MATH. SCAND. 80 (1997), 41–55

ON INDUCTIVE LIMITS OF MATRIX ALGEBRAS OVER HIGHER DIMENSIONAL SPACES, PART I

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Abstract.

Suppose that A is a C*-algebra of real rank zero, and is an inductive limit of $\bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i}))$, where the spaces $X_{n,i}$ are finite CW complexes, and $[n,i]$ are positive integers. In this note, we will prove the following results. (1) If $M_{\mathbb{Q}}$ is the UHF algebra with $K_0(M_{\mathbb{Q}}) = \mathbb{Q}$, then $A \otimes M_{\mathbb{Q}}$ can be expressed as an inductive limit of finite direct sums of matrix algebras over $C(S^1)$. (2) If one supposes further that the cohomology groups $\tilde{H}^*(X_{n,i})$ are torsion free and that $\sup_{n,i}\{\dim(X_{n,i})\} < +\infty$ (one can replace this condition by the condition of slow dimension growth), then A itself can be expressed as an inductive limit of finite direct sums of matrix algebras over $C(S^1)$. Recall that a result of G. Elliott says that the class of C*-algebras of real rank zero, which can be expressed as inductive limits of finite direct sums of matrix algebras over $C(S^1)$, is completely classified by K-theory (graded ordered K-group with dimension range).

1. Introduction.

The study of the inductive limit of

$$A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow \cdots A ,$$

where each $A_n = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i}))$ is a finite direct sum of matrix algebras over finite CW complexes $X_{n,i}$, was first proposed by Effros [Ef], after the complete classification of AF algebras—inductive limits of finite dimensional C^*-algebras, by G. A. Elliott [Ell1] (see also [Glim], [Br]). In the inductive limit, one allows that

$$A_n = \bigoplus_{i=1}^{k_n} P_{n,i} M_{[n,i]}(C(X_{n,i})) P_{n,i} ,$$

where $P_{n,i}$ are projections in $M_{[n,i]}(C(X_{n,i}))$. ,Following the terminology of Blackadar in [Bl1], we call such an inductive limit algebra an AH algebra (strictly speaking, the class of AH algebras also contains the inductive limits of finite direct sums of general homogeneous algebras which may not be of the form $P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}$, i.e., the Dixmier-Douady invariant of homogeneous algebras may not be zero). As pointed out in 4.24 of [EG2], for the purpose of the classification in terms of K-theory, one only needs to study the inductive limits of finite direct sums of full matrix algebras over $X_{n,i}$ (i.e., $M_{[n,i]}(C(X_{n,i}))$), rather than the inductive limits of $\bigoplus_{i=1}^{k_n} P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}$.

Recently, the class of C*-algebras of the above form has been intensively studied by many authors ([Bl1–2], [BBEK], [BDR], [BKR], [D], [DNNP], [DN], [Ell1–5], [EE], [EG1–2], [EGLP1–2], [G1], [GL], [Goo], [Ku], [Li1–2], [Lin1–3], [P1–3], [Phi1–2], [Su1–2], [Th1–2], etc.). In particular, G. A. Elliott classified real rank zero such inductive limit algebras with the spaces $X_{n,i} = S^1$ (Su generalized the result to the case of $X_{n,i}$ = graph), by graded ordered K-group with dimension range. And in [EG2], the classification has been extended to the case that the spaces $X_{n,i}$ are three-dimensional finite CW complexes with one of the following two restrictions: (i) $\tilde{H}^*(X_{n,i})$ are torsion free; or (ii) A is simple.

All these classification results are for the case that the spaces $X_{n,i}$ are CW complexes of lower dimensional (i.e., $\dim X_{n,i} \leq 3$). The classification of inductive limits of matrix algebras over CW complexes with dimensions larger than three is almost completely blank (except for some special cases such as spheres or product of spheres). In this note and the other part of the series (see [G2]),we will fill out this blank.

As proved in [G1], the graded ordered K-group with dimension range is not a complete invariant for general inductive limit C*-algebras of real rank zero, even if it is assumed that $\dim(X_{n,i}) \leq 2$ for all the spaces $X_{n,i}$. So for the classification of real rank zero inductive limit C*-algebras in terms of graded ordered K-group, we need some extra conditions. In this note, we will study the case that A has torsion free K-theory. More precisely, we will prove the following two results:

(1) If A and B are real rank zero AH algebras, then $A \otimes M_{\mathbb{Q}}$ is isomorphic to $B \otimes M_{\mathbb{Q}}$ if and only if

$$\big(K_*(A \otimes M_{\mathbb{Q}}), D_*(A \otimes M_{\mathbb{Q}})\big) \cong \big(K_*(B \otimes M_{\mathbb{Q}}), D_*(B \otimes M_{\mathbb{Q}})\big) ,$$

where $D_*(A) \subset K_*(A)$ is the graded dimension range (see 1.2.1 of [EG2]);

(2) Let A and B be real rank zero inductive limits of algebras $\left(\bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m}\right)$ and $\left(\bigoplus_{i=1}^{l_n} M_{\{n,i\}}(C(Y_{n,i})), \psi_{n,m}\right)$, respectively, with cohomology groups $\tilde{H}^*(X_{n,i})$ $\left(:= \bigoplus_{j=1}^{+\infty} H^j(X_{n,i})\right)$ and $\tilde{H}^*(Y_{n,i})$ torsion free.

