MATH. SCAND. 80 (1997), 125–160

L^2-VERSIONS OF THE HOWE CORRESPONDENCE I

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Abstract

We calculate the explicit decomposition of the metaplectic representation for the dual pairs $\mathrm{SL}(2,\mathbb{R}) \times O(1,1)$ and $U(1,1) \times U(1,1)$ by extending the idea of pluriharmonic functions due to M. Kashiwara and M. Vergne.

§0. Introduction

Since its discovery by A. Weil in connection with number theory and by I. E. Segal working with quantum field theory, the metaplectic representation has played a fundamental role in these and other subjects in mathematics. This is the representation also known as the Segal-Shale-Weil-harmonic-oscillator representation. Inspired by classical invariant theory R. Howe [6] developed the theory of dual reductive pairs, and since then the so-called Howe correspondence has been successfully applied to several problems in representation theory of semisimple Lie groups.

In this paper we study the L^2-version of the Howe correspondence in the sense of finding the explicit measure in the decomposition of the metaplectic representation restricted to the dual pair in question. Specifically, we find for the dual pairs $\mathrm{SL}(2,\mathbb{R}) \times O(1,1)$ and $U(1,1) \times U(1,1)$,

$$\widetilde{L}|_{G\times G'} \cong \int_{\hat{G}}^{\oplus} \pi \otimes \tilde{\pi} d\mu(\pi) \tag{0.1}$$

with $\widetilde{L}$ the metaplectic representation of the symplectic group containing (G, G') as a reductive dual pair (strictly speaking, their double covers). The measure $d\mu$, the intertwining operator, and the Howe correspondence $\pi \mapsto \tilde{\pi}$ in (0.1) is made explicit as well as the corresponding inversion formula. We feel that (0.1) deserves to be studied in its own right but also in view of the role of dual pairs in problems of number theory and representation theory. The main tool we use is a Fourier integral operator which in the case when G' is compact was introduced by M. Kashiwara and M. Vergne. We remark that Rallis and Schiffmann [16] also constructed this intertwining operator

Received January 24, 1994; in revised version May 5, 1994 and April 17, 1995.

inconnection with their study of the discrete spectrum for the dual pair $\mathrm{SL}(2,\mathbf{R}) \times O(p,q)$. Another tool is the calculation of c-functions for line-bundles over the unit disk in the complex plane in [2], [10] and [13]. This is needed for the complex case $U(1,1) \times U(1,1)$. In a subsequent paper we deal with the similar question for larger dual pairs.

We note here further that one of our motivations to undertake this study is a connection and possible application to the Plancherel theorem for affine symmetric spaces of Cayley type introduced by G. Olafsson, see the remarks at the end of §2.

We finally also consider the related question of decomposing the tensor product $\pi_\nu \otimes \bar{\pi}_\nu$ of the analytic continuation π_ν of holomorphic discrete series for the universal covering of $\mathrm{SL}(2,\mathbf{R})$ with its conjugate. By studying the spectral resolution of the Casimir operator we find that when $0 < \nu < \frac{1}{2}$, there is surprisingly a complementary series representation entering into the decomposition as a discrete part. To our knowledge, this problem has not been studied before.

The main results are summarized in Theorems 2, 4-6. We put an appendix in the end of the paper to clarify certain intertwining operators realizing the discrete series of $U(1,1)$ as subrepresentations (or quotients) of some induced representation. We mention that Lemma 5 in §2 may be of special interest showing the analytic continuation of a certain integral operator. This is closely related to the study of Plancherel formulas on sections of line bundles over the unit disk, or more generally, over an Hermitian symmetric space. It will be interesting to make this more precise.

We would like to thank R. Stanton for pointing out the relevance of the work of Rallis and Schiffmann [16] to the problem considered here. We are grateful to the referee for his/her many criticisms of this paper, where some of our earlier arguments were too sketchy, in particular concerning the convergence of integrals. See the remark following the proof of Theorem 1. Furthermore, the treatment of the intertwining operator for the discrete series was greatly clarified through the referee's comments and, indeed, help.

§1. Metaplectic representation of $\mathrm{Sp}(1,\mathbf{R})$

We will follow [7] in the formulation of the metaplectic representation. The group $\mathrm{Sp}(1,\mathbf{R}) = \mathrm{SL}(2,\mathbf{R})$ has the following generators:

$$g(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \ a \in \mathbf{R}^\times,$$

$$t(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \; b \in \mathbf{R},$$

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The metaplectic representation L of the double cover of $\mathrm{SL}(2, \mathbf{R})$ (corresponding to choice of square roots) is defined as follows on $L^2(\mathbf{R})$:

$$L(g(a))f(x) = a^{\frac{1}{2}} f(ax),$$

$$L(t(b))f(x) = e^{-\frac{i}{2}bx^2} f(x),$$

$$L(\sigma)f(x) = \left(\frac{i}{2\pi}\right)^{\frac{1}{2}} \int_{\mathbf{R}} e^{ixy} f(y)dy.$$

We consider $\bar{L}$ given by $\bar{L}f = \overline{L\bar{f}}$. Then the tensor product $L \otimes \bar{L}$ is realized on $L^2(\mathbf{R}^2)$:

$$(L \otimes \bar{L})(g(a))f(x) = |a| f(ax),$$

$$(L \otimes \bar{L})(t(b))f(x) = e^{-\frac{i}{2}b(x_1^2 - x_2^2)} f(x),$$

$$(L \otimes \bar{L})(\sigma)f(x) = \frac{1}{2\pi} \int_{\mathbf{R}^2} e^{i(x_1 y_1 - x_2 y_2)} f(y)dy.$$

On $L^2(\mathbf{R}^2)$ we have the natural unitary action of $O(1,1)$ from the right:

$$hf(x) = f(xh), h \in O(1,1).$$

It is clear that the $O(1,1)$ action commutes with $L \otimes \bar{L}$ of $\mathrm{SL}(2, \mathbf{R})$. Indeed these two groups form a dual reductive pair in the sense of Howe [6]. We denote $\widetilde{L}$ the representation of $\mathrm{SL}(2, \mathbf{R}) \times O(1,1)$ on $L^2(\mathbf{R}^2)$, with $\mathrm{SL}(2, \mathbf{R})$ acting from the left by $L \otimes \bar{L}$ and $O(1,1)$ acting from the right as above.

Our problem is to find the explicit decomposition of $\widetilde{L}$.

The group $O(1,1)$ has generators

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \operatorname{ch} s & \operatorname{sh} s \\ \operatorname{sh} s & \operatorname{ch} s \end{pmatrix}, \; s \in \mathbf{R}.$$

Consider $\mathbf{C}^2$ with basis $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. Let $t > 0$, $\delta = 0, 1$. The following choice determines a unitary representation $\widetilde{\pi}_{\delta, it}$ of $O(1,1)$ on $\mathbf{C}^2$:

$$\widetilde{\pi}_{\delta,it}\left(\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right)=\begin{pmatrix}(-1)^{\delta} & 0\\ 0 & (-1)^{\delta}\end{pmatrix},$$

$$\widetilde{\pi}_{\delta,it}\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

$$\widetilde{\pi}_{\delta,it}\left(\begin{pmatrix}\operatorname{ch} s & \operatorname{sh} s\\ \operatorname{sh} s & \operatorname{ch} s\end{pmatrix}\right)=\begin{pmatrix}e^{ist} & 0\\ 0 & e^{-ist}\end{pmatrix}.$$

Let $t>0$, $\epsilon=\pm 1$. We define the following distributions on R^2,

$$I_0(x,t,\epsilon)=|x_1+\epsilon x_2|^{it},$$

$$I_1(x,t,\epsilon)=sgn(x_1+\epsilon x_2)|x_1+\epsilon x_2|^{it},$$

and the C^2-valued function

$$I_0(x,t)=\begin{pmatrix}I_0(x,t,1)\\ I_0(x,t,-1)\end{pmatrix},\ I_1(x,t)=\begin{pmatrix}I_1(x,t,1)\\ I_1(x,t,-1)\end{pmatrix}.$$

We consider also the representations λ_0, λ_1 of $\mathsf{R}^\times$:

$$\lambda_0(a)=|a|^{it}, \lambda_1(a)=sgn(a)|a|^{it}, a\in\mathsf{R}\setminus\{0\}.$$

The following lemma is then easy to check.

LEMMA 1. *The functions I_δ, $\delta=0,1$ have the following properties:*
(1) $I_\delta(axh^{-1},t,\epsilon)=\widetilde{\pi}_{\delta,it}(h)I_\delta(x,t,\epsilon)\lambda_\delta(a), a\in\mathsf{R}^\times, h\in O(1,1)$;
(2) *I_δ solves the wave equation for $\delta=0,1$:*

$$\left(\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}\right)I(x)=0.$$

Let $\mathscr{S}(\mathsf{R}^2)$ (resp. $\mathscr{S}(\mathsf{R}^2)_0$ or $\mathscr{S}(\mathsf{R}^2)_1$) denote the space of Schwartz (resp. even or odd) functions. Note that $\mathscr{S}(\mathsf{R}^2)_\delta$ is stable under the action of $\widetilde{L}$. We now define the Fourier integral operator $\mathscr{F}_\delta$ on $\mathscr{S}(\mathsf{R}^2)$ by

$$\mathscr{F}_\delta f(\xi,t)=\int_{\mathsf{R}^2}e^{i(x_1^2-x_2^2)\xi/2}I_\delta(x,t)f(x)dx.$$

Note that $\mathscr{F}_\delta f=0$ if $f\in\mathscr{S}(\mathsf{R}^2)_{\delta'}$ and $\delta'\neq\delta$.

PROPOSITION 1. *For all $t>0$, the operator $\mathscr{F}_\delta$ maps $\mathscr{S}(\mathsf{R}^2)_\delta$ into $L^2(\mathsf{R})\otimes\mathsf{C}^2$.*

PROOF. Let $f\in\mathscr{S}(\mathsf{R}^2)_0$, then $\mathscr{F}_1 f=0$. Change variables $\eta=x_1^2-x_2^2$ and

$x_1 \to \sqrt{|\eta|}x_1, x_2 \to \sqrt{|\eta|}x_2$. We find

$$\mathscr{F}_0 f(\xi, t, \epsilon) = \int_{\mathbf{R}} e^{i\eta\xi/2} F(\eta, t, \epsilon) d\eta, \tag{1.1}$$

where

$$F(\eta, t, \epsilon) = |\eta|^{it/2} \int_{x_1^2 - x_2^2 = sgn\eta} |x_1 + \epsilon x_2|^{it} f(\sqrt{|\eta|}x_1, \sqrt{|\eta|}x_2) dh(x) \tag{1.2}$$

and $dh(x)$ is the invariant measure on the hyperbola $x_1^2 - x_2^2 = sgn\eta$, induced from our quadratic form and Lebesgue measure on $\mathbf{R}^2$.

We put $\epsilon = 1$. Let $\eta > 0$. The hyperbola $x_1^2 - x_2^2 = sgn\eta = 1$ can be parameterized by $(\cosh s, \sinh s)$ and $(-\cosh s, \sinh s)$ with $s \in \mathbf{R}$, and $dh(x) = ds$. Thus

$$\begin{aligned} F(\eta, t, 1) = |\eta|^{it/2} &\int_{-\infty}^{\infty} e^{ist} f(\sqrt{|\eta|}\cosh s, \sqrt{|\eta|}\sinh s) ds \\ + |\eta|^{it/2} &\int_{-\infty}^{\infty} e^{ist} f(-\sqrt{|\eta|}\cosh s, \sqrt{|\eta|}\sinh s) ds \end{aligned}$$

and furthermore, since f is a Schwartz class function

$$\begin{aligned} |F(\eta, t, 1)| &\leq C \int_0^{\infty} (1 + |\eta| e^{2s})^{-4} ds \\ &\leq C \int_{|\eta|}^{\infty} (1 + u)^{-4} u^{-1} du \\ &\leq C(1 + |\eta|)^{-2} \ln |\eta| \end{aligned}$$

by an easy calculation. Thus $F(\eta, t, 1)$ as a function of η is in $L^2(\mathbf{R}) \cap L^1(\mathbf{R})$ and therefore (1.1) is everywhere defined and $\mathscr{F}_0 f(\xi, t, 1)$ is in $L^2(\mathbf{R})$. Similarly we show $\mathscr{F}_0 f(\xi, t, -1)$ is in $L^2(\mathbf{R})$.

