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A CHARACTERIZATION OF HARDY-ORLICZ SPACES ON C^n

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1. Introduction.

Recently Stoll [7, Theorem, p. 1032] gave the following result:

THEOREM A. *Let D be a domain in C that has a Green function G, and let φ be a strongly convex function such that $\varphi''(t)$ exists for all $t \in \mathsf{R}$. A holomorphic function $f \in H_\varphi(D)$ if and only if*

$$\iint_D G(t_0, z)\varphi''(\log|f(z)|)\frac{|f'(z)|^2}{|f(z)|^2}\,dx\,dy < \infty$$

for some $t_0 \in D$. Furthermore, if $f \in H_\varphi(D)$ then

$$\lim_{\varepsilon\to 0} \varepsilon \iint_{D'_\varepsilon} \varphi''(\log|f(z)|)\frac{|f'(z)|^2}{|f(z)|^2}\,dx\,dy = 0,$$

where $D'_\varepsilon = \{ z \in D : G(t_0, z) \geq \varepsilon \}$.

Recall that $\psi : \mathsf{R} \to \mathsf{R}$ is strongly convex, if ψ is nonnegative, convex and nondecreasing with $\psi(t)/t \to \infty$ as $t \to \infty$. (As a matter of fact, Stoll does not need this last condition in his proof.) We also recall here the definition of Hardy–Orlicz class. Let $\varphi : \mathsf{R} \to \mathsf{R}$ be a convex, nondecreasing function. Interpret

$$\varphi(-\infty) = \lim_{t\to-\infty} \varphi(t).$$

Let Ω be a domain in C^n, $n \geq 1$. A holomorphic function f in Ω is said to belong to the *Hardy–Orlicz class* $H_\varphi(\Omega)$ if there exists a harmonic function h on Ω for which

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$$\varphi(\log|f(z)|) \leq h(z)$$

for all $z \in \Omega$. Taking $\varphi(t) = e^{pt}$ gives the usual Hardy class $H^p(\Omega)$, $0 < p < \infty$. The *Nevanlinna class* $N(\Omega)$ is the class of holomorphic functions f on Ω for which there is a harmonic function h on Ω for which

$$\log^+|f(z)| \leq h(z) \tag{1}$$

for all $z \in \Omega$. Here we use the notation $\log^+ r = \max\{\log r, 0\}$, when $r > 0$. Observe that (1) is equivalent with the condition that, for any $0 < p < \infty$,

$$\log(1 + |f(z)|^p) \leq h(z)$$

for all $z \in \Omega$. In the sequel we also use the following notation: If D is a domain in R^k, $k \geq 2$, then a subharmonic function u on D is said to belong to the class $S(D)$, if there exists a harmonic function h on D for which $u(x) \leq h(x)$ for all $x \in D$. Observe that we consider the function $-\infty$ as subharmonic. The *Riesz measure* of a subharmonic function u will be denoted by μ_u.

Restricting then attention to domains D in C, whose Green function $G(t,z)$ is comparable to $\delta(z)$, the distance of z to the boundary of D, Stoll got the following corollary to Theorem A:

Let D be a domain in C, $D \neq \mathsf{C}$. A holomorphic function $f \in H_\varphi(D)$ if and only if

$$\iint_D \delta(z)\varphi''(\log|f(z)|)\frac{|f'(z)|^2}{|f(z)|^2}\,dx\,dy < \infty. \tag{2}$$

Furthermore, if $f \in H_\varphi(D)$ then

$$\lim_{s\to 0} s\iint_{D_s} \varphi''(\log|f(z)|)\frac{|f'(z)|^2}{|f(z)|^2}\,dx\,dy = 0, \tag{3}$$

where $D_s = \{z \in D : \delta(z) \geq s\}$.

Since in the case of the unit disc U one has $\delta(z) = 1 - |z|$, the relations (2) and (3) contain Yamashita's results [12, Theorem 1, p. 69] and [13, Theorem 3, p. 116].

Using the invariant gradient Stoll gave in a subsequent paper [8] analogous results to the cited Yamashita's results for functions in the Hardy class $H^p(B)$ on the unit ball B in C^n, $n \geq 1$. Stoll's main result there was [8, Theorem 1, p. 127]:

THEOREM B. *A holomorphic function f belongs to $H^p(B)$, $0 < p < \infty$, if and only if*

$$\int_B (1-|z|^2)^n |f(z)|^{p-2} |\widetilde{\nabla} f(z)|^2 \, d\lambda(z) < \infty.$$

Furthermore, if $f \in H^p(B)$ then

$$\lim_{r\to 1}(1-r^2)^n \int_{B_r} |f(z)|^{p-2} |\widetilde{\nabla} f(z)|^2 \, d\lambda(z) = 0,$$

where $B_r = \{ z \in B : |z| < r \}$. Here $\widetilde{\nabla}$ denotes the invariant gradient on B,

$$|\widetilde{\nabla} f(z)|^2 = \frac{4}{n+1}(1-|z|^2)\left(\sum_{j=1}^{n}\left|\frac{\partial f}{\partial z_j}(z)\right|^2 - \left|\sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z)\right|^2\right),$$

and λ is the invariant measure on B,

$$d\lambda(z) = \frac{n+1}{(1-|z|^2)^{n+1}} \, dm(z).$$

Observe that Beatrous and Burbea [**1**, Theorem 5.2, p. 49, and Corollary 5.3, p. 50] have also obtained related results.