Suppose that $\sup_{n,i}\{\dim(X_{n,i})\} < +\infty$ and $\sup_{n,i}\{\dim(Y_{n,i})\} < +\infty$ (or suppose that A and B satisfy the slow dimension growth condition,in the sense that

$$\lim_{m\to\infty} \min_{\mathrm{rank}(\phi^{i,j}_{n,m}(\mathbf{1}_{A^i_n}))\neq 0} \frac{\mathrm{rank}(\phi^{i,j}_{n,m}(\mathbf{1}_{A^i_n}))}{\dim(X_{m,j})} = +\infty ,$$

where $A^i_n = M_{[n,i]}(C(X_{n,i}))$, and $\phi^{i,j}_{n,m}$ are partial maps of $\phi_{n,m}$ from A^i_n to A^j_m). Then A is isomorphic to B if and only if

$$\big(K_*(A), D_*(A)\big) \cong \big(K_*(B), D_*(B)\big) .$$

In part II [G2] of this series, we will deal with the inductive limit algebras without the condition that the cohomology groups $\tilde{H}^*(X_{n,i})$ are torsion free. But we need the condition that the limit algebras are simple.

The classification results mentioned above are proved by expressing the algebras A (or $A \otimes M_{\mathbb{Q}}$) as inductive limits of finite direct sums of matrix algebras over $C(S^1)$ (therefore the results follow from [Ell2]).

In this note, $\tilde{H}^*(X) = \oplus H^j(X)$ is the reduced cohomology group with $\mathbb{Z}$ coefficients. The term *real rank zero* for a C*-algebra refers to the density of the invertible self adjoint elements in the set of self adjoint elements in the algebra.

ACKNOWLEDGMENT. The work was done while the author was in the university of Toronto during the 1993–1994 academic year.

2. The Proofs of the main results

The following theorem is one of the two main results in this note.

THEOREM 2.1. *Suppose that* $A = \lim_{n\to\infty}\big(\bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m}\big)$ *is a real rank zero* AH *algebra with cohomology groups* $\tilde{H}^*(X_{n,i})\Big(:= \bigoplus H^j(X_{n,i})\Big)$ *torsion free and with* $\sup_{n,i}\dim(X_{n,i}) < +\infty$*(this condition can be replaced by the condition of slow dimension growth of A).Then A can be expressed as an inductive limit of finite direct sums of matrix algebras over* $C(S^1)$*,and hence can be classified by its graded ordered K-group with dimension range.*

2.2. When $\dim(X_{n,i}) \leq 3$, the result was proved in [EG2]. In that proof, we make use of the followingtwo facts:

1. The Chern map $K^*(X) \to \tilde{H}^*(X)$ is an isomorphism;
2. The canonical map $\mathrm{kk}(Y, X) \to \mathrm{KK}(C(X), C(Y))$ is injective (see 2.3 below for the definition of $\mathrm{kk}(Y, X)$).

But these facts are no longer true when the spaces X, Y are higher dimensional CW complexes. Fortunately, we have 3.5.5 of [DN] to replace them.

2.3. Before proving our theorem, let us quote some notations and some known results about the connective K-theory and the homomorphisms between $C(X)$ and $M_k(C(Y))$ (see [DN], [D], [EG2]).

In this note, all finite CW complexes are assumed to be connected.

Let X and Y be finite CW complexes.Let $e_{x_0} : C(X) \to \mathbf{C}$ be the map defined by the evaluation at the base point $x_0 \in X$. The correspondence $\phi \mapsto \phi \oplus e_{x_0}$ induces a map

$$\left[C(X), M_n(C(Y))\right]_1 \longrightarrow \left[C(X), M_{n+1}(C(Y))\right]_1 ,$$

where $[A, B]_1$ denotes the set of homotopy classes of unital homomorphisms from A to B. Taking direct limit over n, we define

$$\mathrm{kk}(Y, X) = \lim_{n \to \infty} \left[C(X), M_n(C(Y))\right]_1 .$$

$\mathrm{kk}(Y, X)$ is a group for which the addition is induced by the direct sum of the homomorphisms. This definition of kk is from [D] which is equivalent to the definition in [DN]:

$$\mathrm{kk}(Y, X) = \lim_{n \to \infty} \left[C_0(X), M_n(C_0(Y))\right] = \left[C_0(X), C_0(Y) \otimes \mathcal{K}\right] ,$$

where $\mathcal{K}$ is the algebra of compact operators on a separable infinite dimensional Hilbert space.

Let $P \in M_n(C(Y))$ be a projection.Let

$$e_{x_0}^{1-P} : C(X) \longrightarrow \mathbf{C} \cdot (1 - P) \quad \left(\subseteq (1 - P) M_n(C(Y))(1 - P) \right)$$

be the map defined by the evaluation at the base point $x_0 \in X$. The correspondence $\phi \mapsto \phi \oplus e_{x_0}^{1-P}$ induces a map

$$\left[C(X), PM_n(C(Y))P\right]_1 \longrightarrow \left[C(X), M_n(C(Y))\right]_1 .$$

If $\mathrm{rank}(P) \geq 3(\dim Y + 1)$, then by [DN], the homomorphisms in the following diagram

$$\left[C(X), PM_n(C(Y))P\right]_1 \longrightarrow \left[C(X), M_n(C(Y))\right]_1 \longrightarrow \mathrm{kk}(Y, X)$$

are isomorphisms.We need the following proposition which is 3.5.5 of [DN].