Let $\pi_{\delta,it}$ be the principal series representation of $\mathrm{SL}(2, \mathbf{R})$ on $L^2(\mathbf{R})$, i. e.

$$\pi_{\delta,it}(g)f(\xi) = |c\xi + d|^{-it-1} sgn^{\delta}(c\xi + d) f\left(\frac{a\xi + b}{c\xi + d}\right), \; g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{R}).$$

THEOREM 1 *The operator* $\mathscr{F}_\delta$ *intertwines the representation* $\widetilde{L}$ *with* $\pi_{\delta,it} \otimes \widetilde{\pi}_{\delta,it}$ *for the* $\mathrm{SL}(2, \mathbf{R}) \times O(1,1)$ *action.*

PROOF Since I_δ transforms as in Lemma 1, the $O(1,1)$ intertwining relation is easy to check. We now prove the $\mathrm{SL}(2, \mathbf{R})$ intertwining relation. We will do this for $\mathscr{F}_0$. The $\mathscr{F}_1$ case is essentially the same.

We need to prove

$$\mathscr{F}_0(L\otimes \bar{L})(g) = \pi_{0,it}(g)\mathscr{F}_0, \;\; g\in \mathrm{SL}(2,\mathsf{R}).$$

It suffices to check this for the generators. We have

$$\begin{aligned}
&\mathscr{F}_0(L\otimes \bar{L})(g(a))f(\xi,t,\epsilon)\\
&= \int_{\mathsf{R}^2} e^{i(x_1^2-x_2^2)\xi/2} I_0(x,t,\epsilon)|a|f(ax)dx\\
&= \int_{\mathsf{R}^2} e^{i(x_1^2-x_2^2)a^{-2}\xi/2} I_0(a^{-1}x,t,\epsilon)f(x)|a|^{-1}dx\\
&= |a^{-1}|^{1+it}\mathscr{F}_0 f(a^{-2}\xi,t,\epsilon)\\
&= \pi_{0,it}(g(a))\mathscr{F}_0 f(\xi,t,\epsilon).
\end{aligned}$$

It is easy to see that

$$\mathscr{F}_0(L\otimes \bar{L})(t(b)) = \pi_{0,it}(t(b))\mathscr{F}_0.$$

We now prove the σ intertwining relation. We state it as a Lemma.

LEMMA 2. For $f\in \mathscr{S}(\mathsf{R}^2)_0$ we have, if $\xi\neq 0$,

$$\begin{aligned}
&\mathscr{F}_0(L\otimes \bar{L})(\sigma)f(\xi,t,\epsilon)\\
&=\frac{1}{2\pi}\int_{\mathsf{R}^2} e^{i(x_1^2-x_2^2)\xi/2} I_0(x,t,\epsilon)\left(\int_{\mathsf{R}^2} e^{i(x_1y_1-x_2y_2)}f(y)dy\right)dx\\
&= |-\xi^{-1}|^{1+it}\int_{\mathsf{R}^2} f(y)e^{-i\xi^{-1}(y_1^2-y_2^2)} I_0(y,t,\epsilon)dy\\
&= \pi_{0,it}(\sigma)\mathscr{F}_0 f(\xi,t,\epsilon).
\end{aligned}$$

PROOF. We let $\epsilon = 1$ and consider the function

$$\psi_{\tau_1,\tau_2}(x) = e^{-\tau_1|x_1+x_2|^2-\tau_2|x_1-x_2|^2},$$

for $\tau_1,\tau_2>0$. Since the dx integral in Lemma 2 is absolute convergent, we have that the LHS in Lemma 2 is

$$\lim_{\tau_2\to 0}\lim_{\tau_1\to 0}\frac{1}{2\pi}\int_{\mathsf{R}^2} e^{i(x_1^2-x_2^2)\xi/2} I_0(x,t,1)\psi_{\tau_1,\tau_2}(x)\left(\int_{\mathsf{R}^2} e^{i(x_1y_1-x_2y_2)}f(y)dy\right)dx.$$

Using Fubini's theorem we get the integral above is

$$\frac{1}{2\pi}\int_{\mathsf{R}^2} f(y)\left(\int_{\mathsf{R}^2} e^{i(x_1^2-x_2^2)\xi/2} I_0(x,t,1)\psi_{\tau_1,\tau_2}(x)e^{i(x_1y_1-x_2y_2)}dx\right)dy.$$

Change variables

$$x = uS,\; y = vS,$$

where $S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. The above integral is

$$\frac{1}{2\pi}\int_{\mathsf{R}^2} f(vS)\int_{\mathsf{R}^2} e^{-2\tau_1 u_1^2-2\tau_2 u_2^2} e^{iu_1u_2\xi}|u_1|^{it}e^{i(u_1v_2+u_2v_1)}dudv \tag{1.3}$$

The inner integral is

$$\int_{\mathsf{R}} e^{-2\tau_1 u_1^2+iu_1v_2}|u_1|^{it}\left(\int_{\mathsf{R}} e^{-2\tau_2 u_2^2+iu_2(u_1\xi+v_1)}du_2\right)du_1$$
$$=\frac{\sqrt{\pi}}{\sqrt{2\tau_2}}\int_{\mathsf{R}} e^{-2\tau_1 u_1^2+iu_1v_2}|u_1|^{it}e^{-2\tau_2\left(\frac{u_1\xi+v_1}{4\tau_2}\right)^2}du_1$$

by the Bochner formula for the Fourier transform of the Gaussian [19]. Taking limit $\tau_1 \to 0$, by the dominated convergence theorem, we see that (1.3) converges to

$$\frac{1}{2\pi}\frac{\sqrt{\pi}}{\sqrt{2\tau_2}}\int_{\mathsf{R}^2}\int_{\mathsf{R}} f(vS)e^{iu_1v_2}|u_1|^{it}e^{-2\tau_2\left(\frac{u_1\xi+v_1}{4\tau_2}\right)^2}du_1dv$$
$$=|\xi|^{-1}\int_{\mathsf{R}}\left\{\frac{1}{\sqrt{8\pi\tau_2}}\int_{\mathsf{R}} e^{-\frac{(u_1+v_1)^2}{8\tau_2}}\left(\int_{\mathsf{R}} f(vS)e^{i\xi^{-1}u_1v_2}|\xi^{-1}u_1|^{it}dv_2\right)du_1\right\}dv_1$$

Now the term in the inner parenthesis, as a function of v_1 is in $L^1(\mathsf{R})$ and take limit $\tau_2 \to 0$ we know that the term in the outer parenthesis tends to this term:

$$\int_{\mathsf{R}} f(vS)e^{-i\xi^{-1}v_1v_2}|\xi^{-1}v_1|^{it}dv_2$$

in the space $L^1(\mathsf{R}, dv_1)$. Thus, the integral becomes

$$|\xi|^{-1}\int_{\mathsf{R}^2} f(vS)e^{-i\xi^{-1}v_1v_2}|\xi^{-1}v_1|^{it}dv$$
$$=|-\xi^{-1}|^{1+it}\int_{\mathsf{R}^2} f(y)e^{-i\xi^{-1}(y_1^2-y_2^2)}I_0(y,t,\epsilon)dy$$
$$=|-\xi^{-1}|^{1+it}\mathscr{F}_0 f(-\xi^{-1},t,\epsilon)$$
$$=\pi_{0,it}(\sigma)\mathscr{F}_0 f(\xi,t,\epsilon).$$

We have thus proved the Lemma.

Consequently we have completed the proof of Theorem 1.

REMARK 1. Formally the LHS of Lemma 2 is

$$\begin{aligned}
&\mathscr{F}_0\widetilde{L}(\sigma)f(\xi,t,\epsilon)\\
&=\frac{1}{2\pi}\int_{\mathsf{R}^2}e^{i(x_1^2-x_2^2)\xi/2}I_0(x,t,\epsilon)\left(\int_{\mathsf{R}^2}e^{i(x_1y_1-x_2y_2)}f(y)dy\right)dx\\
&=\frac{1}{2\pi}\int_{\mathsf{R}^2}f(y)\left(\int_{\mathsf{R}^2}e^{i(x_1y_1-x_2y_2)}e^{i(x_1^2-x_2^2)\xi/2}I_0(x,t,\epsilon)dx\right)dy.
\end{aligned}$$

The inner integral is divergent. We notice that $g(x)=I_0(x,t,1)$ is a distribution on R^2 satisfying the wave equation

$$\left(\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}\right)g=0.$$

And still formally, the inner integral is

$$\begin{aligned}
&\int_{\mathsf{R}^2}e^{i(x_1y_1-x_2y_2)}e^{i(x_1^2-x_2^2)\xi/2}g(x)dx\\
&=\int_{\mathsf{R}^2}e^{i(u_1v_2+u_2v_1)}e^{iu_1u_2\xi}g(u)du\\
&=\int_{\mathsf{R}}e^{iu_1v_2}g(u_1)\int_{\mathsf{R}}e^{i(v_1+u_1\xi)u_2}du_2du_1\\
&=\int_{\mathsf{R}}e^{iu_1v_2}g(u_1)(2\pi)\delta_0(v_1+u_1\xi)du_1\\
&=(2\pi)|\xi|^{-1}\int_{\mathsf{R}}e^{iu_1v_2\xi^{-1}}g(u_1\xi^{-1})\delta_0(v_1+u_1)du_1\\
&=(2\pi)|\xi|^{-1}g(-\xi^{-1}v)e^{-iv_1v_2\xi^{-1}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathscr{F}_0(L\otimes\bar{L})(\sigma)f(\xi,t,\epsilon)\\
&=|\xi|^{-1}\int_{\mathsf{R}^2}f(y)e^{-i\xi^{-1}(y_1^2-y_2^2)/2}I_0(-\xi^{-1}y,t,\epsilon)dy\\
&=|-\xi^{-1}|^{1+it}\mathscr{F}_0f(-\xi^{-1},t,\epsilon)\\
&=\pi_{0,it}(\sigma)\mathscr{F}_0(\xi,t,\epsilon).
\end{aligned}$$

However it seems difficult to make sense out of this calculation so we have chosen the above proof using the Gaussian kernel.

Next we present the Plancherel formula.

THEOREM 2. With the notations as above, we have, the Plancherel formula

$$\|f\|^2 = \frac{1}{8\pi^2}\int_0^\infty \|\mathscr{F}_0 f(t)\|^2_{L^2(\mathsf{R}^1)\otimes\mathsf{C}^2}dt + \frac{1}{8\pi^2}\int_0^\infty \|\mathscr{F}_1 f(t)\|^2_{L^2(\mathsf{R}^1)\otimes\mathsf{C}^2}dt.$$

As representations of $\mathrm{SL}(2,\mathsf{R}) \times O(1,1)$ *on* $L^2(\mathsf{R}^2)$, *we have*

$$\widetilde{L} \cong \int_0^\infty \pi_{0,it}\otimes\widetilde{\pi}_{0,it}\,dt + \int_0^\infty \pi_{1,it}\otimes\widetilde{\pi}_{1,it}\,dt.$$

PROOF. To simplify the argument we prove first the Plancherel formula on the space $\mathscr{D}_\delta(\mathsf{R}^2)$of C^∞ even ($\delta = 0$) or odd ($\delta = 1$) functions with compact supports on R^2. We let $\delta = 0$.

Let $f \in \mathscr{D}_0(\mathsf{R}^2)$, then $\mathscr{F}_1 f = 0$. We follow the proof of Proposition 1. First we notice that the function $F(\eta, t, \epsilon)$ defined there is now having compact support in η and moreover$F(\cdot, t, \epsilon)$ is in $L^2(\mathsf{R}) \cap L^1(\mathsf{R})$, as is shown in the proof of Proposition 1.

Now $\mathscr{F}_0 f$ is the usual Fourier transform of $F(\eta, t, \epsilon)$ in the L^2-sense; $F(\eta, t, \epsilon)$ is essentially the Fourier transform of f along the hyperboloids. See further [20]. It follows from the usual Plancherel formula that

$$\begin{aligned}
\int_{\mathsf{R}^2} |f(x_1, x_2)|^2 dx &= \int_{\mathsf{R}^1}\int_{x_1^2 - x_2^2 = \eta} |f(x_1, x_2)|^2 dh(x)d\eta \\
&= \frac{1}{2\pi}\int_{\mathsf{R}^1}\sum_{\epsilon=\pm1}\int_0^\infty |F(\eta, t, \epsilon)|^2 dt d\eta \\
&= \frac{1}{2\pi}\int_0^\infty\sum_{\epsilon=\pm1}\int_{\mathsf{R}^1} |F(\eta, t, \epsilon)|^2 d\eta dt \\
&= \frac{1}{8\pi^2}\int_0^\infty\sum_{\epsilon=\pm1}\int_{\mathsf{R}^1} |\mathscr{F}_0 f(\xi, t, \epsilon)|^2 d\xi dt \\
&= \frac{1}{8\pi^2}\int_0^\infty \|\mathscr{F}_0 f(t)\|^2_{L^2(\mathsf{R}^1)\otimes\mathsf{C}^2}dt.
\end{aligned}$$

Thus $\mathscr{F}_0$ extends uniquely to an isometric operator from $L^2(\mathsf{R}^2)_0$, the space of even functions in $L^2(\mathsf{R}^2)$, to the direct integral $\int_0^\infty \pi_{0,it}\otimes\widetilde{\pi}_{0,it}\,dt$.