Stoll's proofs to Theorem A and to the results (2) and (3) above rely strongly on the fact that $n = 1$, and cannot thus be generalized for $n > 1$, see [8, p. 136]. In Theorem 1 below we present a proof which applies for all $n \geq 1$. Our proof also avoids the use of Green's formula, unlike Stoll's proof. (Observe that Stoll assumes in Theorem A that $\varphi''(t)$ exists for all $t \in R$, whereas the standard form of Green's formula requires the considered functions to be $\mathscr{C}^2$; however, there exist more general forms of Green's formula!) Our result, Theorem 1, improves Stoll's Theorem A not only by giving the result for all $n \geq 1$, but also by weakening the differentiability assumptions of φ.

Stoll also conjectures [8, p. 136] that for φ as in Theorem A, $f \in H_\varphi(B)$ if and only if

$$\int_B (1-|z|^2)^n \varphi''(\log|f(z)|) \frac{|\widetilde{\nabla} f(z)|^2}{|f(z)|^2} \, d\lambda(z) < \infty. \tag{4}$$

Below in Corollary 2 we verify the if part of this conjecture (and, in fact, for more general φ) by proving, in Corollary 1, that $f \in H_\varphi(\beta)(f \not\equiv 0)$ if and only if

$$(5) \qquad \int_B (1-|z|)\varphi''(\log|f(z)|)\,\frac{\sum_{j=1}^n |\frac{\partial f}{\partial z_j}(z)|^2}{|f(z)|^2}\,dm(z) < \infty.$$

Observe that the condition (4) is stronger than our condition (5). Moreover, our condition (5) applies also to more general domains than just balls: Only replace in $1-|z|$ by $\delta(z)$ in (5).

The notation we use is fairly standard. In addition to the notation already given, we mention here only the following. We use the standard convention $0\cdot\infty = 0$. The Lebesgue measure on R^k is denoted by m_k or just by m. Moreover, if $A \subset \mathsf{R}^k$ is measurable, then we write $|A| = m_k(A)$. The characteristic function of a set A is denoted by χ_A. The α-dimensional Hausdorff measure is denoted by $\mathscr{H}^\alpha$. We write $z = x+iy = (x,y)$ for points of the complex plane C. The terms „harmonic" and „subharmonic" are with regard to the usual laplacian Δ in R^k, $k \geq 2$. Similarly is the Green function for a domain of R^k, $k \geq 2$.

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2. Hardy–Orlicz class and Nevanlinna class.

The following lemma is more or less known; for the proof see [7, Proposition 1, p. 1033].

Lemma 1. *Let D be a domain in R^k, $k \geq 2$, with Green function G. Let $u \not\equiv -\infty$ be subharmonic in D. Then $u \in S(D)$ if and only if there is $w_0 \in D$ such that*

$$(6) \qquad \int_D G(w_0,z)\,d\mu_u(z) < \infty.$$

The next lemma is just a special case for continuous, nondecreasing functions of the classical de la Vallée Poussin's Decomposition Theorem, see [6, Theorem (9.6), p. 127, and Theorem (13.3), p. 100] or also [10, Théorème 53, p. 475].

Lemma 2. *Suppose $F : (a,b) \to \mathsf{R}$ is continuous and nondecreasing. Let*

$$E_\infty = \{\, t \in (a,b) \;:\; F'(t) \text{ exists, and } F'(t) = +\infty \,\}.$$

Then for all a_1 and t, $a < a_1 \leq t < b$,

$$F(t) - F(a_1) = |F([a_1, t] \cap E_\infty)| + \int_{a_1}^{t} F'(t)\, dt.$$

Thus, if $|F(E_\infty)| = 0$, then F is absolutely continuous.

Also the next lemma is due to de la Vallée Poussin, see [10, pp. 467-468] and also [15, p. 48].

LEMMA 3. *Let* $u : (a,b) \to \mathsf{R}$ *and* $g : (c,d) \to \mathsf{R}$ *be absolutely continuous. Suppose that* $u\big((a,b)\big) \subset (c,d)$. *Then* $g \circ u : (a,b) \to \mathsf{R}$ *is absolutely continuous if and only if* $(g' \circ u)u' \in \mathscr{L}^1\big((a,b)\big)$, *and then the chain rule*

$$\frac{d}{dt}\big[g\big(u(t)\big)\big] = g'\big(u(t)\big)u'(t)$$

is valid almost everywhere in (a,b).