PROPOSITION 2.4. *Let X and Y be connected finite* CW *complexes without torsion in cohomology. There is a natural injective map $\theta : \mathrm{kk}(Y, X) \to \mathrm{Hom}(\tilde{H}^*(X), \tilde{H}^*(Y))$, and the image of θ consists of all group homomorphisms* *which preserve both the graduation even-odd of cohomology and the filtration* $F_m\tilde{H}^* = \bigoplus_{q\geq m} \tilde{H}^q$. *That is,* $\theta(\alpha)(H^n) \subset \bigoplus_{\substack{i\geq 0 \\ i\ even}} H^{n+i}$ *for each* n *and* $\alpha \in \mathrm{kk}(Y, X)$.

2.5. In this paper, we use the same notation θ to denote the above map θ from $\mathrm{kk}(Y, X)$ to $\mathrm{Hom}(\tilde{H}^*(X), \tilde{H}^*(Y))$ for different X, Y.

If $x \in \mathrm{kk}(X, X)$ is induced by $\mathrm{id} : C_0(X) \to C_0(X)$, then $\theta(x) \in \mathrm{Hom}(\tilde{H}^*(X), \tilde{H}^*(X))$ is the identity map. Let $x \in \mathrm{kk}(Y, X)$ and $y \in \mathrm{kk}(Z, Y)$. Define $x \times y \in \mathrm{kk}(Z, X)$ to be the composition

$$(y \otimes \mathrm{id}_{\mathcal{K}}) \circ x : C_0(X) \xrightarrow{x} C_0(Y) \otimes \mathcal{K}$$

$$\xrightarrow{y\otimes \mathrm{id}_{\mathcal{K}}} C_0(Z) \otimes \mathcal{K} \otimes \mathcal{K} (= C_0(Z) \otimes \mathcal{K}) .$$

It is true that

$$\theta(x \times y) = \theta(y) \circ \theta(x) \in \mathrm{Hom}\Big(\tilde{H}^*(X), \tilde{H}^*(Z)\Big),$$

where $\theta(y) \circ \theta(x)$ is the composition

$$\tilde{H}^*(X) \xrightarrow{\theta(x)} \tilde{H}^*(Y) \xrightarrow{\theta(y)} \tilde{H}^*(Z) ,$$

(see 3.4.8 and 3.5.4 of [DN]).

2.6 Let X be a finite CW complex with torsion free cohomology groups. One can construct a finite CW complex $\tilde{X} = X_1 \vee X_2 \vee X_3 \ldots \vee X_s$, where each X_j is a sphere S^{n_j}, such that $H^i(X) = H^i(\tilde{X})$for each i. By Proposition 2.4, there exist $[\alpha] \in \mathrm{kk}(\tilde{X}, X)$ and $[\beta] \in \mathrm{kk}(X, \tilde{X})$ such that

$$[\alpha] \times [\beta] = \mathrm{id} \in \mathrm{kk}(X, X) .$$

Suppose that $\alpha : C(X) \to M_{N_1}(C(\tilde{X}))$ and $\beta : C(\tilde{X}) \to M_{N_2}(C(X))$ are representatives of $[\alpha]$ and $[\beta]$. Equivalently, if N_1 and N_2 are large enough, then

$$\beta \otimes \mathrm{id}_{N_1} \circ \alpha : C(X) \longrightarrow M_{N_1N_2}(C(X))$$

is homotopic to $\mathrm{id} \oplus e_{x_0}$ which is defined by

$$(\mathrm{id} \oplus e_{x_0})(f) = \begin{pmatrix} f(x) & & & & \\ & f(x_0) & & & \\ & & f(x_0) & & \\ & & & \ddots & \\ & & & & (x_0) \end{pmatrix}_{N_1N_2\times N_1N_2} .$$

In this note, e_{x_0} denotes the evaluation homomorphism of any given size (in the above case, it is of size $N_1N_2 - 1$).

LEMMA 2.7. *For any fixed finite* CW *complex X with torsion free-cohomology groups, there is an integer N such that, if $\phi : C(X) \to PM_\ell(C(Y))P$ is a unital homomorphism with* $\operatorname{rank}(P) \geq N(\dim Y + 1)$, *then ϕ is homotopic to a homomorphism*

$$\psi : C(X) \longrightarrow PM_\ell(C(Y))P$$

which has a factorization:

$$C(X) \longrightarrow \bigoplus_{j=1}^{t} M_{n_j}(C(X_j)) \longrightarrow PM_\ell(C(Y))P ,$$

where the spaces X_j are spheres or the set $\{\mathrm{pt}\}$ *(from now on, the set* $\{\mathrm{pt}\}$ *is regarded as a sphere).*

PROOF. Let $\tilde{X} = X_1 \vee X_2 \vee \ldots \vee X_s, N_1, N_2, \alpha$, and β be as in 2.6. Consider the composition map

$$X_j \xrightarrow{i_j} \tilde{X} \xrightarrow{r_j} X_j ,$$

where i_j is the inclusion map and r_j is defined by identifying all the points in the subset $X_1 \vee \ldots \vee X_{j-1} \vee X_{j+1} \ldots \vee X_s \subset \tilde{X}$ to a single point. They induce two homomorphisms

$$C(X_j) \xrightarrow{r_j^*} C(\tilde{X}) \xrightarrow{i_j^*} C(X_j) .$$

It is evident that $i_j^* \circ r_j^* = \mathrm{id} \in \operatorname{Hom}(C(X_j), C(X_j))$. Let $d : \oplus C(\tilde{X}) \to M_s(C(\tilde{X}))$ be defined by

$$d(a_1, a_2, \ldots, a_s) = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_s \end{pmatrix} .$$