We now prove that $\mathscr{F}_0$ is onto. Together with Theorem 1 this will prove Theorem 2.

Notice that the space of functions $H(\xi, t, \epsilon)$ on $\mathsf{R}\times\mathsf{R}^+$ that are 1) C^∞ with compact support in t and 2) their Fourier transforms

$$\int_{\mathsf{R}} H(\xi, t, \epsilon)e^{-\frac{i\eta\xi}{2}}d\xi$$

having compact support in $(-\infty, 0)\cup(0,\infty)$ in η for each fixed t and ϵ, is dense in $\int_0^\infty \pi_{0,it}\otimes\widetilde{\pi}_{0,it}\,dt$. We define f via

$$f(x_1,x_2)=\frac{1}{8\pi^2}\sum_{\epsilon=\pm1}\int_0^\infty\int_{-\infty}^\infty e^{-\frac{i(x_1^2-x_2^2)\xi}{2}}|x_1+\epsilon x_2|^{-it}H(\xi,t,\epsilon)d\xi dt.$$

Performing the same calculation as above we know that $f\in L^2(\mathsf{R})_0$ and thus $\mathscr{F}_0 f$ is defined. Moreover the calculation shows that $\mathscr{F}_0 f=H$. Therefore $\mathscr{F}_0$ is onto. Similarly for the odd functions.

§2. Metaplectic representation of $U(1,1)$

§2.1 *Intertwining operators*

Let $U(1,1)$be the group of complex 2×2 matrices g such that

$$g^*Hg=H,$$

here

$$H=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

Later we will use another realization of $U(1,1)$. We write $U(1,1)_L$ for the above realization.

The following elements generate $U(1,1)_L$,

$$g(a)=\begin{pmatrix}a & 0\\ 0 & \bar{a}^{-1}\end{pmatrix}, a\in\mathsf{C}^\times=\mathsf{C}\setminus\{0\},$$

$$t(x)=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}, x\in\mathsf{R},$$

$$\sigma=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

The following choice of $L(g)$ determines a representation of $U(1,1)_L$ on $L^2(\mathsf{C})$,

$$L(g(a))f(w)=af(\bar{a}w),$$

$$L(t(x))f(w)=e^{-ix|w|^2}f(w),$$

$$L(\sigma)f(w)=\frac{i}{2\pi}\int_{\mathsf{C}}e^{2i\Re(w\bar{w}')}f(w')|dw'|^2.$$

Here $\Re z$ denotes the real part of $z\in\mathsf{C}$ and $|dw|^2$ the Lebesgue measure on C.

We consider $L \otimes \bar{L}$ on $L^2(\mathsf{C}) \otimes L^2(\mathsf{C}) = L^2(\mathsf{C}^2)$,

$$(L \otimes \bar{L})(g(a))f(z) = |a|^2 f(\bar{a}z),$$

$$(L \otimes \bar{L})(t(x))f(z) = e^{-ix(|z_1|^2 - |z_2|^2)} f(z),\ z = (z_1, z_2)$$

$$(L \otimes \bar{L})(\sigma)f(z) = \frac{1}{(2\pi)^2} \int_{\mathsf{C}^2} e^{2i\Re(z_1\bar{z}'_1 - z_2\bar{z}'_2)} f(z')|dz'|^2.$$

Now on $L^2(\mathsf{C}^2)$ we have also the unitary action of $U(1,1)$ from the right

$$U \cdot f(z) = f(zU),\ U \in U(1,1).$$

Here $U(1,1)$ is realized as the group of matrices on C^2 keeping $|z_1|^2 - |z_2|^2$ invariant. We write $U(1,1)_R$ to specify this realization. This action commutes with the action of $L \otimes \bar{L}$. We denote $\widetilde{L}$ the representation $U(1,1)_L \times U(1,1)_R$ on $L^2(\mathsf{C}^2)$ with $U(1,1)_L$ acting from the left by $L \otimes \bar{L}$ and $U(1,1)_R$ from the right as above.

Our objective in this section is to give as we did in the real case the decomposition of $L^2(\mathsf{C}^2)$ into irreducibles under the above action of $U(1,1)_L \times U(1,1)_R$.

For $t \in \mathsf{C}$ and n an integer, we denote $\widetilde{\pi}_{it,n}$ the representation of $U(1,1)_R$ on $C^\infty(\mathsf{T})$:

$$\widetilde{\pi}_{it,n}(U)f(b) = |\alpha + b\gamma|^{it-1} \left(\frac{\alpha + b\gamma}{|\alpha + b\gamma|}\right)^n f\left(\frac{\beta + b\delta}{\alpha + b\gamma}\right),\ b \in \mathsf{T}, \tag{2.0}$$

and

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U(1,1)_R.$$

The group $U(1,1)_R$ has the center

$$\left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix},\ 0 \le \theta < 2\pi \right\},$$

which is also the center of $U(1,1)_L$. So first we decompose $L^2(\mathsf{C}^2)$ under the action of the center.

$$L^2(\mathsf{C}^2) = \bigoplus_{n=-\infty}^{\infty} L^2(\mathsf{C}^2)_n,$$

where

$$L^2(\mathsf{C}^2)_n = \{f \in L^2(\mathsf{C}^2) : f(e^{i\theta}z) = e^{in\theta} f(z)\}.$$

It is clear that each $L^2(\mathsf{C}^2)_n$ is a $U(1,1)_L \times U(1,1)_R$ module. Thus

$$\widetilde{L} = \sum_{n=-\infty}^{\infty} \widetilde{L}_n,$$

where

$$\widetilde{L}_n = \widetilde{L}|_{L^2(\mathbf{C}^2)_n}.$$

We notice that if a subspace H of $L^2(\mathbf{C}^2)_n$ is an invariant subspace of $\widetilde{L}_n$ then its complex conjugate $\overline{H} = \{\bar{f}; f \in H\}$ is an invariant subspace of $\widetilde{L}_{-n}$ on $L^2(\mathbf{C}^2)_{-n}$; and that the representations appearing in $\overline{H}$ are just the dual representations of those in H; this follows easily from the formulae of $\widetilde{L}$ in the beginning of this section. We thus need only treat the case $n \leq 0$.

From now on we assume that $n \leq 0$.

We consider now $L^2(\mathbf{C}^2)_n$. Define the following function on $\mathbf{C}^2$,

$$I_n(z, t, b) = |\bar{z}_1 - b\bar{z}_2|^{it-1-n}(\bar{z}_1 - b\bar{z}_2)^n,$$

where $b \in \mathbf{T}$ and $t \in \mathbf{C}$.

Lemma 3. *$I_n(z, t, b)$ has the following properties:*

(1) $I_n(az, t, b) = |a|^{it-1-n}(\bar{a})^n I_n(z, t, b)$, $a \in \mathbf{C}^\times$;

(2) $I_n(zU^{-1}, t, \cdot)(b) = \widetilde{\pi}_{it,n}(U) I_n(z, t, \cdot)(b)$, $U \in U(1,1)_R$, and $\widetilde{\pi}_{it,n}(U)$ acts on b.

(3) $I_n(z, t, b)$ solves the ultra-hyperbolic equation

$$\frac{\partial^2 I_n}{\partial z_1 \partial \bar{z}_1} - \frac{\partial^2 I_n}{\partial z_2 \partial \bar{z}_2} = 0.$$

Proof. (1) follows directly. For the proof of (2) we only need to notice that for U as in (2.0) then

$$U^{-1} = \begin{pmatrix} \bar{\alpha} & -\bar{\gamma} \\ -\bar{\beta} & \bar{\delta} \end{pmatrix}.$$

(3) is easy to check.

The claim (3) above will not be used in our paper; we put it because it might help to compare our intertwining operator (2.1) below with those in [7].

Denote by $\mathscr{S}(\mathbf{C}^2)_n$ the space of Schwartz functions in $L^2(\mathbf{C}^2)_n$. Clearly $\mathscr{S}(\mathbf{C}^2)_n$ is stable under $\widetilde{L}$. We define the following operator on $\mathscr{S}(\mathbf{C}^2)_n$:

$$\mathscr{F}_n f(\xi, t, b) = \int_{\mathbf{C}^2} e^{i(|z_1|^2 - |z_2|^2)\xi} I_n(z, t, b) f(z) |dz|^2 \tag{2.1}$$

Proposition 2. *For all* $t \in \mathbf{R}$ *the operator* $\mathscr{F}_n$ *maps* $\mathscr{S}(\mathbf{C}^2)_n$ *into* $L^2(\mathbf{R}^2) \otimes L^2(\mathbf{T})$, *that is* $\mathscr{F}_n f(\cdot, t, \cdot) \in L^2(\mathbf{R}) \otimes L^2(\mathbf{T})$, *for* $f \in \mathscr{S}(\mathbf{C}^2)_n$.

To prove the Proposition we need the following elementary fact. We omit the proof.

Lemma 4. *If* $f(x_1, x_2)$ *is a Schwartz function of* $(x_1, x_2) \in \mathbf{R}^{2n}$ *then its partial Fourier transform*

$$\widehat{f}_2(x_1, y) = \int_{\mathbf{R}^n} e^{i(y, x_2)} f(x_1, x_2) dx_2$$

is a Schwartz function of (x_1, y).

Proof Proposition 2. We take $b = 1$ in (2.1) and perform change of variables $w_1 = \frac{z_1 + z_2}{2}$, $w_2 = \frac{z_1 - z_2}{2}$. We have

$$\begin{aligned} \mathscr{F}_n f(\xi, t, 1) &= \frac{1}{2} \int_{\mathbf{C}^2} e^{i4\xi \Re(w_2 \bar{w}_1)} |w_2|^{it-1} \left(\frac{w_2}{|w_2|}\right)^n f(w_1 + w_2, w_1 - w_2) |dw|^2 \\ &= \frac{1}{2} \int_{\mathbf{C}} |w_2|^{it-1} \left(\frac{w_2}{|w_2|}\right)^n g(w_2 \xi, w_2) |dw_2|^2, \end{aligned}$$

where $g(u, w_2)$ is the partial Fourier transform of $f(w_1 + w_2, w_1 - w_2)$ in w_1

$$g(u, w_2) = \int_{\mathbf{C}} e^{i4\Re(u\bar{w}_1)} f(w_1 + w_2, w_1 - w_2) |dw_1|^2.$$

It follows from Lemma 4 that $g(u, w_2)$ is in the Schwartz class $\mathscr{S}(\mathbf{C}^2)$, since $f(w_1 + w_2, w_1 - w_2)$ is a Schwartz function of (w_1, w_2). Hence,

$$\begin{aligned} |\mathscr{F}_n f(\xi, t, 1)| &\le C \int_{\mathbf{C}} |w_2|^{-1} |g(w_2 \xi, w_2)| |dw_2|^2 \\ &\le C \int_{\mathbf{C}} |w_2|^{-1} |(1 + |w_2|^2 (1 + \xi^2))^{-N} |dw_2|^2 \end{aligned}$$

for some positive constant C and some integer $N \ge 2$. Changing variables $w_2 \mapsto w_2 (1 + \xi^2)^{-\frac{1}{2}}$ we get

$$|\mathscr{F}_n f(\xi, t, 1)| \le C(1 + \xi^2)^{-\frac{1}{2}}$$

and the function $(1 + \xi^2)^{-\frac{1}{2}}$ is in $L^2(\mathbf{R})$.

Now we note that the integral (2.1) can be written

$$\mathscr{F}_n f(\xi, t, b) = \int_{\mathbf{C}^2} e^{i(|z_1|^2 - |z_2|^2)\xi} I_n(z, t, 1) f(z_1, bz_2) |dz|^2,$$

and $f(z_1, bz_2)$ is a Schwartz function of z; the above estimates are therefore true with $f(z_1, z_2)$ replaced by $f(z_1, bz_2)$ and uniformly for all $b \in \mathsf{T}$. The Proposition follows.

The above Proposition also shows that for $t \in \mathsf{R}$, and $f \in \mathscr{S}(\mathsf{C}^2)_n$ the integral (2.1) is well-defined for all $(\xi, b) \in \mathsf{R} \times \mathsf{T}$. The next Lemma shows that the integral has an analytic continuation in t.