For the next lemma see [7, Proposition 2, p. 1034]:

LEMMA 4. *Let* D *be a domain in* R^k, $k \geq 2$, *with Green function* G. *If* $g \in S(D)$ *and* $g \not\equiv -\infty$, *then*

$$\lim_{\varepsilon \to 0} \varepsilon \mu_g\big(D'_\varepsilon(w_0)\big) = 0$$

for all $w_0 \in D$, *where* $D'_\varepsilon(w_o) = \{\, z \in D \,:\, G(z, w_o) \geq \varepsilon \,\}$.

In the sequel we will use the following notation. Let φ and Ω be as in Theorem 1 below. If f is holomorphic in Ω, then we write $f^\#_\varphi : \Omega \to \mathsf{R}$,

$$f^\#_\varphi(z) = \begin{cases} \varphi''\big(\log|f(z)|\big)\, \dfrac{\sum_{j=1}^n \left|\dfrac{\partial f}{\partial z_j}(z)\right|^2}{|f(z)|^2}, & \text{if this expression is defined,} \\ 0, & \text{otherwise.} \end{cases}$$

In the proof of Theorem 1, we will show that $\Delta[\varphi(\log|f(z)|)]$ exists for a.e. $z \in \Omega$ and is equal to $f^\#_\varphi(z)$ for a.e. $z \in \Omega$.

We then give our main result:

THEOREM 1. *Let* $\varphi : \mathsf{R} \to \mathsf{R}$ *be a nondecreasing, convex function that is bounded from below and differentiable. Let* E_∞ *be the set of points* t *in* R *for which* $\varphi''(t)$ *exists and equals* $+\infty$ *(and which set is of Lebesgue measure zero). Suppose further that* $|\varphi'(E_\infty)| = 0$. *Let* Ω *be a domain in* C^n, $n \geq 1$, *with Green function* G. *Then a holomorphic function* f *on* Ω, $f \not\equiv 0$, *belongs to the Hardy–Orlicz class* $H_\varphi(\Omega)$ *if and only if*

$$\int_{\Omega} G(w_0, z) f_{\varphi}^{\#}(z)\, dm(z) < \infty \tag{7}$$

for some $w_0 \in \Omega$. *Furthermore, if* $f \in H_{\varphi}(\Omega)$, $f \not\equiv 0$, *then*

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\Omega'_{\varepsilon}} f_{\varphi}^{\#}(z)\, dm(z) = 0,$$

where $\Omega'_{\varepsilon} = \{ z \in \Omega : G(w_0, z) \geq \varepsilon \}$.

PROOF. We show first that $f \in H_{\varphi}(\Omega)$ if and only if the condition (7) holds. Recall that $f \in H_{\varphi}(\Omega)$ if and only if $g(z) = \varphi(\log |f(z)|)$ belongs to $S(\Omega)$. Since by Lemma 1 we know that $f \in H_{\varphi}(\Omega)$ if and only if

$$\int_{\Omega} G(w_0, z)\, d\mu_g(z) < \infty$$

for some $w_0 \in \Omega$, it is sufficient to show that

$$\int_{\Omega} G(w_0, z)\, d\mu_g(z) = \int_{\Omega} G(w_0, z) f_{\varphi}^{\#}(z)\, dm(z).$$

Using then the definition of the Riesz measure μ_g of g, we see that it is sufficient to show that

$$\int_{\Omega} g(z) \Delta\psi(z)\, dm(z) = \int_{\Omega} f_{\varphi}^{\#}(z) \psi(z)\, dm(z) \tag{8}$$

for all $\psi \in \mathscr{C}_0^{\infty}(\Omega)$. The method in proving (8) is to use Fubini's theorem, Federer's results and partial integration. Observe already now that the left hand side of (8) is finite, since $\operatorname{spt}\psi$ (support of ψ) is compact.

We first introduce some notation. Write

$$E_0 = \{ z \in \Omega : f(z) = 0 \}, \quad E_j = \left\{ z \in \Omega : \frac{\partial f}{\partial z_j}(z) = 0 \right\}, \quad j = 1, \dots, n,$$

$$E = E_0 \cup E_1 \cup \dots \cup E_n.$$

Since f is holomorphic, then for each $j = 0, 1, \dots, n$ either $E_j = \Omega$ or $\mathscr{H}^{2n-2}(E_j \cap K) < \infty$ for each compact $K \subset \Omega$. If $E_1 = \Omega$, say, then $\dfrac{\partial f}{\partial z_1} \equiv 0$, and so f and thus also g does not depend on z_1. But then

$$\int_\Omega g(z_1,z_2,\dots,z_n)\Delta_{z_1}\psi(z_1,z_2,\dots,z_n)\,dm(z_1,z_2,\dots,z_n)$$

$$= \int_\Omega \varphi''\big(\log|f(z_1,z_2,\dots,z_n)|\big)\frac{\left|\dfrac{\partial f}{\partial z_1}(z_1,z_2,\dots,z_n)\right|^2}{|f(z_1,z_2,\dots,z_n)|^2}\psi(z_1,z_2,\dots,z_n)\, dm(z_1,z_2,\dots,z_n) = 0.$$

Therefore, remembering also that $f \not\equiv 0$, we may suppose that $\mathscr{H}^{2n-2}(E\cap K) < \infty$ for all $K \subset \Omega$ compact.