Then $\gamma_1 := d \circ \bigoplus_{j=1}^{s} r_j^* \circ \bigoplus_{j=1}^{s} i_j^*$, defined by

$$C(\tilde{X}) \xrightarrow{\bigoplus_{j=1}^{s} i_j^*} \bigoplus_{j=1}^{s} C(X_j) \xrightarrow{\bigoplus_{j=1}^{s} r_j^*} \underbrace{C(\tilde{X}) \oplus \cdots \oplus C(\tilde{X})}_{s\text{-copies}} \xrightarrow{d} M_s(C(\tilde{X})) ,$$

is homotopic to $\mathrm{id} \oplus e_{x_0} \in \operatorname{Hom}\big(C(\tilde{X}), M_s(C(\tilde{X}))\big)$ which is defined by

$$\mathrm{id}\oplus e_{x_0}(f) \;=\; \begin{pmatrix} f(x) & & & \\ & f(x_0) & & \\ & & \ddots & \\ & & & f(x_0) \end{pmatrix}_{s\times s} .$$

That is, $[\gamma_1] = \left[d \circ \bigoplus_{j=1}^{s} r_j^* \circ \bigoplus_{j=1}^{s} i_j^*\right] = [\mathrm{id}] \in \mathrm{kk}(\tilde{X}, \tilde{X})$.

Let α_i be the composition

$$C(X) \xrightarrow{\alpha} M_{N_1}(C(\tilde{X})) \xrightarrow{i_j^*\otimes \mathrm{id}_{N_1}} M_{N_1}(C(X_j)) ,$$

where $i_j^* \otimes \mathrm{id}_{N_1}$ is the canonical map from $M_{N_1}(C(\tilde{X}))$ to $M_{N_1}(C(X_j))$.

Let β_j be the composition

$$C(X_j) \xrightarrow{\text{inclusion}} C(\tilde{X}) \xrightarrow{\beta} M_{N_2}(C(X)) .$$

Let $\gamma : C(X) \to M_{sN_1N_2}(C(X))$ be the composition

$$C(X) \xrightarrow{\alpha} M_{N_1}(C(\tilde{X})) \xrightarrow{\gamma_1\otimes \mathrm{id}_{N_1}} M_{sN_1}(C(\tilde{X})) \xrightarrow{\beta\otimes \mathrm{id}_{sN_1}} M_{sN_1N_2}(C(X)) .$$

Then $[\gamma] = [\mathrm{id}] \in \mathrm{kk}(X, X)$. Furthermore, γ factors through $\bigoplus_{i=1}^{s} M_{N_1}(C(X_i))$ since γ_1 factors through $\bigoplus_{i=1}^{s} C(X_i)$. Set $N = 6SN_1N_2$. Let $\phi : C(X) \to PM_l(C(Y))P$ (with $\mathrm{rank}(P) \geq N(\dim Y + 1)$) be a unital homomorphism. Then there exists a trivial projection $Q < P$ with $\mathrm{rank}(Q) = 3sN_1N_2(\dim Y + 1)$. Also,

$$\left[C(X), M_{3(\dim Y+1)}(C(Y))\right]_1 = \left[C(X), PM_l(C(Y))P\right]_1 = \mathrm{kk}(Y, X) .$$

Hence there exists a homomorphism $\phi' : C(X) \to M_{3(\dim Y+1)}(C(Y))$ with $[\phi'] = [\phi] \in \mathrm{kk}(Y, X)$. Let

$$\phi_1 = (\phi' \otimes \mathrm{id}_{sN_1N_2}) \circ \gamma : C(X) \longrightarrow QM_l(C(Y))Q \;=\; M_{3sN_1N_2(\dim Y+1)}(C(Y))$$

be defined by the composition

$$C(X) \xrightarrow{\gamma} M_{sN_1N_2}(C(X)) \xrightarrow{\phi'\otimes \mathrm{id}_{sN_1N_2}} M_{3sN_1N_2(\dim Y+1)}(C(Y)) .$$

Define $\phi_2 : C(X) \to (P - Q)M_l(C(Y))(P - Q)$ by

$$\phi_2(f)(x_0) \;=\; f(x_0)(P - Q) .$$

It follows that $\psi := \phi_1 \oplus \phi_2$ defines the same element as the one defined by ϕ in $\mathrm{kk}(Y, X)$. Hence ϕ and ψ are homotopic. Furthermore, ϕ_1 factors through $\bigoplus_{j=1}^{s} M_{N_1}(X_i)$ and ϕ_2 factors through C. This ends the proof.

REMARK 2.8. With the same N as in 2.7, if $\phi : M_k(C(X)) \to PM_l(C(Y))P$ is a unital homomorphism with $\text{rank}(P) \geq kN(\dim Y + 1)$, then ϕ is homotopic to a homomorphism ψ which factors through $\bigoplus_{i=1}^{t} M_{k_i} C(X_i)$, where the spaces X_i are spheres. Furthermore, for any $\varepsilon > 0$ and any spaces X, there is $N > 0$ such that, if $\phi : M_k(C(X)) \to PM_l(C(Y))P$ is a unital homomorphism with $\text{rank}(P) \geq kN(\dim Y + 1)$, then ϕ is homotopic to a homomorphism ψ which factors through a finite direct sum of matrix algebras over spheres with an additional property $\text{SPV}(\psi) < \varepsilon$. (See the proof of Lemma 3.27 of [EG2].) As in 5.15 of [EG2], one can prove the following:

