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CONSTRUCTION OF VACUUM FOR THE POSITIVE-ENERGY REPRESENTATION AND ITS BOSE COUNTERPART

OLE RASK JENSEN

Abstract.

In [3] Lundberg constructed a vacuum for the positive energy representation of CAR (cf. [1]) using the GNS-construction as a main tool. In the present paper we produce the same representation using the embedding-in-the-dual technique, as proposed in [8]. The advantage of this way of approach lies in the fact that both the original representation and its Dirac (and Schrödinger) version (cf. [8]) act within a common domain named the master extension. In the master extension we construct two vacuum-representations. Each one can be the positive energy representation using the other representation as reference. Since an almost identical procedure can be used with respect to the representation of CCR given by Wakimoto [9], we show the two procedures simultaneously.

1. The Fock algebra and its master extension.

Let us fix a number $q \in \{1, -1\}$. To a given complex Hilbert space $\mathscr{H}, <, >$ we attach an algebra $\Gamma_0\mathscr{H}$ generated by $\mathscr{H}$ (*one-particle space*) (cf. [7]) and a unity which will be denoted by $\emptyset$ and called the *vacuum*. We call $\Gamma_0\mathscr{H}$ a *Fock algebra* if the scalar product from $\mathscr{H}$ is extended over $\Gamma_0\mathscr{H}$ in such a way that for every $x \in \mathscr{H}$ the operator $a^+(x)$ of multiplication by x admits the adjoint $a(x)$ defined on the whole $\Gamma_0\mathscr{H}$ and if the adjoint fulfils the q-Leibniz rule, i.e.

$$\langle xf, g\rangle = \langle f, a(x)g\rangle,$$

$$[a(x), a^+(y)]_q = \langle x, y\rangle I$$

and

$$a(x)\emptyset = 0,$$

where $[A, B]_q := AB - qBA$. Finally, we assume that $\emptyset$ is a unit vector.

In the case of $q = 1$, the algebras $\Gamma_0\mathscr{H}$ are commutative and are called

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Bose algebras (cf. [6]) and in the case of $q = -1$, the generators of $\Gamma_0\mathcal{H}$ anti-commute and $\Gamma_0\mathcal{H}$ are then called *Fermi algebras* (cf. [8]).

We shall write $\Gamma\mathcal{H}$ for the completion of $\Gamma_0\mathcal{H}, <, >$.

Denote by $\widetilde{\Gamma}\mathcal{H}$ the conjugate linear algebraic dual to $\Gamma_0\mathcal{H}$, i.e. the linear space of all conjugate linear functionals over $\Gamma_0\mathcal{H}$. We write

$$\langle y, x\rangle = \overline{\langle x, y\rangle},$$

for the value of $x \in \widetilde{\Gamma}\mathcal{H}$ in $y \in \Gamma_0\mathcal{H}$. We make the space $\Gamma\mathcal{H}$ a linear subspace of $\widetilde{\Gamma}\mathcal{H}$, using the identification

$$\Gamma\mathcal{H} \ni x \to \langle\cdot, x\rangle \in \widetilde{\Gamma}\mathcal{H}.$$

The space $\widetilde{\Gamma}\mathcal{H}$ shall be called the *master extension* assigned to the Fock algebra. Incidentally, multiplication in $\Gamma_0\mathcal{H}$ can be naturally extended to $\widetilde{\Gamma}\mathcal{H}$ making out of it an algebra (cf. [8]).

2. Exponentials of quadratic forms.

A conjugate-linear operator $\iota : \mathcal{H} \to \mathcal{H}$ such that $\iota^2 = qI$ and $\iota' = q\iota$ shall be called a *q-conjugation*. It should be noticed that prime stands for real transpose. We can write any q-conjugation in the form $\iota = SJ$, where S and J commute, J is a conjugation and S is linear with $S^2 = qI$ and $S^* = qS$.

Given an orthonormal basis $\{e_n\}$, we consider on $\Gamma_0\mathcal{H}$ the operators

$$a(h_\iota) = \sum_{n=1}^{\infty} a(\iota e_n)a(e_n)$$

and

$$a^+(h_\iota) = \sum_{n=1}^{\infty} a^+(e_n)a^+(\iota e_n).$$

For infinite dimensional ι, the element $h_\iota = a^+(h_\iota)\emptyset$ does not belong to $\Gamma\mathcal{H}$. Note that if, for $q = -1$, we used an ordinary conjugation, instead of a q-conjugation, the element corresponding to h_ι would depend on the choice of orthonormal basis, which should be avoided. The element

$$\delta_\iota = \exp\left(-\frac{1}{2}h_\iota\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{1}{2}h_\iota\right)^n$$

exists in $\widetilde{\Gamma}\mathcal{H}$. Let us introduce the operator

$$a(\delta_{q\iota}) = \sum_{n=0}^{\infty} \frac{1}{n!}(-\frac{1}{2}a(h_{q\iota}))^n.$$

Since sufficiently high powers $(-\frac{1}{2}a(h_{q\iota}))^n$ annihilate any given element of $\Gamma_0\mathscr{H}$, the series above are finite on $\Gamma_0\mathscr{H}$.