For the sake of clarity we divide the rest of the proof into separate steps.

Step 1. $\varphi''\big(\log|f(z)|\big) \geq 0$ *for* a.e. $z \in \Omega$.
Write

$\Omega_1 = \{ z\in\Omega : \varphi''\big(\log|f(z)|\big)$ is defined, and $\varphi''\big(\log|f(z)|\big) \geq 0 \}$, $\quad F = \Omega\setminus\Omega_1$.
Since φ is convex (and differentiable), φ'' exists and $\varphi'' \geq 0$ in $\mathsf{R}\setminus K_1$, where $|K_1| = 0$. Write

$$K_2 = \{ e^x \in \mathsf{R} : x\in K_1 \},\quad K_3 = \{ w \in \mathsf{C} : |w| \in K_2 \},\quad F_1 = f^{-1}(K_3).$$

Then $F\subset F_1$. One sees easily that $|K_2| = 0$ and $|K_3| = 0$. To see that $|F_1| = 0$, we proceed as follows.

Take an arbitrary domain Ω_1 such that $\overline{\Omega_1}\subset\Omega$ is compact. Write $F_2 = F_1\cap\Omega_1$, $E_1 = E\cap\Omega_1$, and

$$H = \{ (z_2,\dots,z_n) \in \mathsf{C}^{n-1} \ :\ (\mathsf{C}\times\{(z_2,\dots,z_n)\})\cap E_1 \text{ is finite} \}.$$

Since $\mathscr{H}^{2n-2}(E_1) < \infty$, it follows from [2, 2.10.25, p. 188] that $\mathscr{H}^{2n-2}(\mathsf{C}^{n-1}\setminus H) = 0$. Since F_2 is measurable, it follows from Fubini's theorem that

$$|F_2| = \int \chi_{F_2}(z)\,dm(z) = \int_H \left[\int_{\mathsf{C}} \chi_{F_2}(z_1,z_2,\dots,z_n)\,dm(z_1)\right] dm(z_2,\dots,z_n).$$

If we show that for all $(z_2,\dots,z_n)\in H$,

$$\int_{\mathsf{C}} \chi_{F_2}(z_1,z_2,\dots,z_n)\,dm(z_1) = 0,$$

it follows that $|F_2| = 0$, and thus also $|F_1| = 0$, since Ω_1 was arbitrary. So take $(z_2,\dots,z_n)\in H$ arbitrarily, and write

$\Omega_{11} = \{ z_1 \in \mathbf{C} : (z_1, z_2, \dots, z_n) \in \Omega_1 \}$, $F_{21} = \{ z_1 \in \mathbf{C} : (z_1, z_2, \dots, z_n) \in F_2 \}$, $E_{11} = \{ z_1 \in \mathbf{C} : (z_1, z_2, \dots, z_n) \in E_1 \}$.

Since $\chi_{F_{21}}(z_1) = \chi_{F_2}(z_1, z_2, \dots, z_n)$ for all $z_1 \in \mathbf{C}$, it is sufficient to show that $|F_{21}| = 0$. Since $F_{21} \subset \Omega_{11}$ and $E_{11} \subset \Omega_{11}$ is finite, it is sufficient to show that each point $z_1^o \in \Omega_{11} \setminus E_{11}$ has a neighborhood $U_{z_1^o} \subset \Omega_{11}$ such that $|F_{21} \cap U_{z_1^o}| = 0$. For this purpose take $z_1^o \in \Omega_{11} \setminus E_{11}$ arbitrarily, and define $h : \Omega_{11} \to \mathbf{C}$, $h(z_1) = f(z_1, z_2, \dots, z_n)$. Since $z_1^o \notin E_{11}$,
$h'(z_1^o) = \dfrac{\partial f}{\partial z_1}(z_1^o, z_2, \dots, z_n) \neq 0.$
Thus there is a neighborhood $U_{z_1^o}$ of z_1^o such that $h_1 = h|U_{z_1^o}$ is biholomorphic. Since $F_{21} \cap U_{z_1^o} = h_1^{-1}(K_3 \cap h(U_{z_1^o}))$, we see that indeed $|F_{21} \cap U_{z_1^o}| = 0$, concluding the proof of Step 1.