COROLLARY 2.9. *Let A be an inductive limit of* $(\bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m})$ *with torsion free* $\tilde{H}^*(X_{n,i})$ *and with slow dimension growth (or* $\sup_{n,i} \dim X_{n,i} < +\infty$*, resp.). If A is of real rank zero, then A can be written as an inductive limit of algebras* $(\bigoplus_{i=1}^{l_n} M_{\{n,i\}}(C(Y_{n,i})), \psi_{n,m})$*, where the spaces $Y_{n,i}$ are spheres. Furthermore, the inductive limit system* $\{\bigoplus_{i=1}^{l_n} M_{\{n,i\}}(C(Y_{n,i})), \psi_{n,m}\}$ *can be chosen to satisfy the slow dimension growth condition (or* $\sup_{n,i}\{\dim Y_{n,i}\} < +\infty$*, resp.).*

Theorem 2.1 follows from the above corollary and Theorem 5.17 of [EG2]. Our next task is to prove the following theorem.

THEOREM 2.10. *Let A be an arbitrary inductive limit of algebras* $\bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i}))$ *with the spaces $X_{n,i}$ arbitrary finite* CW *complexes. Suppose that A is of real rank zero. Then $A \otimes M_{\mathbb{Q}}$ can be written as an inductive limit of finite direct sums of matrix algebras over $C(S^1)$.*

(We do not need any restriction on $\dim(X_{n,i})$, since the inductive limit system for $A \otimes M_{\mathbb{Q}}$ can always be made to satisfy the slow dimension growth condition.)

2.11. The following result is the Proposition 2 of [D] (see also 3.4.7 of [DN]).

For any finite CW complexes X, Y, there exists a canonical injective homomorphism

$$\theta : \text{kk}(Y, X) \otimes \mathbb{Q} \longrightarrow \text{Hom}_{\mathbb{Q}}(\tilde{H}^*(X) \otimes \mathbb{Q}, \tilde{H}^*(Y) \otimes \mathbb{Q})$$

such that the image of θ consists of all the group homomorphisms preserving both the graduation even-odd of cohomology and the filtration $F_m\tilde{H}^* = \bigoplus_{q\geq m} \tilde{H}^*$.This result can be interpreted as follows.

(1) Let $\alpha : \tilde{H}^*(X)\otimes \mathbb{Q} \to \tilde{H}^*(Y)\otimes \mathbb{Q}$ be a homomorphism satisfying

$$\alpha(H^n(X)\otimes \mathbb{Q}) \subset \bigoplus_{i\geq n} H^i(X)\otimes \mathbb{Q}\ ,$$

$$\alpha(H^{\mathrm{odd}}(X)\otimes \mathbb{Q}) \subseteq H^{\mathrm{odd}}(X)\otimes \mathbb{Q}, \qquad \text{and}$$

$$\alpha(H^{\mathrm{even}}(X)\otimes \mathbb{Q}) \subseteq H^{\mathrm{even}}(X)\otimes \mathbb{Q}.$$

Then there exist an element $[\phi]\in \mathrm{kk}(Y,X)$ (represented by a homomorphism $\phi : C(X)\to M_k(C(Y))$) and a positive integer m such that

$$\theta([\phi]) \;=\; m\cdot\alpha \in \mathrm{Hom}_{\mathbb{Q}}\big(\tilde{H}^*(X)\otimes \mathbb{Q}, \tilde{H}^*(Y)\otimes \mathbb{Q}\big)\ .$$

(2) If two homomorphisms $\phi_1,\phi_2 : C(X)\to M_k(C(Y))$ $(k\geq 3(\dim Y+1))$ satisfy

$$\theta([\phi_1]) \;=\; \theta([\phi_2]) \in \mathrm{Hom}_{\mathbb{Q}}\big(\tilde{H}^*(X)\otimes \mathbb{Q},\ \tilde{H}^*(Y)\otimes \mathbb{Q}\big)\ ,$$

then there is an integer m such that

$$m[\phi_1] \;=\; m[\phi_2] \in \mathrm{kk}(Y,X)\ .$$

That is, ϕ_1^m is homotopic to ϕ_2^m, where $\phi_i^m : C(X)\to M_{km}(C(Y))$ are defined by

$$\phi_i^m(f) \;=\; \mathrm{diag}\big(\underbrace{\phi_i(f),\cdots,\phi_i(f)}_{m\,\text{copies}}\big), \qquad i=1,2.$$

2.12. Given any finite CW complex X. One can construct a CW complex $\tilde{X} = X_1\vee X_2\vee\cdots\vee X_s$, where the spaces X_s are spheres, with $H^i(X)\otimes \mathbb{Q} = H^i(\tilde{X})\otimes \mathbb{Q}$. Consider the isomorphism

$$\alpha : H^i(X)\otimes \mathbb{Q}\longrightarrow H^i(\tilde{X})\otimes \mathbb{Q}$$

and its inverse

$$\alpha^{-1} : H^i(\tilde{X})\otimes \mathbb{Q}\longrightarrow H^i(X)\otimes \mathbb{Q}\ .$$

Applying 2.11 (1), there exist homomorphisms $\tau_1 : C(X)\to M_{k_1}C(\tilde{X})$ and $\tau_2 : C(\tilde{X})\to M_{k_2}(C(X))$, and integers m_1 and m_2, such that

$$\theta([\tau_1]) \;=\; m_1\cdot\alpha \qquad \text{and} \qquad \theta([\tau_2]) \;=\; m_2\cdot\alpha^{-1}\ .$$