LEMMA 5. *The integral (2.1) for fixed $b \in \mathsf{T}$ and $\xi \in \mathsf{R}$ has analytic continuation in t in the half-plane*

$$\{t \in \mathsf{C}, \Re(it) > -(|n| + 1)\}.$$

Moreover when t is in this half-plane $\mathscr{F}_n f(\xi, t, b)$ as a function of ξ, for every fixed $b \in \mathsf{T}$, is in $L^2(\mathsf{R}, (1 + \xi^2)^{\Re(it)} d\xi)$.

PROOF. We follow the proof of Proposition 2 and we put $b = 1$ (same argument for general values). We get

$$\mathscr{F}_n f(\xi, t, 1) = \frac{1}{2} \int_{\mathsf{C}} |w_2|^{it-1} (\frac{w_2}{|w_2|})^n g(w_2 \xi, w_2) |dw_2|^2.$$

We write further this integral using polar coordinates $w_2 = re^{i\theta}$,

$$\mathscr{F}_n f(\xi, t, 1) = \frac{1}{2} \int_0^\infty r^{it} h(\xi, r) dr,$$

where

$$h(\xi, r) = \int_0^{2\pi} g(\xi re^{i\theta}, re^{i\theta}) e^{in\theta} d\theta.$$

It is clear that $\mathscr{F}_n f(\xi, t, 1)$ is absolute convergent since g is a Schwartz function. Thus $\mathscr{F}_n f(\xi, t, 1)$ has analytic continuation on $\Re(it) > -1$. This proves the Lemma for $n = 0$. Now let $n \neq 0$, we have

$$h(\xi, 0) = g(0, 0) \int_0^{2\pi} e^{in\theta} d\theta = 0.$$

Furthermore,

$$\frac{\partial h(\xi, 0)}{\partial r} = \int_0^{2\pi} (\xi e^{i\theta} \partial_1 g(0,0) + \xi e^{-i\theta} \bar{\partial}_1 g(0,0) + e^{i\theta} \partial_2 g(0,0) + e^{-i\theta} \bar{\partial}_2 g(0,0)) e^{in\theta} d\theta,$$

where ∂_1, ∂_2 is the usual partial complex derivative with respect to the first and the second argument, respectively. Thus

$$\frac{\partial h(\xi,0)}{\partial r}=0$$

if $|n|>1$. It is now not difficult to see that

$$\frac{\partial^k h(\xi,0)}{\partial r^k}=0$$

for all $k=0,1,,|n|-1$. It follows from the standard argument ([1], Chapter I, §4.2, or our calculation below) that the integral $\mathscr{F}_n f(\xi,t,1)$ has analytic continuation to the region $\{t\in\mathsf{C},\Re(it)>-(1+|n|)\}$.

Now, by partial integration, we have

$$\mathscr{F}_n f(\xi,t,1)=\frac{1}{4}\frac{(-1)^{|n|-1}}{(it+1)(it+2)\cdots(it+|n|-1)}\int_0^\infty r^{it+|n|-1}\frac{\partial^{|n|-1}h(\xi,r)}{\partial^{|n|-1}r}dr.$$

The function g is of Schwartz class and we thus have, using the formula for h,

$$\left|\frac{\partial^{|n|-1}h(\xi,r)}{\partial^{|n|-1}r}\right|\le C(1+|\xi|)^{|n|-1}(1+|r|^2(1+|\xi|^2))^{-N}$$

for some sufficient large $N\ge 2$. Thus, if $it\ne -k$, $k=1,2,,|n|-1$ we have

$$\begin{aligned}|\mathscr{F}_n f(\xi,t,1)|&\le C(1+|\xi|)^{|n|-1}\int_0^\infty r^{\Re(it)+|n|-1}(1+|r|^2(1+|\xi|^2))^{-N}dr\\ &\le C(1+|\xi|)^{|n|-1}(1+|\xi|^2)^{\frac{-\Re(it)-|n|+1}{2}}(1+|\xi|^2)^{-\frac{1}{2}}\\ &\le C(1+|\xi|^2)^{\frac{-\Re(it)-1}{2}}\end{aligned}$$

Thus $\mathscr{F}_n f(\xi,t,1)$ is in $L^2(\mathsf{R},(1+\xi^2)^{\Re(it)}d\xi)$. If $it=-k$ for some $k=1,2,,|n|-1$ then by taking residue we have

$$\mathscr{F}_n f(\xi,t,1)=C\int_0^\infty r^{it+|n|-1}\log r\frac{\partial^{|n|-1}h(\xi,r)}{\partial^{|n|-1}r}dr.$$

for some constant C. A similar estimate gives

$$\begin{aligned}|\mathscr{F}_n f(\xi,t,1)|&\le C(1+\log(1+|\xi|^2))(1+|\xi|)^{|n|-1}(1+|\xi|^2)^{\frac{-\Re(it)-|n|+1}{2}}(1+|\xi|^2)^{-\frac{1}{2}}\\ &\le C(1+\log(1+|\xi|^2))(1+|\xi|^2)^{\frac{-\Re(it)-1}{2}}\end{aligned}$$

which is still in the space $L^2(\mathsf{R},(1+\xi^2)^{\Re(it)}d\xi)$.

For $t\in\mathsf{C}$, we let $\pi_{it,n}$ denote the representation of $U(1,1)_L$ on $L^2(\mathsf{R},(1+|\xi|^2)^{\Re(it)}d\xi)$ defined by:

(2.2) $$\pi_{it,n}(g)f(\xi) = |\gamma\xi+\delta|^{-it-1}\left(\frac{\gamma\xi+\delta}{|\gamma\xi+\delta|}\right)^n f\left(\frac{\alpha\xi+\beta}{\gamma\xi+\delta}\right),$$

where

$$g^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U(1,1)_L.$$

(Indeed, direct calculation shows that $L^2(\mathsf{R}, (1+|\xi|^2)^{\Re(it)}d\xi)$ is invariant under the above action.) Note that when $t \in \mathsf{R}$ the representation space is $L^2(\mathsf{R})$.

THEOREM 3. *For $t \in \mathsf{R}$, the operator $\mathscr{F}_n$ intertwines the action $\widetilde{L}_n$ with $\pi_{it,n} \otimes \widetilde{\pi}_{it,n}$ of $U(1,1)_L \times U(1,1)_R$.*

PROOF. Proposition 2 shows that $\mathscr{F}_n$ maps $\mathscr{S}(\mathsf{C}^2)_n$ into $L^2(\mathsf{R}) \otimes L^2(\mathsf{T})$ so that the statement makes sense. The $U(1,1)_R$ right intertwining relation is an immediate consequence of the formula (2) in Lemma 3. We now prove the left intertwining relation on the generators $g(a), t(x), \sigma$ of $U(1,1)_L$.

We have

$$\begin{aligned}
&\mathscr{F}_n(L\otimes\bar{L})(g(a))f(\xi,t,b)\\
&= \int_{\mathsf{C}^2} e^{i(|z_1|^2-|z_2|^2)\xi} I_n(z,t,b)|a|^2 f(\bar{a}z)|dz|^2\\
&= \int_{\mathsf{C}^2} e^{i(|z_1|^2-|z_2|^2)|a^{-1}|^2\xi} I_n(\bar{a}^{-1}z,t,b)|a^{-1}|^2 f(z)|dz|^2\\
&= |a^{-1}|^{it+1-n}(a^{-1})^n \mathscr{F}f(|a^{-1}|^2\xi,t,b)\\
&= \pi_{it,n}(g(a))\mathscr{F}_n f(\xi,t,b).
\end{aligned}$$

It is also clear that

$$\mathscr{F}_n(L\otimes\bar{L})(t(x)) = \pi_{it,n}(t(x))\mathscr{F}_n.$$

Now we prove the σ-intertwining relation. Let $f \in \mathscr{S}_n(\mathsf{C}^2)$, then

$$\begin{aligned}
&\mathscr{F}_n(L\otimes\bar{L})(\sigma)f(\xi,t,b)\\
&= \int_{\mathsf{C}^2} e^{i(|z_1|^2-|z_2|^2)\xi} I_n(z,t,b)(L\otimes\bar{L})(\sigma)f(z)|dz|^2\\
&= \frac{1}{(2\pi)^2}\int_{\mathsf{C}^2} e^{i(|z_1|^2-|z_2|^2)\xi} I_n(z,t,b)\left(\int_{\mathsf{C}^2} e^{2i\Re(z_1\bar{z}_1'-z_2\bar{z}_2')}f(z')|dz'|^2\right)|dz|^2
\end{aligned}$$

Notice (3) of Lemma 3. We can argue similarly as in the proof of Lemma 2 to get that this integral is (we omit the details)

$$|\xi|^{-it-1-n}(-\xi)^n \int_{\mathbf{C}^2} e^{-i\xi^{-1}(|z_1'|^2-|z_2'|^2)} I_n(z',t,b)f(z')|dz'|^2$$
$$= \pi_{it,n}(\sigma)\mathscr{F}_n f(\xi,t,b)$$

This finishes the proof of Theorem 3.

Now using Lemma 5, we have, by analytic continuation

PROPOSITION 3 *The operator $\mathscr{F}_n$ intertwines $\widetilde{L}_n$ with the representation $\pi_{it,n} \otimes \widetilde{\pi}_{it,n}$ for all t such that $|\Re(it)| < |n| + 1$.*

Let $D_{\nu-2l,-\nu}$ and $\overline{D_{\nu-2l,-\nu}}$ be the representations of $U(1,1)_L$ and define $P_{|n|-2l,n}$ and $\overline{P_{|n|-2l,n}}$ as in the Appendix. Here $\nu = -n$. (Recall that this last operator is a certain extension from the real axis of a function there to a holomorphic function in the lower half plane.) From Lemmas A1 and A2 in the Appendix we have the next claim, where with abuse of notation we also denote $D_{\nu-2l,-\nu}$ (resp. $\overline{D_{\nu-2l,-\nu}}$) the actions of $U(1,1)_L$ on the *space of all holomorphic functions* in the upper half plane (resp. lower halfplane). This will be enough for our purpose as we will later see that only the actions of $U(1,1)_L$ on the corresponding *Hilbert spaces of holomorphic functions* (see Appendix) will be involved in the L^2-decomposition.

PROPOSITION 4 *For each fixed $b \in \mathbf{T}$ the operator $P_{|n|-2l,n}\mathscr{F}_n(i(|n|-1-2l),b)$ intertwines $U(1,1)_L$ action $\widetilde{L}_n$ with the representation $D_{\nu-2l,-\nu}$(acting on the space of holomorphic functions on the upper half-plane); the operator $\overline{P_{|n|-2l,n}}\mathscr{F}_n(-i(|n|-1-2l),b)$ intertwines the $U(1,1)_L$ action $\widetilde{L}_n$ with the representation $\overline{D_{\nu-2l,-\nu}}$(acting on the space of holomorphic functions in the lower half-plane; recall that by Lemma 5 and Lemma A2 $\mathscr{F}_n$ in this case maps into a subspace of the principal series representation).*

§2.2 *Plancherel formula*

Before presenting the Plancherel formula on $L^2(\mathbf{C}^2)_n$, we recall some facts about irreducible decomposition of a space of L^2-sections of a line bundle over the unit disk in the complex plane. See e. g. [2], [10] and [13].

Let $D = \{w \in \mathbf{C} : |w| < 1\}$ be the unit disk in the complex plane with $|dw|^2$ the Lebesgue measure on D and let ν be a real number. Consider the weighted measure $d\mu_\alpha(w) = (1-|w|^2)^\alpha |dw|^2$ on D, where $\alpha = \nu - 2$. Then the universal covering group of $U(1,1)_R$ acts unitarily on $L^2(D, d\mu_\alpha)$ via

$$(2.3)\qquad g: f(z) \mapsto f\left(\frac{\beta+\delta w}{\alpha+\gamma w}\right)(\alpha+\gamma w)^{-\nu},\ g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U(1,1)_R$$

Here we tacitly subsume the choice of logarithm corresponding to the covering group.

When ν is a nonnegative integer (or real number), the explicit irreducible decomposition of this action is given in [2], [10] and [13]; we summarize here briefly the result.