Before going to Step 2, we observe, that using Fubini's theorem, our decisive claim (8) above can be written into the following form:

$$\begin{aligned}
&\int \cdots \int \left[\int g(z) \Delta_{z_1} \psi(z)\, dm(z_1) \right] dm(z_2, \dots, z_n) + \dots \\
&\quad + \int \cdots \int \left[\int g(z) \Delta_{z_n} \psi(z)\, dm(z_n) \right] dm(z_1, \dots, z_{n-1}) \\
&= \int \cdots \int \left[\int \varphi''(\log|f(z)|) \frac{\left| \frac{\partial f}{\partial z_1}(z) \right|^2}{|f(z)|^2} \psi(z)\, dm(z_1) \right] dm(z_2, \dots, z_n) + \dots \\
&\quad + \int \cdots \int \left[\int \varphi''(\log|f(z)|) \frac{\left| \frac{\partial f}{\partial z_n}(z) \right|^2}{|f(z)|^2} \psi(z)\, dm(z_n) \right] dm(z_1, \dots, z_{n-1}).
\end{aligned}$$

Using moreover [2, 2.10.25, p. 188] (in the way indicated above in the proof of Step 1), we may therefore suppose that $n = 1$ and that E is locally finite in Ω (and also by Step 1 above, $\varphi''(\log|f(z)|) \geq 0$ for a.e. $z \in \Omega$). Since g is bounded from below and $\operatorname{spt}\psi$ is compact, we see that

$$\iint_\Omega g(z) \Delta\psi(z)\, dm(z) = \iint_\Omega \varphi''(\log|f(z)|) \frac{|f'(z)|^2}{|f(z)|^2} \psi(z)\, dm(z)$$

if and only if

$$(9) \qquad \iint_{\Omega} g(x,y)\psi_{xx}(x,y)\,dx\,dy = \iint_{\Omega} g_{xx}(x,y)\psi(x,y)\,dx\,dy,$$
$$\iint_{\Omega} g(x,y)\psi_{yy}(x,y)\,dx\,dy = \iint_{\Omega} g_{yy}(x,y)\psi(x,y)\,dx\,dy$$

for each $\psi \in \mathscr{C}_0^\infty(\Omega)$, and if moreover

$$(10) \qquad \Delta g(z) = g_{xx}(x,y) + g_{yy}(x,y) = \varphi''(\log|f(z)|)\,\frac{|f'(z)|^2}{|f(z)|^2}$$

for a.e. $z \in \Omega$. It is clearly sufficient to show that (9) and (10) hold.

Write

$$b_1 = \inf\{\, y \in \mathsf{R} : (\mathsf{R} \times \{y\}) \cap \text{ spt } \psi \neq \emptyset\,\},$$
$$b_2 = \sup\{\, y \in \mathsf{R} : (\mathsf{R} \times \{y\}) \cap \text{ spt}\psi \neq \emptyset\,\}.$$

Step 2. *If*

$$B_1 = \{\, y \in [b_1, b_2] : (\mathsf{R} \times \{y\}) \cap E = \emptyset\,\},$$

then $m_1([b_1, b_2] \setminus B_1) = 0$.

This is clear, since E is locally finite in Ω.

For $y \in \mathsf{R}$, set $A(y) = \{\, x \in \mathsf{R} : (x,y) \in \text{ spt } \psi\}$ and $\Omega(y) = \{\, x \in \mathsf{R} : (x,y) \in \Omega\,\}$.

Step 3. *If*

$$B_2 = \{\, y \in [b_1, b_2] : \varphi''(\log|f(x,y)|) \geq 0 \textit{ for a.e. } x \in \Omega\ (\mathrm{y})\,\},$$

then $m_1([b_1, b_2] \setminus B_2) = 0$.

This follows by Fubini's theorem and by Step 1 above.

Step 4. *If*

$$B_3 = \Big\{\, y \in [b_1, b_2]: \textit{ there is a bounded open set } A'(y) \textit{ such that } A(y) \subset A'(y)$$
$$\subset \overline{A'(y)} \subset \Omega(y) \quad \textit{and} \int_{A'(y)} \varphi''(\log|f(x,y)|)\,\frac{|f'(x,y)|^2}{|f(x,y)|^2}\,dx < \infty \Big\},$$

then $m_1([b_1, b_2] \setminus B_3) = 0$.

In the sequel we will use the following convention: If $y \in B_3$, *then* $A'(y)$ *means always a set described in the above definition of* B_3.

For the proof of Step 4, take an arbitrary $y_0 \in B_1 \cap B_2$. Since $A(y_0)$ is compact and E is closed in Ω, there are real numbers $b_1' = b_1'(y_0)$, $b_2' = b_2'(y_0)$ such that $b_1' < y_0 < b_2'$ and that

$$K := ([b'_1, b'_2] \times \mathrm{R}) \cap \operatorname{spt}\psi \subset \Omega \setminus E.$$