3. The Schrödinger and Dirac Versions.

It is well known that every bounded linear operator $T:\mathscr{H}_1 \to \mathscr{H}_2$, has a unique extension to a homomorphism $\Gamma T:\Gamma_0\mathscr{H}_1 \to \Gamma_0\mathscr{H}_2$. Notice that $(\Gamma T)\emptyset = \emptyset$ for all T.

Given a q–conjugation ι and an $x \in \mathscr{H}$, define the operators

$$a_\iota(x) = \frac{1}{\sqrt{2}}(a(x) + a^+(\iota x))$$

and

$$a_\iota^+(x) = \frac{1}{\sqrt{2}}(a^+(x) - qa(\iota x))$$

transforming $\widetilde{\Gamma}\mathscr{H}$ into itself. Consider the mapping

$$\mathrm{B}_\iota = \delta_\iota\left(\Gamma 2^{\frac{1}{2}} I\right)a(\delta_{q\iota})$$

of $\Gamma_0\mathscr{H}$ into $\widetilde{\Gamma}\mathscr{H}$ (the Segal-Bargmann transformation) and define

$$\mathscr{H}_\iota = \{\mathrm{B}_\iota x \mid x \in \mathscr{H}\} = \left\{\sqrt{2}\delta_\iota x \mid x \in \mathscr{H}\right\}.$$

It easily follows from the identities

$$a(\delta_\iota)a^+(x) = (a^+(x) - a(\iota x))a(\delta_\iota)$$

and

$$\left(\Gamma 2^{\frac{1}{2}}\right)a^+(x) = a^+\left(2^{\frac{1}{2}}x\right)\left(\Gamma 2^{\frac{1}{2}}\right)$$

and the duals, that

$$\mathrm{B}_\iota a^+(x) = a_\iota^+(x)\mathrm{B}_\iota$$

and

$$\mathrm{B}_\iota a(x) = a_\iota(x)\mathrm{B}_\iota$$

on $\Gamma_0\mathscr{H}$. δ_ι is the vacuum for $a_\iota(x)$ since $\mathrm{B}_\iota\emptyset = \delta_\iota$.

Introduce a scalar product in $\Gamma_0\mathscr{H}_\iota$ setting

$$\langle f, g\rangle_\iota = \langle \mathrm{B}_\iota^{-1} f, \mathrm{B}_\iota^{-1} g\rangle,$$

where $f, g \in \Gamma_0\mathscr{H}_\iota := \{\delta_\iota h : h \in \Gamma_0\mathscr{H}\}$. It is simple to check that considered on $\Gamma_0\mathscr{H}_\iota, <,>_\iota$, the operators $a_\iota^+(x)$ and $a_\iota(x)$ are dual to each other. Moreover,

$$[a_\iota(x), a_\iota^+(y)]_q = \langle x, y\rangle I = \langle a_\iota^+(x)\delta_\iota, a_\iota^+(y)\delta_\iota\rangle_\iota I.$$

It is easily seen that we have defined a functor assigning to a given Fock algebra $\Gamma_0\mathscr{H}, <,>$ and q-conjugation ι, a new Fock algebra $\Gamma_0\mathscr{H}_\iota, <,>_\iota$ which, for $q = 1$, we call the Schrödinger version, and, for $q = -1$, the Dirac version of $\Gamma_0\mathscr{H}, <,>$ with q-conjugation ι (cf. [8]).

4. Construction of the vacuums.

Let in the sequel $\mathscr{H}, <,>$ be a Hilbert space and $\Gamma_0\mathscr{H}$ the associated Fock algebra. By easy computations we obtain the following

LEMMA 1. *Consider an orthogonal projection P and two commuting q-conjugations ι and υ, such that $P = \frac{1}{2}(I - q\iota\upsilon)$. Then for $x \in \mathscr{H}$*

$$a_\iota^+(x) = a_\upsilon^+((I - P)x) - qa_\upsilon(\iota Px),$$

$$a_\iota(x) = a_\upsilon^+(\iota Px) + a_\upsilon((I - P)x),$$

and $\delta_\iota \in \Gamma_0\mathscr{H}_\iota$ is the vacuum for $a_\iota(x)$, while

$$a_\upsilon^+(x) = a_\iota^+((I - P)x) - qa_\iota(\upsilon Px),$$

$$a_\upsilon(x) = a_\iota^+(\upsilon Px) + a_\iota((I - P)x),$$

and $\delta_\upsilon \in \Gamma_0\mathscr{H}_\upsilon$ is the vacuum for $a_\upsilon(x)$.

Since we use Nelson's notations $d\Gamma$ and Γ for the functor of the second quantization and its exponential (cf. [5]), we shall write J for the involution Γ considered in [4]. J is by definition a conjugation commuting with both ι and υ.

Let $V = (I - P) + qJ\iota P$. It is straightforward to check that V is unitary and that $V^2 = I - (1 - q)P$. Notice that if $q = 1$ we might use $J = \iota$.