Define the following transform, for f a C^∞-function on D with compact support,

$$\tilde{f}(t,b) = \int_D \left(\frac{1-|w|^2}{|1-b\bar{w}|^2}\right)^{\frac{-it+1-\nu}{2}} \frac{1}{(1-b\bar{w})^\nu} f(w)\,d\mu_\alpha(w),\ b\in \mathrm{T} \tag{2.4}$$

Then we have the inversion formula

$$\begin{aligned} f(0) &= \frac{2^{-2\nu}}{\pi^2}\int_0^\infty \left(\int_\mathrm{T} \tilde{f}(t,b)\,db\right)|C(t)|^{-2}dt \\ &+ \sum_{\nu-2l>1} \frac{\nu-1-2l}{\pi}\int_\mathrm{T} \tilde{f}(i(\nu-1-2l),b)\,db, \end{aligned}$$

and the Plancherel formula

$$\begin{aligned} \int_D |f(w)|^2 d\mu_\alpha(w) &= \frac{2^{-2\nu}}{\pi^2}\int_0^\infty \left(\int_\mathrm{T} |\tilde{f}(t,b)|^2 db\right)|C(t)|^{-2}dt+ \\ &+ \sum_{\nu-2l>1} \frac{\nu-1-2l}{\pi}\|\tilde{f}(i(\nu-1-2l),\cdot)\|^2_{\tilde{D}_{\nu-2l,-\nu}}, \end{aligned}$$

where $C(t)$ is the generalized Harish-Chandra C-function

$$C(t) = \frac{2^{-\nu+1-it}\Gamma(it)}{\Gamma(\frac{-\nu+1+it}{2})\Gamma(\frac{\nu+1+it}{2})},$$

and $\tilde{D}_{\nu-2l,-\nu}$ are the holomorphic discrete series. Here the summation is over all the nonnegative integers l such that $\nu-2l>1$.

As representations of the universal covering of $U(1,1)_R$,

$$L^2(D, d\mu_\alpha) \cong \int_0^\infty \tilde{\pi}_{it,-\nu}dt + \sum_{\nu-2l>1} \tilde{D}_{\nu-2l,-\nu}, \tag{2.5}$$

where $\tilde{\pi}_{it,-\nu}$ are the principal series of $U(1,1)_R$ as defined in (2.0).

More explicitly, the space $\tilde{D}_{\nu-2l,-\nu}$ in the above formula consists of all the distributions $f(b) = \sum_{m=-l}^\infty \hat{f}(m)b^m$ on T such that

$$\|f\|^2_{\tilde{D}_{\nu-2l,-\nu}} = \sum_{m=-l}^\infty \frac{\Gamma(\nu-l+m)}{(l+m)!}|\hat{f}(m)|^2 < \infty.$$

The group action is

$$\widetilde{D}_{\nu-2l,-\nu}(g)f(b) = |\alpha + b\gamma|^{2l}(\alpha + b\gamma)^{-\nu} f\left(\frac{\beta + b\delta}{\alpha + b\gamma}\right), b \in \mathsf{T},$$

and

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U(1,1)_R.$$

This is just the action (2.0) with $t = i(\nu - 1 - 2l)$ and $n = -\nu$. One can derive those results from [13], we omit the details. (With certain normalization, writing the complexified Lie algebra $su(1,1)^{\mathsf{C}}$ of $SU(1,1)$ as $\mathsf{C}H + \mathsf{C}X^+ + \mathsf{C}X^-$, with $[H, X^\pm] = \pm 2X^\pm = \pm 2H^*(H)X^\pm$, then $\widetilde{D}_{\nu-2l,-\nu}$ is a lowest weight module of $su(1,1)^{\mathsf{C}}$,with lowest weight $(\nu - 2l)H^*$ and b^{-l} is the lowest weight vector. The map $b^m \mapsto w^{m+l}$is a $su(1,1)^{\mathsf{C}}$-intertwining operator identifying $\widetilde{D}_{\nu-2l,-\nu}$ with the weighted Bergman space of holomorphic functions on the unit disk $D = \{w \in \mathsf{C}; |w| < 1\}$ with weight $(1 - |w|^2)^{\underline{\alpha - 2l - 2}}|dw|^2$. We omit the routine calculation.) Similarly we introduce $\overline{\widetilde{D_{\nu-2l,-\nu}}}$of $U(1,1)_R$. It consists of all the distributions of the form $f(b) = b^{-\nu}\sum_{m=-\infty}^{l} \widehat{f}(m)b^m$ on T such that

$$\|f\|^2_{\overline{\widetilde{D_{\nu-2l,-\nu}}}} = \sum_{m=-\infty}^{l} \frac{\Gamma(\nu - l - m)}{(l - m)!} |\widehat{f}(m)|^2 < \infty.$$

The group action is

$$\overline{\widetilde{D_{\nu-2l,-\nu}}}(g)f(b) = (\alpha + b\gamma)^{-1-l}\left(\overline{\alpha + b\gamma}\right)^{\nu-1-l} f\left(\frac{\beta + b\delta}{\alpha + b\gamma}\right), \ b \in \mathsf{T},$$

and

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U(1,1)_R.$$

This is just the action (2.0) with $t = -i(\nu - 1 - 2l)$ and $n = -\nu$. (Infinitesimally, $\overline{\widetilde{D_{\nu-2l,-\nu}}}$ is realized as a quotient of the whole representation (2.0) on smooth functions on the circle.)

Let $\mathscr{D}_n^0$ be the subspace of $L^2(\mathsf{C}^2)_n$ of C^∞-functions on $\{z \in \mathsf{C}^2; |z_1|^2 - |z_2|^2 \neq 0\}$ with compact supports. It is clear that $\mathscr{D}_n^0$ is dense in $L^2(\mathsf{C}^2)_n$.

THEOREM 4. *Let $f \in \mathscr{D}_n^0$. We have the following Plancherel formula*

$$\begin{aligned}
&\int_{\mathbf{C}^2} |f(z)|^2 |dz|^2 \\
&= \frac{2^{-2|n|-1}}{\pi^4} \int_0^\infty \|\mathcal{F}_n f(\cdot, t, \cdot)\|^2_{L^2(\mathbf{R})\otimes L^2(\mathbf{T})} |C(t)|^{-2} dt \\
&\quad + \sum_{|n|-2l>1} \frac{|n|-1-2l}{\pi^2} \|P_{|n|-2l,n}\mathcal{F}_n f(\cdot, i(|n|-1-2l), \cdot)\|_{D_{|n|-2l,n}\otimes \widetilde{D}_{|n|-2l,n}} \\
&\quad + \sum_{|n|-2l>1} \frac{|n|-1-2l}{\pi^2} \|\overline{P_{|n|-2l,n}}\mathcal{F}_n f(\cdot, -i(|n|-1-2l), \cdot)\|^2_{\overline{D_{|n|-2l,n}}\otimes \overline{\widetilde{D_{|n|-2l,n}}}}.
\end{aligned}$$

As representations of $U(1,1)_L \times U(1,1)_R$,

$$\widetilde{L}_n \cong \int_0^\infty \pi_{it,n} \otimes \widetilde{\pi}_{it,n}\, dt \oplus \sum_{|n|-2l>1} D_{|n|-2l,n} \otimes \widetilde{D}_{|n|-2l,n} \oplus \sum_{|n|-2l>1} \overline{D_{|n|-2l,n}} \otimes \overline{\widetilde{D_{|n|-2l,n}}}.$$

PROOF. Step 1. Plancherel formula. We start with (2.1). Since f is in $\mathcal{D}_n^0$ we see that $\mathcal{F}_n f$ is well-defined for all $t \in \mathbf{C}$. By parametrizing the space $\mathbf{C}^2$ by the hyperboloids $\sqrt{|\eta|}z$, with z such that $|z_1|^2 - |z_2|^2 = sgn\,\eta$, and $\eta \in \mathbf{R}$, we see that (2.1) can be written as

$$\mathcal{F}_n f(\xi, t, b) = \int_{\mathbf{R}} e^{i\eta\xi} F(\eta, t, b) d\eta, \tag{2.7}$$

where

$$\begin{aligned}
F(\eta, t, b) = \frac{1}{2}|\eta|^{\frac{it+1}{2}} \int_{|z_1|^2-|z_2|^2=sgn\eta} & |\bar{z}_1 - b\bar{z}_2|^{it-1-n}(\bar{z}_1 - b\bar{z}_2)^n \\
& \times f(\sqrt{|\eta|}z_1, \sqrt{|\eta|}z_2) dh(z)
\end{aligned}$$

and $dh(z)$ is a measure on the hyperboloid $|z_1|^2 - |z_2|^2 = \pm 1$. Similar calculation as in the proof of Proposition 2 shows that $F(\eta, t, b)$ as a function of η is in $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ uniformly for all $b \in \mathbf{T}$ and it vanishes near $\eta = 0$ and $\eta = \pm\infty$. Thus the integral (2.7) makes sense.

Assume $\eta > 0$. Let

$$G(\eta, w) = f(\sqrt{\eta}z_1, \sqrt{\eta}z_2) z_1^{-n}, \tag{2.8}$$

where

$$w = \frac{z_2}{z_1}, \; |z_1|^2 - |z_2|^2 = 1. \tag{2.9}$$

The map (2.9) clearly maps the hyperboloid $|z_1|^2 - |z_2|^2 = 1$ onto the unit disk $D = \{w \in \mathbf{C} : |w| < 1\}$ and since the function f satisfies $f(e^{i\theta}z) =$

$e^{in\theta}f(z)$, we see that $G(\eta, w)$ is well-defined. We can also invert (2.8) and get that, if $|z_1|^2 - |z_2|^2 > 0$,

$$f(|z_1|, z_2) = \left(\frac{|z_1|^2 - |z_2|^2}{|z_1|^2}\right)^{-\frac{n}{2}} G(|z_1|^2 - |z_2|^2, \frac{z_2}{|z_1|}). \tag{2.10}$$

Writing $z_1 = e^{i\theta} r$ we see that

$$\begin{aligned}
F(\eta, t, b) &= \frac{1}{2}|\eta|^{\frac{it+1}{2}} \int_0^{2\pi} \int_{|r|^2 - |z_2|^2 = 1} |r - be^{-i\theta}\bar{z}_2|^{it-1-n} e^{-in\theta}(r - be^{-i\theta}\bar{z}_2)^n \\
&\quad \times f\left(\sqrt{|\eta|}re^{i\theta}, \sqrt{|\eta|}z_2\right) dh(r, z_2) d\theta \\
&= \frac{1}{2}|\eta|^{\frac{it+1}{2}} \int_0^{2\pi} |\eta|^{\frac{it+1}{2}} \int_{|r|^2 - |z_2|^2 = 1} |r - b\bar{z}_2|^{it-1-n}(r - b\bar{z}_2)^n \\
&\quad \times f\left(\sqrt{|\eta|}r, \sqrt{|\eta|}z_2\right) dh(r, z_2) d\theta \\
&= \pi|\eta|^{\frac{it+1}{2}} \int_{|r|^2 - |z_2|^2 = 1} |r - b\bar{e}^{-i\theta} z_2|^{it-1-n}(r - b\bar{z}_2)^n \\
&\quad \times f\left(\sqrt{|\eta|}r, \sqrt{|\eta|}z_2\right) dh(r, z_2)
\end{aligned} \tag{2.11}$$

where the second equality is obtained by performing the change of variables $z_2 \mapsto e^{i\theta} z_2$ and the fact that $f(e^{i\theta}z) = e^{in\theta}f(z)$, and where $dh(r, z_2) = |dz_2|^2$. Now we use (2.9), we have that $dh(r, z_2)$ becomes the invariant measure $(1 - |w|^2)^{-2}|dw|^2$ on D, and (2.11) can be written as

$$\begin{aligned}
\pi^{-1}\eta^{-\frac{it+1}{2}} F(\eta, t, b) &= \int_D \left(\frac{1 - |w|^2}{|1 - b\bar{w}|^2}\right)^{\frac{-it+1+n}{2}} (1 - b\bar{w})^n \\
&\quad \times G(\eta, w)(1 - |w|^2)^{-n-2}|dw|^2,
\end{aligned} \tag{2.12}$$

This is just the transform (2.4) with $\nu = -n$. Now since f is in $\mathscr{D}_n^0$ it is clear that $G(\eta, w)$ is a function of $(\eta, w) \in \mathrm{R}^+ \times D$ with compact support and is a C^∞ in w.