Hence

$$\theta\big([\tau_2 \otimes \mathbf{1}_{k_1} \circ \tau_1]\big) \;=\; m_1 m_2[\mathrm{id}] \;=$$

$$m_1 m_2 \theta\big([\mathrm{id}]\big) \in \mathrm{Hom}_{\mathbb{Q}}(\tilde{H}^*(X) \otimes \mathbb{Q} \;, \tilde{H}^*(X) \otimes \mathbb{Q}) \;,$$

where in the equation $[\mathrm{id}] = \theta[\mathrm{id}]$, the first $[\mathrm{id}]$ is in $\mathrm{Hom}_{\mathbb{Q}}(\tilde{H}^*(X) \otimes \mathbb{Q}, \tilde{H}^*(X) \otimes \mathbb{Q})$ and the second $[\mathrm{id}]$ is in $\mathrm{kk}(X, X)$.By (2) of 2.11, there exists m_3 such that

$$m_3[\tau_2 \otimes \mathbf{1}_{k_1} \circ \tau_1] \;=\; m_1 m_2 m_3[\mathrm{id}] \in \mathrm{kk}(X, X) \;.$$

Set $m = m_1 \cdot m_2 \cdot m_3$. Set $l_1 = m_1$ and $l_2 = m_2 m_3$. Without loss of generality, we assume that $l_1 \geq 3(\dim(X) + 1)$ and $l_2 \geq 3(\dim(X) + 1)$. Then there exist two homomorphisms $\tau_3 : C(X) \to M_{l_1}(C(\tilde{X}))$ and $\tau_4 : C(\tilde{X}) \to M_{l_2}(C(X))$ such that $[\tau_3] = [\tau_1] \in \mathrm{kk}(\tilde{X}, X)$and $[\tau_4] = m_3[\tau_2] \in \mathrm{kk}(X, \tilde{X})$. Hence

$$\tau_4 \otimes \mathbf{1}_l \circ \tau_3 : C(X) \longrightarrow M_{l_1 l_2}(C(X)) = M_m(C(X))$$

satisfies $[\tau_4 \otimes \mathbf{1}_{l_1} \circ \tau_3] = m[\mathrm{id}] \in \mathrm{kk}(X, X)$.That is, $\tau_4 \otimes \mathbf{1}_{l_1} \circ \tau_3$ is homotopic to id^m which is defined by

$$\mathrm{id}^m(f) \;=\; \begin{pmatrix} f & & \\ & \ddots & \\ & & f \end{pmatrix}_{m \times m} .$$

So we have proved the following statement: Given any finite CW complex X, there exists $m > 0$ such that $\mathrm{id}^m : C(X) \to M_m(C(X))$ is homotopic to a homomorphism which factors through a matrix algebra over $C(\tilde{X})$. (Note that once the conclusion holds for m, it holds for any multiple of m). Furthermore, for any $\varepsilon > 0$, the above $m > 0$ can be chosen such that the map $\mathrm{id}^m : C(X) \to M_m(C(X))$ is homotopic to a homomorphism $\phi : C(X) \to M_m(C(X))$ with the following two properties: (1) ϕ factors through a matrix algebra over $C(\tilde{X})$; (2) $\mathrm{SPV}(\phi) < \varepsilon$. (In the constructions of τ_3 and τ_4 above, m_3 can be changed to a large multiple of m_3, and τ_4 can be chosen suitably to make $\mathrm{SPV}(\tau_4)$ small, see 3.27 of [EG2]).

LEMMA 2.13. *If A is an inductive limit of algebras $\bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m}$, where the spaces $X_{n,i}$ are arbitrary finite* CW *complexes, and if A is of real rank zero, then $A \otimes M_{\mathbb{Q}}$ can be written as an inductive limit of finite direct sums of matrix algebras over wedges of spheres.*

PROOF: Notice that, for any l,

$$A \otimes M_l \otimes M_{\mathbb{Q}} = A \otimes M_{\mathbb{Q}},$$

$$A \otimes M_l = \lim_{n\to\infty} \left(\bigoplus_{i=1}^{l_n} M_{[n,i]}(C(X_{n,i})) \otimes M_l \ , \ \phi_{n,m} \otimes \mathbf{1}_l \right) .$$

Furthermore, for any UHF algebra M_U,

$$A \otimes M_U \otimes M_{\mathbb{Q}} = A \otimes M_{\mathbb{Q}}.$$

Let $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_m \cdots$ be a sequence of positive numbers. For A_1, applying 2.12 to every block of A_1, there exists m_1 such that the map

$$\mathrm{id}^{m_1} : A_1 \longrightarrow A_1 \otimes M_{m_1}(\mathbb{C})$$

is homotopic to a homomorphism $\psi_1 : A_1 \to A_1 \otimes M_{m_1}(\mathbb{C})$ with the properties that $\mathrm{SPV}(\psi_1) < \varepsilon_1$ and ψ_1 factors through $\bigoplus_{i=1}^{k_1} M_{\{1,i\}}(C(Y_{1,i}))$, where the spaces $Y_{1,i} = \tilde{X}_{1,i}$ are as in 2.12.

Set $B_1 = A_1$, $B_2 = A_2 \otimes M_{m_1}$. Let $\psi_{1,2} : B_1 \to B_2$ be defined by $\psi_{1,2} = \phi_{1,2} \otimes \mathbf{1}_{m_1} \circ \psi_1$.