Since K is compact, we find a finite number of discs $B(z_j, r_j)$, $j = 1, \ldots, m$, such that

$$K \subset U := B(z_1, r_1) \cup \ldots \cup B(z_m, r_m) \subset \overline{U} \subset \Omega \setminus E, \tag{11}$$

and that $f|B(z_j, r_j)$ is biholomorphic, $j = 1, \ldots, m$. By (11) we find constants c^* and C^* such that

$$0 < c^* \leq |f(z)| \leq C^* < \infty$$

whenever $z \in \overline{U}$. We have then for each $j = 1, \ldots, m$,

$$\begin{aligned} \iint_{B(z_j,r_j)} \varphi''(\log|f(z)|)\frac{|f'(z)|^2}{|f(z)|^2}\,dx\,dy &= \iint_{f(B(z_j,r_j))} \varphi''(\log|w|)\frac{1}{|w|^2}\,d\xi\,d\eta \\ &\leqq \int_{c^*}^{C^*}\int_0^{2\pi} \varphi''(\log r)\frac{1}{r}\,dr\,d\theta = 2\pi\int_{c^*}^{C^*}\varphi''(\log r)\frac{dr}{r} \\ &= 2\pi\int_{\log c^*}^{\log C^*}\varphi''(t)\,dt \leq 2\pi[\varphi'(\log C^*) - \varphi'(\log c^*)] < \infty. \end{aligned}$$

Above we have used the fact that φ' is nondecreasing, and also the notation $w = \xi + i\eta$. Hence

$$\iint_U \varphi''(\log|f(z)|)\frac{|f'(z)|^2}{|f(z)|^2}\,dx\,dy < \infty.$$

But then we see with the aid of Fubini's theorem that for a.e. $y \in [b'_1, b'_2]$ there is a bounded open set $A'(y)$ such that $A(y) \subset A'(y) \subset \overline{A'(y)} \subset \Omega(y)$ and

$$\int_{A'(y)} \varphi''(\log|f(x,y)|)\frac{|f'(x,y)|^2}{|f(x,y)|^2}\,dx < \infty.$$

(For example, the set $U(y) = \{x \in R : (x,y) \in U\}$ is such an open bounded set.) Since $y_0 \in B_1 \cap B_2$ was arbitrary, it follows that for a.e. $y \in [b_1, b_2]$ there is a bounded open set $A'(y)$ such that $A(y) \subset A'(y) \subset \overline{A'(y)} \subset \Omega(y)$ and

$$\int_{A'(y)} \varphi''(\log|f(x,y)|)\frac{|f'(x,y)|^2}{|f(x,y)|^2}\,dx < \infty.$$

Thus $m_1([b_1, b_2] \setminus B_3) = 0$.

Write then $B^* = B_1 \cap B_2 \cap B_3$. It follows from Steps 2, 3 and 4 that $m_1([b_1, b_2] \setminus B^*) = 0$.

Step 5. *The function*

$$t \mapsto \varphi'(t)$$

is absolutely continuous on each finite interval $I \subset \mathsf{R}$.

This follows from Lemma 2, because of the facts that by [5, Corollary 25.5.1, p. 246] φ' is continuous, that φ' is clearly nondecreasing and that $|\varphi'(E_\infty)| = 0$, by the assumption of our theorem. (Another, rather short possibility to see the absolute continuity of φ' is just to refer to [9, Theorem (45.2), p. 105, and Corollary (43.4), p. 103].)

Let $y \in B^*$ be fixed. Write $A = A(y)$ and $A' = A'(y)$. Since A' is open in R, one can write $A' = \cup_{j=1}^{\infty} A'_j$, where A'_j, $j = 1, 2, \ldots$, are pairwise disjoint and open intervals in R.

Step 6. *The function*

$$x \mapsto g_x(x,y) = \varphi'\left(\tfrac{1}{2}\log[u(x,y)^2 + v(x,y)^2]\right) \frac{u(x,y)u_x(x,y) + v(x,y)v_x(x,y)}{u(x,y)^2 + v(x,y)^2} \tag{12}$$

is absolutely continuous on each A'_j, $j = 1, 2, \ldots$. *Moreover, for* a.e. $x \in A'$,

$$g_{xx} = \{\varphi''(\log|f|)(uu_x + vv_x)^2 + \varphi'(\log|f|)[(uu_{xx} + vv_{xx})(u^2 + v^2) - (u_x^2 + v_x^2)(u^2 + v^2) + 2(uu_x - vv_x)^2]\}/(u^2 + v^2)^2. \tag{13}$$

Fix j arbitrarily and write $A'' = A'_j$. Since the function

$$x \mapsto \frac{u(x,y)u_x(x,y) + v(x,y)v_x(x,y)}{u(x,y)^2 + v(x,y)^2}$$

is $\mathscr{C}^\infty$ and the set $\overline{A'}$ is compact, this function is also absolutely continuous on A'. Hence, in order to prove that the function (12) is absolutely continuous on A'', it is sufficient to prove that the function

$$x \mapsto \varphi'\left(\tfrac{1}{2}\log[u(x,y)^2 + v(x,y)^2]\right) \tag{14}$$

is absolutely continuous on A''.