By the Plancherel formula for the integral transform (2.4), we have

$$\int_D |G(\eta, w)|^2 (1-|w|^2)^{-n-2} |dw|^2$$

$$= \frac{2^{-2|n|}}{\pi^4} \int_0^\infty \left(\int_{\mathrm{T}} |\eta^{-\frac{it+1}{2}} F(\eta, t, b)|^2 db \right) |C(t)|^{-2} dt$$

$$+ \sum_{|n|-2l>1} \frac{|n|-1-2l}{\pi^3} \|\eta^{\frac{-n-1-2l-1}{2}} F(\eta, i(-n-1-2l), \cdot)\|^2_{\widetilde{D}_{-n-2l,n}}.$$

Thus,

$$\int_{|z_1|^2-|z_2|^2=1} |f(\sqrt{\eta} z_1, \sqrt{\eta} z_2)|^2 dh(z)$$

$$= \pi \int_{r^2-|z_2|^2=1} |f(\sqrt{\eta} r, \sqrt{\eta} z_2)|^2 |dz_2|^2$$

$$= \pi \int_D |G(\eta, w)|^2 (1-|w|^2)^{-n-2} |dw|^2$$

$$= \frac{2^{-2|n|}}{\pi^3} \int_0^\infty \left(\int_{\mathrm{T}} |\eta^{-\frac{it+1}{2}} F(\eta, t, b)|^2 db \right) |C(t)|^{-2} dt$$

$$+ \sum_{|n|-2l>1} \frac{|n|-1-2l}{\pi^2} \eta^{(-n-1-2l-1)} \|F(\eta, i(-n-1-2l), \cdot)\|^2_{\widetilde{D}_{-n-2l,n}}.$$

Similarly we get the integral formula over $|z_1|^2 - |z_2|^2 = -1$ with $i(-n-1-2l)$ replaced by $-i(-n-1-2l)$ and $\widetilde{D}_{-n-2l,n}$ by $\overline{D_{-n-2l,n}}$,

$$\int_{|z_1|^2-|z_2|^2=-1} |f\left(\sqrt{|\eta|} z_1, \sqrt{|\eta|} z_2\right)|^2 dh(z)$$

$$= \frac{2^{-2|n|}}{\pi^3} \int_0^\infty \left(\int_{\mathrm{T}} ||\eta|^{-\frac{it+1}{2}} F(\eta, t, b)|^2 db \right) |C(t)|^{-2} dt$$

$$+ \sum_{|n|-2l>1} \frac{|n|-1-2l}{\pi^2} |\eta|^{(n+1+2l-1)} \|F(\eta, -i(-n-1-2l), \cdot)\|^2_{\widetilde{\overline{D_{-n-2l,n}}}}.$$

(We need only to notice that,on the hyperboloid $|z_1|^2 - |z_2|^2 = -1$, by changing variables as in (2.8) and (2.9) with z_1^{-n} replaced by $z_2^{-n}$and $w = \frac{z_1}{z_2}$, formula (2.12) becomes instead

$$(2.12') \qquad \pi^{-1}|\eta|^{-\frac{it+1}{2}}F(\eta,t,b) = b^n\int_D\left(\frac{1-|w|^2}{|1-\bar{b}\bar{w}|^2}\right)^{\frac{-it+1+n}{2}}(1-\bar{b}\bar{w})^n \times G(\eta,w)(1-|w|^2)^{-n-2}|dw|^2.$$

So most calculation will follow from the previous one.)

Now we calculate the norm $\|f\|_{L^2(\mathsf{C}^2)}$. We know that

$$\|f\|^2_{L^2(\mathsf{C}^2)} = \int_{\mathsf{R}}\left(\int_{|z_1|^2-|z_2|^2=sgn\eta}|\eta||f(\sqrt{|\eta|}z_1,\sqrt{|\eta|}z_2)|^2dh(z)\right)d\eta$$

Dividing R into the positive and negative half lines $\mathsf{R}=(-\infty,0)\cap[0,\infty)$ and using our formulas (2.12) and (2.12'), we see that the above becomes

$$\frac{2^{-2|n|}}{\pi^3}\int_{-\infty}^0\int_0^\infty \|F(\eta,t,\cdot)\|^2_{L^2(\mathsf{T})}|C(t)|^{-2}dtd\eta$$
$$+\frac{2^{-2|n|}}{\pi^3}\int_0^\infty\int_0^\infty \|F(\eta,t,\cdot)\|^2_{L^2(\mathsf{T})}|C(t)|^{-2}dtd\eta$$
$$+\sum_{|n|-2l>1}\frac{|n|-1-2l}{\pi^2}\int_0^\infty \eta^{-n-1-2l}\|F(\eta,i(|n|-1-2l),\cdot)\|^2_{\widetilde{D}_{|n|-2l,n}}d\eta$$
$$+\sum_{|n|-2l>1}\frac{|n|-1-2l}{\pi^2}\int_{-\infty}^0 |\eta|^{n+1+2l}\|F(\eta,-i(|n|-1-2l),\cdot)\|^2_{\widetilde{\overline{D_{|n|-2l,n}}}}d\eta.$$

Since $F(\eta,t,\cdot)$ is vanishing near $\eta=0$ we see that the first two integrals are

$$\frac{2^{-2|n|}}{\pi^3}\int_{\mathsf{R}}\int_0^\infty \|F(\eta,t,\cdot)\|^2_{L^2(\mathsf{T})}|C(t)|^{-2}dtd\eta$$
$$=\frac{2^{-2|n|-1}}{\pi^4}\int_0^\infty\int_{\mathsf{R}} \|F(\eta,t,\cdot)\|^2_{L^2(\mathsf{T})}|C(t)|^{-2}dtd\eta$$
$$=\frac{2^{-2|n|-1}}{\pi^4}\int_0^\infty \|\mathscr{F}_nf(\cdot,t,\cdot)\|^2_{L^2(\mathsf{R})\otimes L^2(\mathsf{T})}|C(t)|^{-2}dt,$$

using the Fubini theorem and Fourier Plancherel formula.Using (A.4) and (A.6) in the Appendix we see that the last two integrals are

$$\int_0^\infty \eta^{-n-1-2l}\|F(\eta,i(|n|-1-2l),\cdot)\|^2_{\widetilde{D}_{|n|-2l,n}}d\eta$$
$$=\|P_{|n|-2l,n}\mathscr{F}_nf(\cdot,i(|n|-1-2l),\cdot)\|^2_{D_{|n|-2l,n}\otimes\widetilde{D}_{|n|-2l,n}},$$

and

$$\int_{-\infty}^{0} |\eta|^{n+1+2l} \|F(\eta, -i(|n|-1-2l), \cdot)\|^2_{\widetilde{\overline{D_{|n|-2l,n}}}} d\eta$$
$$= \|\overline{P_{|n|-2l,n}}\mathscr{F}_n f(\cdot, -i(|n|-1-2l), \cdot)\|^2_{\overline{D_{|n|-2l,n}}\otimes\widetilde{\overline{D_{|n|-2l,n}}}}.$$

Substituting these into above formula we get the formula as claimed in the Theorem.

Step 2. The intertwining relation. For $U(1,1)_L$ action, this is proved previously in Propositions 3, 4. For $U(1,1)_R$ action this is proved in Proposition 3 and in Step 1.

Step 3. Decomposition. Step 1 shows that the mapping that maps f to

$$\mathscr{F}_n f(\xi, t, b) \oplus \sum_{|n|-2l>1} P_{|n|-2l,n}\mathscr{F}_n(i(|n|-2l-1), b)f(w)$$
$$\oplus \sum_{|n|-2ln>1} \overline{P_{|n|-2l,n}}\,\mathscr{F}_n(-i(|n|-2l-1), b)f(w)$$

can be uniquely extended to an isometric operator from $L^2(\mathsf{C}^2)_n$ into the direct integral

$$\int_0^\infty \pi_{it,n}\otimes\tilde{\pi}_{it,n}|C(t)|^{-2}dt \oplus \sum_{|n|-2l>1} D_{|n|-2l,n}\otimes\widetilde{D}_{|n|-2l,n} \oplus \sum_{|n|-2l>1} \overline{D_{|n|-2l,n}}\otimes\widetilde{\overline{D_{|n|-2l,n}}}.$$

We prove now that it is onto.

The fact that the discrete series is in the image of $\mathscr{F}_n$ on $L^2(\mathsf{C}^2)_n$ can be obtained from our proof Plancherel formula, see also Theorem 5 below. We will not duplicate the argument.

Suppose now that there is a function $H(\xi, t, b)$ in $\int_0^\infty \pi_{it,n}\otimes\tilde{\pi}_{it,n}|C(t)|^{-2}dt$ (i.e. the space $L^2(\mathsf{R}\times\mathsf{R}^+\times\mathsf{T}, |C(t)|^{-2}d\xi dt db)$) such that

$$\int_{\mathsf{R}}\int_0^\infty\int_{\mathsf{T}} \mathscr{F}_n f(\xi, t, b)\overline{H(\xi, t, b)}|C(t)|^{-2}db\,dt\,d\xi = 0$$

for all f in $\mathscr{D}_n^0$. Thus for almost all $(t, b)\in\mathsf{R}^+\times\mathsf{T}$ the function

$$h(\eta, t, b) = \int_{\mathsf{R}} H(\xi, t, b)e^{-i\xi\eta}d\xi$$

is in $L^2(\mathsf{R}, d\eta)$. Thus $h(\eta, t, b)$is in $L^2(\mathsf{R}\times\mathsf{R}^+\times\mathsf{T}, |C(t)|^{-2}d\eta dt db)$ and we have

$$\int_{\mathsf{R}}\int_0^\infty\int_{\mathsf{T}} F(\eta, t, b)\overline{h(\eta, t, b)}|C(t)|^{-2}db\,dt\,d\eta = 0,$$

where $F(\eta, t, b)$ is, as before, defined via (2.7). Let $p(\eta)$ be a C^∞-function that has compact support in $(0,\infty)$ and let $q(w)$ be a C^∞-function of $w \in D$ that has compactsupport on D. We define

$$G(\eta, w) = p(\eta)q(w)$$

and define $f(z_1, z_2)$ via (2.10). Then f is C^∞ and having compact support on $\{(z_1, z_2)||z_1|^2 - |z_2|^2 > 0\}$. Moreover (2.11) gives us

$$F(\eta, t, b) = \pi\eta^{\frac{n+1}{2}}p(\eta)\widetilde{q}(t, b),$$

where $\widetilde{q}(t, b)$ is the transform (2.4). The above orthogonality relation now reads

$$\int_{\mathsf{R}}\left(\int_0^\infty \int_{\mathsf{T}} \eta^{\frac{n+1}{2}}\widetilde{q}(t, b)\overline{h(\eta, t, b)}|C(t)|^{-2}dbdt\right)p(\eta)d\eta = 0$$

However the functions $p(\eta)$ are dense in the space $L^1(\mathsf{R}, d\eta)$ thus we have for almost all $\eta > 0$

$$\int_0^\infty \int_{\mathsf{T}} \eta^{\frac{n+1}{2}}\widetilde{q}(t, b)\overline{h(\eta, t, b)}|C(t)|^{-2}dbdt = 0,$$

or

$$\int_0^\infty \int_{\mathsf{T}} \widetilde{q}(t, b)\overline{h(\eta, t, b)}|C(t)|^{-2}dbdt = 0.$$

Similarly for $\eta < 0$. Now by the decomposition (2.5) we see that $h(\eta, t, b) = 0$, consequently $H(\xi, t, b) = 0$. This completes the proof of Theorem 4.

§2.3 *Discrete series and decomposition of $\widetilde{L}_n$*

The following Theorem gives the K-finite functions in the discrete series of $\widetilde{L}_n$. It can be derived from Theorem 4 and its proof and the Appendix. We present here only the results and intend to give the full details (for larger classes of other dual pairs) in a future paper.

THEOREM 5. *Each discrete series of $\widetilde{L}_n$ is of the type $D_{|n|-2l,n} \otimes \widetilde{D}_{|n|-2l,n}$ or $\overline{D_{|n|-2l,n}} \otimes \overline{\widetilde{D}_{|n|-2l,n}}$. Moreover the space of K-finite functions in $D_{|n|-2l,n} \otimes \widetilde{D}_{|n|-2l,n}$ is the span of all functions $f(z_1, z_2)$ on C^2 of the form*

$$\chi_{(0,\infty)}(|z_1|^2 - |z_2|^2)(|z_1|^2 - |z_2|^2)^{|n|-1-l}e^{-(|z_1|^2-|z_2|^2)}p(|z_1|^2 - |z_2|^2)g(\frac{z_2}{z_1})z_1^n,$$

where $\chi_{(0,\infty)(x)}$ is the characteristic function of R^+, p is a polynomial on R^+ and *g is a* $(U(1)\times U(1))_L$*-finite vector in* $\widetilde{D}_{|n|-2l,n}$ *realized as a discrete series of the space* $L^2(D,(1-|w|^2)^{|n|-2}|dw|^2)$ *(see [13] for a calculation of basis vectors of* $\widetilde{D}_{|n|-2l,n}$*). The space of K-finite functions in* $\overline{D_{|n|-2l,n}}\otimes\overline{\widetilde{D_{|n|-2l,n}}}$ *is the span of all functions f on* $\mathbb{C}^2$ *of the form*

$$\chi_{(0,\infty)}(|z_2|^2-|z_1|^2)\,(|z_2|^2-|z_1|^2)^{|n|-1-l}e^{-(|z_2|^2-|z_1|^2)}p(|z_2|^2-|z_1|^2)g\left(\frac{z_1}{z_2}\right)z_2^n,$$

where p is a polynomial on $\mathbb{R}^+$*, g is a* $(U(1)\times U(1))_R$*-finite vector in* $\overline{\widetilde{D_{|n|-2l,n}}}$ *realized as a discrete series of the space* $L^2(D,(1-|w|^2)^{|n|-2}|dw|^2)$.