Define on $\widetilde{\Gamma}\mathscr{H}$

$$a_P(x) = a_\iota(Vx) \text{ and } a_P^+(x) = a_\iota^+(Vx).$$

Hence

$$a_P(x) = a_\upsilon^+(JPx) + a_\upsilon((I - P)x)$$

and

$$a_P^+(x) = a_\upsilon^+((I - P)x) - qa_\upsilon(JPx).$$

Since

$$[a_P(x), a_P^+(y)]_q = \langle x, y\rangle I,$$

the operators $a_P(x)$ and $a_P^+(y)$, $x, y \in \mathcal{H}$, generate a Fock algebra with one-particle space $\mathcal{H}_\iota$. Notice that the operators $a_P(e_i)$ appear in [2] as ψ_i.

As an example, consider $L^2(S^1)$ as a one-particle space, choose $\{e_n, \widehat{e}_n\}$, where $e_n = e^{2ni\theta}$ and $\widehat{e}_n = e^{(2n+1)i\theta}$, $n \in \mathbb{Z}$, as the orthonormal basis.

Setting

$$\iota = \sum_{n=-\infty}^{\infty} (\langle\cdot, e_n\rangle\widehat{e}_n + q\langle\cdot, \widehat{e}_n\rangle e_n),$$

$$J = \sum_{n=-\infty}^{\infty} (\langle\cdot, e_n\rangle e_n + \langle\cdot, \widehat{e}_n\rangle\widehat{e}_n)$$

and

$$P = \sum_{n=-\infty}^{-1} (\langle e_n, \cdot\rangle e_n + \langle\widehat{e}_n, \cdot\rangle\widehat{e}_n),$$

we get $\iota P = P\iota$.

Then

$$V = \sum_{n=0}^{\infty}(\langle e_n, \cdot\rangle e_n + \langle\widehat{e}_n, \cdot\rangle\widehat{e}_n) + \sum_{n=-\infty}^{-1} (\langle\widehat{e}_n, \cdot\rangle e_n + q\langle e_n, \cdot\rangle\widehat{e}_n),$$

and $\upsilon = (I - 2P)\iota$ concludes the construction of the model.

Following [3], consider the gauge-invariant quasi-free state ω_P setting

$$\begin{aligned}&\omega_P\big(a_\upsilon^+(y_m)\cdots a_\upsilon^+(y_1)a_\upsilon(x_1)\cdots a_\upsilon(x_n)\big)\\ &= \big\langle\delta_\iota, a_\upsilon^+(y_m)\cdots a_\upsilon^+(y_1)a_\upsilon(x_1)\cdots a_\upsilon(x_n)\delta_\iota\big\rangle_\iota,\end{aligned}$$

for P, υ and ι as in lemma 1 and $x_i, y_j \in \mathcal{H}$. We get the following result.

THEOREM 2. *Let P, υ and ι be as in lemma 1, $q = -1$ and $x_i, y_j \in \mathcal{H}$, then*

$$\omega_P\big(a_\upsilon^+(y_m)\cdots a_\upsilon^+(y_1)a_\upsilon(x_1)\cdots a_\upsilon(x_n)\big) = \delta_{nm}\det\big(\langle x_j, Py_i\rangle\big).$$

PROOF. We make the direct computations:

$$\begin{aligned}&\big\langle\delta_\iota, a_\upsilon^+(y_m)\cdots a_\upsilon^+(y_1)a_\upsilon(x_1)\cdots a_\upsilon(x_n)\delta_\iota\big\rangle_\iota\\ &= \langle a_\upsilon(y_1)\cdots a_\upsilon(y_m)\mathrm{B}_\iota\emptyset, a_\upsilon(x_1)\cdots a_\upsilon(x_n)\mathrm{B}_\iota\emptyset\rangle_\iota\end{aligned}$$

since

$$a_v(x)\mathrm{B}_\iota = \big(a_\iota((I-P)x) + a_\iota^+(vPx)\big)\mathrm{B}_\iota$$
$$= \mathrm{B}_\iota(a((I-P)x) + a^+(vPx))$$

and

$$[a((I-P)x), a^+(vPx)] = 0,$$

then

$$= \langle \mathrm{B}_\iota a^+(vPy_1)\cdots a^+(vPy_m)\emptyset, \mathrm{B}_\iota a^+(vPx_1)\cdots a^+(vPx_n)\emptyset\rangle_\iota$$
$$= \langle a^+(vPy_1)\cdots a^+(vPy_m)\emptyset, a^+(vPx_1)\cdots a^+(vPx_n)\emptyset\rangle$$
$$= \delta_{nm}\det\big(\langle vPy_i, vPx_j\rangle\big)$$

Noticing that we write $P, a_v^+(x)$ and $a_v(x)$ instead of $T, a(f)^*$ and $a(f)$ as in [3], we can see that our ω_P coincides with that of [3]. An analogous result as theorem 2 can be obtained for $q = 1$, by use of an identical argument.

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MATHEMATISK INSTITUT
AARHUS UNIVERSITEIT
BYGNING 530
8000 ÅRHUS C
DENMARK