To see this, it is by Lemma 3 sufficient to show that the function

$$x \mapsto \varphi''\left(\tfrac{1}{2}\log[u(x,y)^2 + v(x,y)^2]\right) \frac{u(x,y)u_x(x,y) + v(x,y)v_x(x,y)}{u(x,y)^2 + v(x,y)^2}$$

is integrable over A'. (Observe that this function is defined a.e. in A', since $y \in B_2$.) The other assumptions of Lemma 3 are namely satisfied: the function

$$x \mapsto \tfrac{1}{2}\log[u(x,y)^2 + v(x,y)^2]$$

is absolutely continuous on A', since it is $\mathscr{C}^\infty$ and $\overline{A'}$ is compact; also the function $t \mapsto \varphi'(t)$ is absolutely continuous on each finite interval $I \subset \mathsf{R}$, by Step 5.

Since $y \in B^* \subset B_1 \cap B_3$, we know that

$$\int_{A'} \varphi''(\log|f(x,y)|)\frac{|f'(x,y)|^2}{|f(x,y)|^2}\,dx < \infty, \tag{15}$$

and that there is m_1 such that

$$|f'(x,y)| \geq m_1 > 0 \tag{16}$$

for all $x \in A'$. Using moreover the facts that the functions

$$x \mapsto u(x,y), \quad x \mapsto u_x(x,y), \quad x \mapsto v(x,y), \quad x \mapsto v_x(x,y)$$

are $\mathscr{C}^\infty$ and the set $\overline{A'}$ is compact, we see with the aid of (15) and (16) that

$$\int_{A'} \varphi''(\log|f(x,y)|)\frac{|u(x,y)u_x(x,y) + v(x,y)v_x(x,y)|}{u(x,y)^2 + v(x,y)^2}\,dx < \infty,$$

so the function (14) and thus also the function (12) is absolutely continuous on A'' by Lemma 3.

By Lemma 3 one can use the chain rule when computing the derivative of the function (14). Using this and other standard rules of differentation, we see that (13) holds, thus concluding the proof of Step 6.

Step 7. $\iint g(x,y)\Delta\psi(x,y)\,dx\,dy = \iint \Delta g(x,y)\psi(x,y)\,dx\,dy$, *and*

$$\Delta g(z) = \varphi''(\log|f(z)|)\frac{|f'(z)|^2}{|f(z)|^2}$$

for a.e. $z \in \Omega \cap \operatorname{spt}\psi$.

To prove the first claim, we integrate by parts twice. For the first partial integration, we observe that the function

$$x \mapsto g(x,y) = \varphi(\tfrac{1}{2}\log[u(x,y)^2 + v(x,y)^2])$$

is in fact a $\mathscr{C}^1$ function. (As above in the proof of Step 5, we see that the function $t \mapsto \varphi'(t)$ is continuous.) Therefore we get for each $j = 1, 2, \ldots,$

$$\int_{A'_j} g(x,y)\psi_{xx}(x,y)\,dx = -\int_{A'_j} g_x(x,y)\psi_x(x,y)\,dx.$$

Because of Step 6 we can once more integrate by parts, and then we get for each $j = 1, 2, \ldots,$

$$\int_{A'_j} g(x,y)\psi_{xx}(x,y)\,dx = \int_{A'_j} g_{xx}(x,y)\psi(x,y)\,dx.$$

Thus also

$$\int_{A'} g(x,y)\psi_{xx}(x,y)\,dx = \int_{A'} g_{xx}(x,y)\psi(x,y)\,dx.$$

Since here $y \in B^*$ was arbitrary, and $m_1([b_1,b_2] \setminus B^*) = 0$, we get, by integrating this with respect to y and using Fubini's theorem,

$$\iint g(x,y)\psi_{xx}(x,y)\,dx\,dy = \iint g_{xx}(x,y)\psi(x,y)\,dx\,dy. \tag{17}$$

Proceeding as in Steps 2 – 6 and interchanging the roles of x and y, one can similarly show that

$$\iint g(x,y)\psi_{yy}(x,y)\,dx\,dy = \iint g_{yy}(x,y)\psi(x,y)\,dx\,dy. \tag{18}$$

Adding then (17) and (18), and using also (13) and a corresponding formula for g_{yy}, one sees that Step 7 holds, thus concluding the proof of the first part of Theorem 1.

To prove the second part of Theorem 1, suppose that $f \in H_\varphi(\Omega)$ and $f \not\equiv 0$. But then $g \in S(\Omega)$ and $g \not\equiv -\infty$, and so by the above Lemma 4,

$$\lim_{\varepsilon\to 0} \varepsilon\mu_g\big(\Omega'_\varepsilon(w_0)\big) = 0 \tag{19}$$

for all $w_0 \in \Omega$, where $\Omega'_\varepsilon(w_0) = \{\, z \in \Omega : G(w_0,z) \geq \varepsilon \,\}$. On the other hand, we know by the above proof that

$$\mu_g\big(\Omega'_\varepsilon(w_0)\big) = \int_{\Omega'_\varepsilon(w_0)} f_\varphi^{\#}(z)\,dm(z).$$

So combining this with (19), we see that the proof of Theorem 1 is complete.