For $B_2 (= A_2 \otimes M_{m_1})$, similarly, there exists m_2 such that

$$\mathrm{id}^{m_2} : B_2 \longrightarrow B_2 \otimes M_{m_2}$$

is homotopic to a homomorphism $\psi_2 : B_2 \to B_2 \otimes M_{m_2}$ with the properties that $\mathrm{SPV}(\psi_2) < \varepsilon_2$ and ψ_2 factors through $\bigoplus_{i=1}^{k_2} M_{\{2,i\}}(C(Y_{2,i}))$, where the spaces $Y_{2,i} = \tilde{X}_{2,i}$. Set $B_3 = A_3 \otimes M_{m_1} \otimes M_{m_2}$ and $\psi_{2,3} = \phi_{2,3} \otimes \mathbf{1}_{m_1 m_2} \circ \psi_2 : B_2 \to B_3$.

We continue in this manner to get an inductive limital gebra $B = \lim_{n\to\infty} (B_n, \psi_{n,m})$. By making suitable choices of ε_i, one makes B of real rank zero. Also, we need to change each m_i to a larger one to make B of slow dimension growth.

Recall that

$$\psi_i : B_i (= A_i \otimes M_{m_1 m_2 \dots m_{i-1}}) \longrightarrow B_i \otimes M_{m_i}$$

is homotopic to id^{m_i}. It follows that $\psi_{i,i+1}$ is homotopic to $(\phi_{i,i+1} \otimes \mathbf{1}_{m_1 m_2 \dots m_i}) \circ \mathrm{id}^{m_i}$:

$$A_i \otimes M_{m_1 m_2 \dots m_{i-1}} \xrightarrow{\mathrm{id}^{m_i}} A_i \otimes M_{m_1 m_2 \dots m_i} \xrightarrow{\phi_{i,i+1} \otimes \mathbf{1}_{m_1 m_2 \dots m_i}} A_{i+1} \otimes M_{m_1 m_2 \dots m_i} .$$

Hence $B = \lim(B_n, \psi_{nm})$ is shape equivalent to $A \otimes M_U = \lim(A_n \otimes M_{m_1 m_2 \cdots m_{n-1}}, \tilde{\phi}_{n,m})$, where M_U is the UHF algebra $M_U =$

$M_{m_1} \otimes M_{m_2} \otimes \cdots \otimes \cdots$, and $\tilde{\phi}_{n,m}$ are compositions of the maps $\tilde{\phi}_{i,i+1} = (\phi_{i,i+1} \otimes \mathbf{1}_{m_1 m_2 \ldots m_i}) \circ \mathrm{id}^{m_i}, n \le i \le m-1$.

(Here the shape equivalence is the shape equivalence of inductive limit systems in the sense of 2.1 of [EG2].) By Theorem 2.2 and Remark 2.32 of [EG2], $A \otimes M_U \cong B$. Therefore, $A \otimes M_{\mathbb{Q}} \cong B \otimes M_{\mathbb{Q}}$. On the other hand, B can also be regarded as an inductive limit of algebras $\bigoplus_{i=1}^{k_n} M_{\{n,i\}}(C(Y_{n,i}))$ (notice that each $\psi_{n,n+1}$ factors through $\bigoplus_{i=1}^{k_n} M_{\{n,i\}}(C(Y_{n,i}))$). This ends the proof.

Theorem 2.10 follows from Lemma 2.13 and Theorem 2.1.

REMARK 2.14. In 3.5.5 of [DN] (see 2.4), the injective map $\theta : \mathrm{kk}(Y, X) \to \mathrm{Hom}(\tilde{H}^*(X), \tilde{H}^*(Y))$depends on the identification of $k^q(X)$ with $\bigoplus_{j \ge 0} H^{q+2j}(X)$ in 3.4.8 of [DN]. And this identification is not natural since the splitting of the sequence

$$0 \longrightarrow k^{q+2}(X) \longrightarrow k^q(X) \longleftrightarrow H^q(X) \longrightarrow 0$$

is not natural. By θ being natural, in 2.4, we mean that $\theta(\mathrm{id}) = \mathrm{id}$ and $\theta(x \times y) = \theta(y) \circ \theta(x)$ for all $x \in \mathrm{kk}(Y, X)$ and all $y \in \mathrm{kk}(Z, Y)$. Our θ is natural after we fix an identification of $k^q(X)$ with $\bigoplus_{j \ge 0} H^{q+2j}(X)$ in 3.4.8 of [DN]) for each X, since the map $\mathrm{kk}(Y, X) \to \mathrm{Hom}\big(k^*(X), k^*(Y)\big)$ is natural. On the other hand, if $f : Y \to X$ is a continuous map, then f induces a homomorphism of C^*-algebras $C(f) : C(X) \to C(Y)$ and a homomorphism of cohomology groups $f^* : \tilde{H}^*(X) \to \tilde{H}^*(Y)$. It may not be true that

$$\theta(C(f)) = f^* .$$

But we do not need this equality.

For the homomorphism $\phi : \mathrm{kk}(Y, X) \otimes \mathbb{Q} \to \mathrm{Hom}_{\mathbb{Q}}(\tilde{H}^*(X) \otimes \mathbb{Q}, \tilde{H}^*(Y) \otimes \mathbb{Q})$ in 2.11, we also need an identification of $k^q(X) \otimes \mathbb{Q}$ with $\bigoplus_i H^{q+2i}(X) \otimes \mathbb{Q}$. Here the identification can be chosen to be the one induced by the Chern map: $K^*(X) \otimes \mathbb{Q} \to \tilde{H}^*(X) \otimes \mathbb{Q}$, which is natural. Hence in this case,

$$\theta(C(f)) = f^*$$

if $C(f) \in \mathrm{kk}(Y, X) \otimes \mathbb{Q}$ and $f^* \in \mathrm{Hom}(\tilde{H}^*(X) \otimes \mathbb{Q}, \tilde{H}^*(Y) \otimes \mathbb{Q})$ are induced by a same map $f : Y \to X$.

