note that $f(M)$ is an ANR, since $f(M)$ is obtained from the disjoint union $M \cup B$ of two ANR's by identifying the points in A and B using $f|A : A \to B$ and A is an ANR (see [6, Chapter V, Theorem (9.1)]). Thus we see that $\check{H}^m(f(M))$ is isomorphic to the singular cohomology $H^m(f(M))$ (see [22]), which is isomorphic to $H_m(f(M))$ by the universal coefficient theorem. Thus we have shown that $\beta_0(N - f(M)) = \dim H_m(f(M)) + 1$.

We also note that $\theta_1(f)$ vanishes, since $H^1(N) = 0$ implies $v(f) = 0$ and M and N are orientable.

Thus, using Lemma 6.3 together with the remark just after the lemma, we see that we have only to show that the class $u \in H_{m-1}(A)$ does not vanish. Let $p \in f(M)$ be a normal crossing point of f. Then we may assume that the generic map g which we used in the proof of Theorem 6.1 satisfies that $g|g^{-1}(U)$ is an immersion with normal crossings having a double point at p for some neighborhood U of p in N. Choosing U smaller if necessary, we may assume that $(U, U \cap g(M))$ is homeomorphic to $(\mathrm{R}^{m+1}, H_1 \cup H_2)$, where H_1 and H_2 are distinct hyperplanes of R^{m+1}. Furthermore, choosing g sufficiently close to f in $f^{-1}(U)$, we may assume that $(f^{-1}(U), f^{-1}(U) \cap A)$ is homeomorphic to $(f^{-1}(U), f^{-1}(U) \cap \overline{M(g)})$ by a homeomorphism which is close to the identity map of $f^{-1}(U)$. Since the class $u \in H_{m-1}(A)$ is constructed from $[\overline{M(g)}]$, we see easily that $u \neq 0$, since the image of u in $H_{m-1}(A, A - p')$ is non-zero, where p' is a point in A which corresponds to p in $f(A)$. This completes the proof.

We note that, as a corollary to Proposition 6.4, we obtain a converse of the Jordan-Brouwer theorem for codimension-1 generic maps, i.e., Corollary 4.6.

REFERENCES

1. C. Biasi, W. Motta and O. Saeki, *A note on separation properties of codimension-1 immersions with normal crossings*, Topology Appl. 52 (1993), 81–87.
2. C. Biasi, W. Motta and O. Saeki, *A remark on the separation by immersions in codimension-1*, Topology Appl. 61 (1995), 179–186.
3. C Biasi and M. C. Romero Fuster, *A converse of the Jordan-Brouwer theorem*, Illinois J. Math. 36 (1992), 500–504.
4. C. Biasi and O. Saeki, *On the Betti number of the image of a codimension-k immersion with normal crossings*, Proc. Amer. Math. Soc. 123 (1995), 3549–3554.
5. C. Biasi and O. Saeki, *On the Betti number of the image of a generic map*, to appear in Comment. Math. Helv.
6. K. Borsuk, *Theory of Retracts*, Monografie Matematyczne, Tom 44, PWN – Polish Scientific Publishers, Warszawa, 1967.
7. P. E. Conner and E. E. Floyd, *Differentiable Periodic Maps*, Ergeb. d. Math. Grenzgeb., band 33, 1964.
8. M. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. 17 (1982), 357–453.

9. C. G. Gibson, K. Wirthmüller, A. A. du Plessis and E. J. N. Looijenga, *Topological Stability of Smooth Mappings*, Lecture Notes in Math. 552, 1976.
10. R. M. Goresky, *Triangulation of stratified objects*, Proc. Amer. Math. Soc. 72 (1978), 193–200.
11. M. J. Greenberg, *Lectures on Algebraic Topology*, Mathematics Lecture Notes, W. A. Benjamin, Inc., New York, Amsterdam, 1967.
12. R. Herbert, *Multiple Points of Immersed Manifolds*, Mem. Amer. Math. Soc. 34 (250), 1981.
13. M. D. Hirsch, *The complement of a codimension-k immersion*, Proc. Camb. Philos. Soc. 107 (1990), 103–107.
14. S. Izumiya and W. L. Marar, *On topologically stable singular surfaces in a 3-manifold*, J. Geometry 52 (1995), 108–119.
15. K. Kuga, *Representing homology classes of* $S^2 \times S^2$, Topology 23 (1984). 133–137.
16. R. Lashof and S. Smale, *Self-intersections of immersed manifolds*, J. Math. and Mech. 8 (1959), 143–157.
17. J. Milnor and J. Stasheff, *Characteristic Classes*, Ann. of Math. Stud. no. 76, 1974.
18. J. J. Nuño Ballesteros and M. C. Romero Fuster, *Separation properties of continuous maps in codimension 1 and geometrical applications*, Topology Appl. 46 (1992), 107–111.
19. F. Ronga, *'La classe duale aux points doubles' d'une application*, Compositio Math. 27 (1973), 223–232.
20. O. Saeki, *Separation by a codimension-1 map with a normal crossing point*, Geom. Dedicata 57 (1995), 235–247.
21. O. Saeki, *A converse of the Jordan-Brouwer Theorem for quasi-regular immersions*, Illinois J. Math 40 (1996), 108–114.
22. E. H. Spanier, *Algebraic Topology*, TATA McGraw-Hill Publishing Company Ltd., Bombay, New Delhi, 1966.
23. M. Spivak, *A comprehensive Introduction to Differential Geometry*, volume one, second edition, Publish or Perish, Inc., Berkeley, 1979.
24. H. Whitney, *The general type of singularity of a set of* $2n-1$ *smooth functions of n variables*, Duke Math. J. 10 (1943), 161–172.
25. H. Whitney, *The singularities of a smooth n-manifold in* $(2n-1)$*-space*, Ann. of Math. 45 (1944), 247–293.

DEPARTAMENTO DE MATHEMÁTICA
ICMSC-USP
CAIXA POSTAL 668
13560-970 SÃO CARLOS, SP,
BRAZIL
E-mail: biasi@ICMSC.SC.USP.BR

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
HIROSHIMA UNIVERSITY
HIGASHI-HIROSHIMA 739
JAPAN
E-mail: seaki@math.sci.hiroshima-u.ac.jp