COROLLARY 1. *Let φ be as in Theorem 1. Let Ω be a domain in $\mathbf{C}^n$, $\Omega \neq \mathbf{C}^n$, $n \geq 1$, with Green function G satisfying the following condition: For each $w_0 \in \Omega$ there are positive constants $c_1 = c_1(w_0)$ and $c_2 = c_2(w_0)$ such that*

$$c_1\delta(z) \leq G(w_0,z) \leq c_2\delta(z) \tag{20}$$

for all $z \in \Omega \setminus B\big(w_0, \frac{1}{2}\delta(w_0)\big)$. *Then a holomorphic function* f *on* Ω, $f \not\equiv 0$, *belongs to the Hardy–Orlicz class* $H_\varphi(\Omega)$ *if and only if*

$$\int_\Omega \delta(z) f_\varphi^\#(z)\, dm(z) < \infty.$$

Furthermore, if $f \in H_\varphi(\Omega)$ *and* $f \not\equiv 0$, *then*

$$\lim_{s\to 0} s \int_{\Omega_s} f_\varphi^\#(z)\, dm(z) = 0,$$

where $\Omega_s = \{ z \in \Omega : \delta(z) \geq s \}$.

The proof follows immediately from Theorem 1 (where the Riesz measure of the function $g(z) = \varphi(\log|f(z)|)$ was computed) and from [7, Proposition 3, p. 1035], since its proof extends verbatim to the general case of R^k, $k \geq 2$.

REMARK. There are a lot of important examples of domains which satisfy the condition (20) in Corollary 1. To name just a few of such domains, we mention $\mathscr{C}^2$ domains and Liapunov–Dini domains, see [3, pp. 325–331], [11, Theorems 2.3 and 2.5, pp. 21, 27], [14, Lemma, p. 117] and also [4, Theorem 1, p. 269, and (vi), p. 270].

Next we give another corollary to Theorem 1, a special case of which verifies the if part of the cited conjecture of Stoll [8, p. 136]:

COROLLARY 2. *Let* φ *be as in Theorem 1. Then a holomorphic function* f *on* B, $f \not\equiv 0$, *belongs to the Hardy–Orlicz class* $H_\varphi(B)$ *if*

$$\int_B (1 - |z|^2)^n \varphi''(\log|f(z)|) \frac{|\widetilde{\nabla} f(z)|^2}{|f(z)|^2}\, d\lambda(z) < \infty. \tag{21}$$

The integrand in (21) *is defined to be* 0 *in the case when its expression is not defined.*

PROOF. Using the Cauchy–Schwarz inequality we get

$$|\widetilde{\nabla} f(z)|^2 = \frac{4}{n+1}(1 - |z|^2)\left(\sum_{j=1}^n \left|\frac{\partial f}{\partial z_j}(z)\right|^2 - \left|\sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)\right|^2\right)$$

$$\geq \frac{4}{n+1}(1 - |z|^2)^2 \left(\sum_{j=1}^n \left|\frac{\partial f}{\partial z_j}(z)\right|^2\right).$$

By this inequality, by (21) and by the definition of the invariant measure λ, one obtains easily

$$\int_B (1 - |z|^2) f_\varphi^\#(z)\, dm(z) < \infty.$$

But then it follows from Corollary 1 that $f \in H_\varphi(B)$, concluding the proof.

As the last result we give a characterization for the Nevanlinna class, see also [7, Proposition 4, p. 1037] and [8, Proposition 6, p. 136]:

THEOREM 2. *Let Ω be a domain in $\mathbb{C}^n$, $n \geq 1$. A holomorphic function f in Ω belongs to the Nevanlinna class $N(\Omega)$ if and only if for some $w_0 \in \Omega$,*

$$\int_\Omega G(w_0, z) \frac{\sum_{j=1}^n |\frac{\partial f}{\partial z_j}(z)|^2}{(1 + |f(z)|^2)^2}\, dm(z) < \infty. \tag{22}$$

PROOF. As observed before, $f \in N(\Omega)$ if and only if the subharmonic function $g_1(z) = \log(1 + |f(z)|^2)$ has a harmonic majorant in Ω. Since g_1 is a $\mathscr{C}^\infty$ function, one sees by a simple computation that

$$d\mu_{g_1}(z) = \frac{1}{c_n} \Delta\left[\log(1 + |f(z)|^2)\right] dm(z) = C_n \frac{1}{(1 + |f(z)|^2)^2} \sum_{j=1}^n \left|\frac{\partial f}{\partial z_j}(z)\right|^2 dm(z).$$

Here c_n and C_n are constants depending only on n. Using this equality and Lemma 1 we see that $f \in N(\Omega)$ if and only if (22) holds.

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