REMARK 1. From the result of Kashiwara-Vergne [7] we know that the limit of holomorphic discrete series of $U(1,1)$ is $\pi_1=L_0$, where L_0 is the subspace of functions f in $L^2(\mathbb{C})$, with $U(1,1)$ acting on by L, satisfying

$$f(e^{i\theta}z)=f(z).$$

Therefore the tensor product $\pi_1\otimes\bar{\pi}_1$ consists of those functions in $L\otimes\bar{L}$ transforming according to

$$f(e^{i\theta}z_1,e^{i\phi}z_2)=f(z_1,z_2),$$

and thus $\pi_1\otimes\bar{\pi}_1$ is a $U(1,1)_L$ submodule of $L^2(\mathbb{C}^2)_0$. From our Theorem 4 we can then easily read off

$$\pi_1\otimes\bar{\pi}_1\cong\int_0^\infty \pi_{it,0}\,dt,$$

abstractly. This gives another proof of the result than that of [17].

REMARK 2. We consider the affine symmetric space $X=SU(1,1)/SO(1,1)$. See [4]. This space can be realized as $\mathrm{T}\times\mathrm{T}\setminus$ diagonal and the space $L^2(X)$ is then $SU(1,1)_L$-isomorphic to

$$\begin{aligned}L^2(\mathrm{T})\otimes L^2(\mathrm{T})&=(\pi_1+\bar{\pi}_1)\otimes(\pi_1+\bar{\pi}_1)\\&=\pi_1\otimes\pi_1+\bar{\pi}_1\otimes\bar{\pi}_1+\pi_1\otimes\bar{\pi}_1+\bar{\pi}_1\otimes\pi_1.\end{aligned}$$

By [17] we know that

$$\pi_1\otimes\pi_1\cong\sum_{n=1}^{\infty}\pi_{2n}.$$

Using this and the decomposition in Remark 1, we now get

$$L^2(X) \cong \sum_{n=1}^{\infty} \pi_{2n} \oplus \sum_{n=1}^{\infty} \bar{\pi}_{2n} \oplus 2 \int_0^{\infty} \pi_{0,it}\, dt.$$

§3. Tensor products of analytic continuations of the holomorphic discrete series of SU(1, 1).

As it is proved in [7], the metaplectic representation L of $SL(2, \mathrm{R})$ is the direct sum of $\pi_{\frac{1}{2}}$ and $\pi_{\frac{3}{2}}$, and $L^{O(1)} = \pi_{\frac{1}{2}}$. Here π_ν is the analytic continuation of the discrete series and $L^{O(1)}$ stands for the space of $O(1)$ invariant vectors in L. It then follows from our Theorem 2 that

$$\pi_{\frac{1}{2}} \otimes \bar{\pi}_{\frac{1}{2}} \cong (L \otimes \bar{L})^{O(1)\times O(1)} \cong \int_0^{\infty} \pi_{it}\, dt,$$

as SL(2, R) representations, see also [5].

It is a natural question to study what happens in this decomposition when we move to more singular values than $\frac{1}{2}$. In this section we thus give the irreducible decomposition of the tensor product $\pi_\nu \otimes \bar{\pi}_\nu$ with $0 < \nu < \frac{1}{2}$. When $\nu > 1$, π_ν is the holomorphic discrete series and this decomposition is done in [17] from the representation point of view, and in [14] from the analytic point of view. π_ν has analytic continuation to all $\nu \geq 0$. The representations $\pi_\nu \otimes \bar{\pi}_\nu$ are all equivalent when $\nu \geq \frac{1}{2}$. (For $\nu = \frac{1}{2}$ this is mentioned in [5] and when $\frac{1}{2} \leq \nu \leq 1$ this can acturally be done using our methods below.) When $\nu < \frac{1}{2}$, there is surprisingly a complementary series representation in the irreducible decomposition as a discrete summand.

We will first study the spectrum of the Casimir operator on K-fixed vectors in $\pi_\nu \otimes \bar{\pi}_\nu$.To get the desired decompostion from the knowledge of the spectrum one needs the classical reduction theory. See for example [15].

The representation π_ν can be realized on the space of holomorphic functions f on the unit disk D in the complex plane such that

$$f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$$

and

$$\|f\|_{\pi_\nu}^2 = \sum_{n=0}^{\infty} |\widehat{f}(n)|^2 \frac{\Gamma(n+1)\Gamma(\nu)}{\Gamma(n+\nu)} < \infty.$$

The group action is (again, subsuming universal cover)

$$g : f(z) \mapsto f(\frac{\beta + \delta w}{\alpha + \gamma w})(\alpha + \gamma w)^{-\nu},\ g = \alpha\beta\gamma\delta \in \mathrm{SU}(1,1).$$

Denote C_t the complementary series of $SL(2, \mathsf{R})$ with index t as in [9].

THEOREM 6 *Let* $0 < \nu < \frac{1}{2}$. *Then*

$$\pi_\nu \otimes \bar{\pi}_\nu \cong \int_0^\infty \pi_{it}\, dt \oplus C_{1-2\nu}.$$

PROOF. The representation $\pi_\nu \otimes \bar{\pi}_\nu$ can be realized as the space of functions $F(z, w)$ analytic in z and anti-analytic in w with orthogonal basis $\{z^n \bar{w}^m\}$ and

$$\|z^n\bar{w}^m\|^2_{\pi_\nu\otimes\bar{\pi}_\nu} = \frac{\Gamma(n+1)\Gamma(\nu)}{\Gamma(n+\nu)}\frac{\Gamma(m+1)\Gamma(\nu)}{\Gamma(m+\nu)}. \tag{3.1}$$

The Casimir operator C on $\pi_\nu \otimes \bar{\pi}_\nu$ is calculated in [12]. It is

$$C = (1 - z\bar{w})^2 \frac{\partial^2}{\partial z \partial \bar{w}} - \nu\bar{w}(1 - z\bar{w})\frac{\partial}{\partial \bar{w}} - \nu z(1 - z\bar{w})\frac{\partial}{\partial z} + \nu^2 z\bar{w}.$$

Let $(\pi_\nu \otimes \bar{\pi}_\nu)_0$ be the space of K-invariant vectors in $\pi_\nu \otimes \bar{\pi}_\nu$. Then $(\pi_\nu \otimes \bar{\pi}_\nu)_0$ consists of functions of the form $F(z, w) = f(z\bar{w})$ in $\pi_\nu \otimes \bar{\pi}_\nu$. By writing $t = z\bar{w}$, the Casimir operator then takes the form

$$C_0 = (1 - t)^2(t\frac{d^2}{dt^2} + \frac{d}{dt}) - 2\nu t(1 - t)\frac{d}{dt} + \nu^2 t \tag{3.2}$$

on $(\pi_\nu \otimes \bar{\pi}_\nu)_0$.

The space $(\pi_\nu \otimes \bar{\pi}_\nu)_0$ has an orthonormal basis $\{e_n, n = 0, 1, 2 \cdots\}$:

$$e_n(t) = \frac{t^n}{\Gamma_n} = \frac{(z\bar{w})^n}{\Gamma_n},$$

where

$$\Gamma_n = \frac{\Gamma(n+1)\Gamma(\nu)}{\Gamma(n+\nu)}.$$

The Casimir operator C_0 takes the following form relative to the basis$\{e_n, n = 0, 1, 2 \cdots\}$:

$$C_0 e_n = a_n e_{n+1} + b_n e_n + c_n e_{n-1}, \tag{3.3}$$

with

$$a_n = (n+1)(n+\nu),\ b_n = -n(2n+2\nu),\ c_n = a_{n-1}.$$

For this, see [12], and [14].

The following function is an eigenfunction of C_0:

$$\phi_\lambda(t) = (1-t)^{-(\nu-\frac{1-i\lambda}{2})}{}_2F_1\left(\frac{1-i\lambda}{2}, \frac{1-i\lambda}{2}; 1; t\right)$$

with eigenvalue

$$\Lambda = \nu - \frac{1}{4}(1+\lambda^2).$$

See [23]. (Note that here we are using different parameters.) It is proved in [14] and [23] that the function ϕ_λ has the following series expansion

$$\phi_\lambda(t) = \sum_{n=0}^{\infty} p_n(\Lambda)t^n,$$

where

$$p_n(\Lambda) = \frac{1}{(n!)^2} S_n\left(-\left(\frac{\lambda}{2}\right)^2; \frac{1}{2}, \frac{1}{2}, \frac{2\nu-1}{2}\right).$$

Here S_n are the continuous dual Hahn polynomials,

$$S_n(x^2; a, b, c) = (a+b)_n(a+c)_n\ {}_3F_2(-n, a+ix, a-ix; a+b, a+c; 1).$$

Furthermore, we have

$$\text{(3.4)}\qquad \Lambda p_n(\Lambda) = (n+1)^2 p_{n+1}(\Lambda) - 2n(n+\nu)p_n(\Lambda) + (n-1+\nu)^2 p_{n-1}(\Lambda).$$

It follows from the complex orthogonality relation of Wilson [22], p. 697 that

$$\int_C \frac{\Gamma(\frac{1}{2}+z)\Gamma(\frac{1}{2}-z)\Gamma(\frac{1}{2}+z)\Gamma(\frac{1}{2}-z)\Gamma(\nu-\frac{1}{2}+z)\Gamma(\nu-\frac{1}{2}-z)}{\Gamma(2z)\Gamma(-2z)} S_n(z^2)S_n(z^2)dz$$
$$= 2(n!)^2\Gamma^2(\nu+n)\delta_{n,m},$$

where the contour C is the imaginary axis deformed so as to separate the sequences $\{\nu-\frac{1}{2}+k\}_{k=0}^{\infty}$, $\{\frac{1}{2}-\nu-k\}_{k=0}^{\infty}$. Performing the residue calculation at $\pm(\frac{1}{2}-\nu)$, we find out that the real orthogonality relation reads as follows:

$$(3.5)\qquad \int_0^\infty \left|\frac{\Gamma(\frac{1+i\lambda}{2})\Gamma(\frac{1+i\lambda}{2})\Gamma(\frac{2\nu-1+i\lambda}{2})}{\Gamma(i\lambda)}\right|^2 S_n\left(-\left(\frac{\lambda}{2}\right)^2\right) S_m\left(-\left(\frac{\lambda}{2}\right)^2\right) d\lambda +$$
$$+ 4\pi \frac{\Gamma(1-\nu)\Gamma^2(\nu)}{\Gamma(1-2\nu)} S_n\left(\left(\frac{1-2\nu}{2}\right)^2\right) S_m\left(\left(\frac{1-2\nu}{2}\right)^2\right)$$
$$= 4\pi 2(n!)^2\Gamma^2(\nu+n)\delta_{n,m}.$$

So (3.5) is equivalent to saying that the functions

$$\widetilde{e_n}(\Lambda) = \frac{n!}{\Gamma(\nu+n)} p_n(\Lambda) = \frac{1}{n!\Gamma(\nu+n)} S_n\left(-\left(\frac{\lambda}{2}\right)^2\right)$$

are orthonormal in $L^2(\mathsf{R}\cup\{\pm i(1-2\nu)\}, d\mu)_0$, the subspace of even functions in $L^2(\mathsf{R}\cup\{\pm i(1-2\nu)\}, d\mu)$ (see [11] for the proof); here $d\mu$ is a measure on $(\mathsf{R}\cup\{\pm i(1-2\nu)\}$ obtained from the above orthogonality relation.

Now from (3.4), by direct calculation, we know that the multiplication operator by Λ on $L^2(\mathsf{R}\cup\{\pm i(1-2\nu)\}, d\mu)_0$ has matrix form on $\widetilde{e}_n$:

$$\Lambda\widetilde{e_n}(\Lambda) = a_n\widetilde{e_{n+1}}(\Lambda) + b_n\widetilde{e_n}(\Lambda) + c_n\widetilde{e_{n-1}}(\Lambda).$$

This is just the same recursion formula as (3.4). So the operator C_0 is unitarily equivalent to the multiplication operator by Λ; consequently, the spectrum of C_0 is equal to that of the multiplication operator by Λ on $L^2(\mathsf{R}\cup\{\pm i(1-2\nu)\}, d\mu)_0$, which by the standard argument is $(-\infty, \nu-\frac{1}{4})\cup\{\nu^2\}$.

It follows that the representation $\pi_\nu\otimes\bar{\pi}_\nu$ is decomposed into the sum of direct integral of the principal series π_{it} and the complementary series $C_{1-2\nu}$. Our Theorem in this section is therefore proved.