2.15. Following [EfK], the definition of shape equivalence below is given by Blackadar (see Theorem 4.8 and Definition 4.10 of [Bl3]). Suppose that A

and B are two separable C*-algebras. A is shape equivalent to B if there are two inductive limit systems $A = \varinjlim(A_n, \phi_{n,m})$ and $B = \varinjlim(B_n, \psi_{n,m})$ which are shape equivalent, in the sense that there are two subsequences $\{A_{k_n}\}_{n=1}^{\infty}$ and $\{B_{l_n}\}_{n=1}^{\infty}$, and homomorphisms $\xi_n : A_{k_n} \to B_{l_n}$ and $\eta_n : B_{l_n} \to A_{k_{n+1}}$ with the properties that $\eta_n \circ \xi_n$ is homotopic to $\phi_{k_n,k_{n+1}}$ and $\eta_{n+1} \circ \eta_n$ is homotopic to $\psi_{l_n,l_{n+1}}$ for every n. Notice that the shape equivalence of A and B does not imply that any inductive limit systems of A and B are shape equivalent. (One needs to choose the inductive limit systems of A and B carefully to make them shape equivalent.)

By the result in [EG2], Dadarlat proved the following: If X, Y are connected finite CW complexes, then $C(X) \otimes M_{\mathbb{Q}}$ is shape equivalent to $C(Y) \otimes M_{\mathbb{Q}}$ if and only if $K_*(X) \otimes \mathbb{Q} \cong K_*(Y) \otimes \mathbb{Q}$. The following corollary is a generalization of this result.

COROLLARY 2.16. *Suppose that A and B are* AH *algebras (not necessarily of real rank zero). Then $A \otimes M_{\mathbb{Q}}$ is shape equivalentto $B \otimes M_{\mathbb{Q}}$ if and only if*

$$\big(K_*(A \otimes M_{\mathbb{Q}}), D_*(A \otimes M_{\mathbb{Q}})\big) \cong \big(K_*(B \otimes M_{\mathbb{Q}}), D_*(B \otimes M_{\mathbb{Q}})\big) .$$

PROOF. By 3.16 of [EG2] and the idea used in 3.26 of [EG2], one can construct real rank zero AH algebras $\tilde{A}$ and $\tilde{B}$ which are shape equivalent to $A \otimes M_{\mathbb{Q}}$ and $B \otimes M_{\mathbb{Q}}$, respectively (see the proof of 2.13). Then $\tilde{A} \otimes M_{\mathbb{Q}}$ (or $\tilde{B} \otimes M_{\mathbb{Q}}$) is shape equivalent to $A \otimes M_{\mathbb{Q}} \otimes M_{\mathbb{Q}} = A \otimes M_{\mathbb{Q}}$ (or $B \otimes M_{\mathbb{Q}}$). By Theorem 2.10,

$$\begin{aligned}
&\big(K_*(A \otimes M_{\mathbb{Q}}), D_*(A \otimes M_{\mathbb{Q}})\big) \cong \big(K_*(B \otimes M_{\mathbb{Q}}), D_*(B \otimes M_{\mathbb{Q}})\big)\\
&\Leftrightarrow \big(K_*(\tilde{A} \otimes M_{\mathbb{Q}}), D_*(\tilde{A} \otimes M_{\mathbb{Q}})\big) \cong \big(K_*(\tilde{B} \otimes M_{\mathbb{Q}}), D_*(\tilde{B} \otimes M_{\mathbb{Q}})\big)\\
&\Leftrightarrow \tilde{A} \otimes M_{\mathbb{Q}} \cong \tilde{B} \otimes M_{\mathbb{Q}} .
\end{aligned}$$

On the hand, the last isomorphism implies that $A \otimes M_{\mathbb{Q}}$ is shape equivalent to $B \otimes M_{\mathbb{Q}}$.

REMARK 2.17. In this note, we use the fact that each cocycle in $K_*(C(X)) = K^*(X)$ can be realized as a finite sum of push forward of elements in $K_*(C(S^{n_i}))$, via some homomorphisms from $C(S^{n_i})$ to $M_k(C(X))$.

This idea was also used in [G3], [G4]. In [G3]-[G4], the author proved (see Theorem 4 of [G3] and Theorem 1.2 of [G4]) that each element of $K^1(X)$ can be realized as a sum of pull backs of finitely many elements in $K^1(S^{n_i})$ via some maps from X to S^{n_i} (equivalently, each element in $K_1(C(X))$ can be realized as a sum of push forwards of finitely many elements in $K_1(C(S^{n_i}))$ via homomorphisms from $C(S^{n_i})$ to $C(X)$). The proof of this result is much more difficult, since it does not allow passing to matrix algebras over $C(X)$.

But for the purpose of [G3] and [G4] (the characterization of C_p-smooth extension of Ext(X) up to modulo torsion), it is enough to use the weaker result (allowing passing to matrix algebras). So the proof of the main results in [G3] and [G4] can be simplified (we omit the detail). However, the above stronger result in [G3] and [G4] can be used to prove that an element in Ext(X) is completely determined up to torsion by its pairing with n-tuples of functions via Curto index [G5]. .

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