We note that in the course of proof of Theorem 6 we actually found explicit bases and measures giving the isomorphism in the Theorem.

Remark. We can also prove directly that the complementary series $C_{1-2\nu}$ enters as a discrete part into their reducible decomposition of $\pi_\nu\otimes\bar{\pi}_\nu$. It follows from [14] that the operator

$$\pi_\nu\otimes\bar{\pi}_\nu \to C^\infty(D), f(z,w)\mapsto f(z,z)(1-|z|^2)^\nu$$

intertwines the $\pi_\nu\otimes\bar{\pi}_\nu$ with the regular action on $C^\infty(D)$. Now the function $(1-|z|^2)^{\frac{1-i\lambda}{2}} {}_2F_1(\frac{1-i\lambda}{2}, \frac{1-i\lambda}{2}; 1; |z|^2)$ is an eigenfunction of the Casimir operator on $C^\infty(D)$. Thus its inverse under R,

$$\phi_\lambda(z,w) = (1-z\bar{w})^{-(\nu-\frac{1-i\lambda}{2})} {}_2F_1\left(\frac{1-i\lambda}{2}, \frac{1-i\lambda}{2}; 1; z\bar{w}\right)$$

is an eigenfunction of the Casimir operator C on $\pi_\nu \otimes \bar{\pi}_\nu$. Now when $\lambda = -i(1-2\nu)$ then $(1-|z|^2)^{\frac{1-i\lambda}{2}} {}_2F_1(\frac{1-i\lambda}{2}, \frac{1-i\lambda}{2}; 1; |z|^2)$ is the matrix coefficient of the complementary series $C_{1-2\nu}$. Its inverse under R is

$$\phi_{-i(1-2\nu)}(z,w) = {}_2F_1(\nu,\nu;1;z\bar{w}) = \sum_{n=0}^{\infty} \left(\frac{(\nu)_n}{n!}\right)^2 (z\bar{w})^n.$$

We calculate its norm in $\pi_\nu \otimes \bar{\pi}_\nu$.

$$\begin{aligned}
\|\phi_{-i(1-2\nu)}\|^2_{\pi_\nu\otimes\bar{\pi}_\nu} &= \sum_{n=0}^{\infty} \left(\frac{(\nu)_n}{n!}\right)^4 \|(z\bar{w})^n\|^2_{\pi_\nu\otimes\bar{\pi}_\nu} \\
&= \sum_{n=0}^{\infty} \left(\frac{(\nu)_n}{n!}\right)^4 \left(\frac{n!}{(\nu)_n}\right)^2 \\
&= \sum_{n=0}^{\infty} \left(\frac{(\nu)_n}{n!}\right)^2 \\
&\approx \sum_{n=0}^{\infty} \frac{1}{n^{2(1-\nu)}} \\
&< \infty
\end{aligned}$$

since $2(1-\nu) > 1$ if $\nu < \frac{1}{2}$. Thus $\phi_{-i(1-2\nu)}(z,w)$ is indeed in $\pi_\nu \otimes \bar{\pi}_\nu$, that is $C_{1-2\nu}$ is embedded into $\pi_\nu \otimes \bar{\pi}_\nu$ as a discrete series.

Appendix

We present here some realizations of the discrete series for $U(1,1)_L$. Again, the parameters satisfy $\nu - 2l > 1$ with ν and l non-negative integers.

Let $D_{\nu-2l,-\nu}$be the space of holomorphic functions h on the upper-half plane $U = \{w = u + iv \in \mathbf{C}; v > 0\}$ such that

$$\|h\|^2_{D_{\nu-2l,-\nu}} = 2^{\nu-2l}\pi \int_U |h(w)|^2 v^{\nu-2l-2} |dw|^2 < \infty$$

and $U(1,1)_L$ acts on it via

(A.1) $$D_{\nu-2l,-\nu}(g)h(w) = (\gamma w + \delta)^{-(\nu-2l)} h\left(\frac{\alpha w + \beta}{\gamma w + \delta}\right),$$

if

$$g^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SU}(1,1)_L,$$

and

(A.2) $$D_{\nu-2l,-\nu}(g)h(w) = e^{-i\nu\theta}h(w),$$

if

$$g = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}.$$

(Here $\mathrm{SU}(1,1)_L = \mathrm{SL}(2,\mathsf{R})$ consists of those g with $\alpha, \beta, \gamma, \delta \in \mathsf{R}$ and $\alpha\delta - \beta\gamma = 1$.)

The space of $(U(1) \times U(1))_L$- finite vectors in $D_{\nu-2l,-\nu}$ is spanned by the vectors

$$\left(\frac{w-i}{w+i}\right)^k (w+i)^{-(\nu-2l)}, k = 0, 1, .$$

Recall the representation $\pi_{it,-\nu}$ introduced in (2.2) with $t = i(\nu - 1 - 2l)$. The space of K-finite vectors in the representation space is spanned by (k an integer)

$$\left(\frac{\xi-i}{\xi+i}\right)^k (\xi+i)^{\nu-2l-2}$$

Lemma A1 *The following operator*

$$P_{\nu-2l,-\nu}\Phi(w) = \frac{(\nu-2l-1)!}{2\pi i^{\nu-2l}} \int_{\mathsf{R}} \frac{\Phi(\xi)}{(\xi-w)^{\nu-2l}} d\xi$$

is an $U(1,1)_L$-intertwining operator between the representation $\pi_{it,-\nu}$ and $D_{\nu-2l,-\nu}$ (acting on the space of all holomorphic functions on U), where $t = i(\nu - 1 - 2l)$. The map $P_{\nu-2l,-\nu}$ establishes a $u(1,1)_L$-unitary equivalence between a quotient of $\pi_{it,-\nu}$ and $D_{\nu-2l,-\nu}$.

It is clear that if Φ is in $\pi_{it,-\nu}$ (as in Lemma 5 and the remark after (2.2)) then the integral defines a holomorphic function on U. The intertwining formula follows from change of variables. The $u(1,1)_L$-unitary equivalence follows from explicit calculation of $P_{\nu-2l,-\nu}\Phi$ when Φ is a $(U(1) \times U(1))_L$-finite vector.

We will use another realization of $D_{\nu-2l,-\nu}$. Each function $h(w)$ can be expressed as

$$h(w) = \int_0^\infty e^{iw\eta}\eta^{\nu-1-2l}F(\eta)d\eta,$$

where $F(\eta)$ is a function on R^+, and that

$$\|h\|^2_{D_{\nu-2l,-\nu}} = \int_{\mathrm{R}^+} |F(\eta)|^2 \eta^{\nu-1-2l} d\eta$$

Thus the map $F \mapsto h$ is a unitary operator from $L^2(\mathrm{R}^+, \eta^{\nu-1-2l}d\eta)$ onto $D_{\nu-2l,-\nu}$ and the group $U(1,1)_L$ is acting on $L^2(\mathrm{R}^+, \eta^{\nu-1-2l}d\eta)$. The space of $(U(1) \times U(1))_L$-finite functions in $L^2(\mathrm{R}^+, \eta^{\nu-1-2l}d\eta)$ is spanned by

$$p(\eta)e^{-\eta},$$

where p is a polynomial on R^+.

Now, if $F(\eta)$ is a C^∞-function with compact support on R, we take

$$\Phi(\xi) = \int_{\mathrm{R}} e^{i\eta\xi} F(\eta) d\eta.$$

Then

$$P_{\nu-2l,-\nu}\Phi(w) = \int_0^\infty e^{iw\eta}\eta^{\nu-1-2l}F(\eta)d\eta \tag{A.3}$$

and

$$\|P_{\nu-2l,-\nu}\Phi\|^2_{D_{\nu-2l,-\nu}} = \int_0^\infty |F(\eta)|^2\eta^{\nu-1-2l}d\eta. \tag{A.4}$$

(Note we take only integrals of F over the half line R^+.) All this can be calculated directly. See [13].

Let $\overline{D_{\nu-2l,-\nu}}$ be the space of holomorphic functions $h(w)$ on the lower half plane $\overline{U} = \{w = u + iv \in \mathrm{C}; v < 0\}$ such that

$$\|f\|^2_{\overline{D_{\nu-2l,-\nu}}} = 2^{\nu-2l}\pi \int_{\overline{U}} |h(w)|^2 |v|^{\nu-2l-2} |dw|^2 < \infty.$$

The group $U(1,1)_L$ is acting on it via the same formulae as in (A.1) and (A.2).

When $t = -i(\nu - 2l - 1)$ the representation $\pi_{it,-\nu}$ defined in (2.2) is dual to $\pi_{-it,-\nu}$. There is a $u(1,1)$-invariant subspace of $\pi_{it,-\nu}$ isomorphic to $\overline{D_{\nu-2l,-\nu}}$.

We make this explicit as follows. We define the operator

$$\overline{P_{\nu-2l,-\nu}}\Phi(w) = \frac{1}{2\pi i}\int_{\mathrm{R}} \frac{\Phi(\xi)}{w-\xi} d\xi$$

for a function $\Phi(\xi)$ on R. Note that when we have the decay as in Lemma 5 corresponding to the present value of t, then the integral converges, and it gives a certain extension of a function to a holomorphic function in the lower halfplane. Similar to Lemma A1 (except now the principal series has infinite dimensional subrepresentations and a finite dimensional quotient) we have now

LEMMA A2. *The operator* $\overline{P_{\nu-2l,-\nu}}$ *is an* $U(1,1)_L$*-intertwining operator between an invariant subspace of the representation* $\pi_{it,-\nu}$ *and* $\overline{D_{\nu-2l,-\nu}}$ *(acting on the space of all holomorphic functions on* $\overline{U}$*), where* $-t = i(\nu - 1 - 2l)$*. The map* $\overline{P_{\nu-2l,-\nu}}$ *establishes a* $u(1,1)_L$*-unitary equivalence between an invariant subspace of* $\pi_{it,-\nu}$ *and* $\overline{D_{\nu-2l,-\nu}}$ *(by the extension).*

We can also realize $D_{\nu-2l,-\nu}$ on $L^2(-\mathsf{R}^+, |\eta|^{-(\nu-1-2l)}d\eta)$. Each function $h(w)$ can be expressed as

$$h(w) = \int_{-\infty}^{0} e^{iw\eta}F(\eta)d\eta,$$

where $F(\eta)$ is a function on $-\mathsf{R}^+$, and that

$$\|h\|^2_{D_{\nu-2l,-\nu}} = \int_{-\infty}^{0} |F(\eta)|^2|\eta|^{-(\nu-1-2l)}d\eta.$$

Thus the group $U(1,1)_L$ is acting on $L^2(-\mathsf{R}^+, |\eta|^{-(\nu-1-2l)}d\eta)$. The space of $(U(1)\times U(1))_L$-finite functions in $L^2(-\mathsf{R}^+, |\eta|^{-(\nu-1-2l)}d\eta)$ is spanned by

$$p(\eta)e^{\eta},$$

where p is a polynomial on $-\mathsf{R}^+$.

If $F(\eta)$ is a C^∞-function with compact support on R, we take

$$\Phi(\xi) = \int_{\mathsf{R}} e^{i\eta\xi}F(\eta)d\eta.$$

Then

$$\overline{P_{\nu-2l,-\nu}}\Phi(w) = \int_{-\infty}^{0} e^{iw\eta}F(\eta)d\eta \tag{A.5}$$

and

$$\|\overline{P_{\nu-2l,-\nu}}\Phi\|^2_{\overline{D_{\nu-2l,-\nu}}} = \int_{-\infty}^{0} |F(\eta)|^2|\eta|^{-(\nu-1-2l)}d\eta \tag{A.6}$$

(The representation $\overline{D_{\nu-2l,-\nu}}$ can also be realized on the space of all the distributions Φ on R with Fourier transform

$$\widehat{\Phi}(\eta) = \int_{\mathsf{R}} e^{-i\eta\xi}\Phi(\xi)d\xi$$

supported on $-\mathsf{R}^+$, and such that

$$\|\Phi\|^2_{\overline{D_{\nu-2l,-\nu}}} = \int_{-\infty}^{0} |\widehat{\Phi}(\eta)|^2 |\eta|^{-(\nu-1-2l)} d\eta < \infty.$$

The group action is

$$\overline{D_{\nu-2l,-\nu}}(g)\Phi(\xi) = |\gamma\xi + \delta|^{-\nu+2l} \left(\frac{\gamma\xi+\delta}{|\gamma\xi+\delta|}\right)^{-\nu} \Phi\left(\frac{\alpha\xi+\beta}{\gamma\xi+\delta}\right)$$

and

$$g^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U(1,1)_L.$$

See, e.g. [18] where a generalization of $\overline{D_{\nu-2l,-\nu}}$ for higher rank groups is studied.)

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