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ADJUNCTION MAPPINGS ON NUMERICAL QUADRICS

Y. S. POON

Abstract.

Using Reider's method, we analyse the adjunction mappings on general-type surfaces with $p_g = 0, q = 0, c_2 = 4$ and even intersection form. In particular, we prove that when the fundamental group of such a surface does not have any irreducible SU(2) or SO(3)-representations, there exists a divisor numerically equivalent to the canonical divisor so that its adjunction map is a regular bimeromorphism.

§1. Introduction.

When F is a line bundle on a compact complex surface X, the adjunction map is the rational map associated to the line bundle KF where K is the canonical line bundle on X [16]. Applying a construction of Serre [5] [8] [15] and Bogomolov's criterion of determining stability of holomorphic vector bundles on compact complex surfaces [3] [13], Reider developed his famous method of studying adjunction maps on algebraic surfaces [14]. In particular, he re-proved and improved Bombieri's results on pluricanonical maps on general-type surfaces [4].

Recently, Kotschick applied Reider's method to study pluricanonical maps of numerical Godeaux surfaces and numerical Campedelli surfaces [9] [10] . Among other results, he found that the space of irreducible SO(3) and SU(2)-representations of the fundamental group of such a surface often contains obstructions to a certain pluricanonical map from being an embedding. The representations of the fundamental group appear because of Donaldson's Stable Bundle Theorem [7] stating that a holomorphic vector bundle on an algebraic surface is stable if and only if there is a Hermitian-Einstein metric.

We use Kotschick's idea of applying Reider's method and Donaldson's Stable Bundle Theorem to study adjunction maps on numerical quadric surfaces. A numerical quadric surface is a general-type surface having the same set of numerical topological data as the complex quadric surface. Let

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b_n be the nth Betti number and χ the Euler number of a compact complex surface X. Let Q be the intersection form on the torsion-free part of the second integral cohomology group. When the intersection form is odd, we set $\epsilon = 1$. When the intersection form is even, we set $\epsilon = 0$. Let b_+ be the number of positive eigenvalues of the intersection form.

DEFINITION. A general-type surface X is a *numerical quadric surface* if

$$b_+(X) = 1, \qquad b_1(X) = 0, \qquad \chi(X) = 4, \qquad \epsilon(X) = 0.$$

Let p_g be the geometric genus of a compact complex surface X, q the irregularity, c_1^2 the self-intersection of the first Chern class and c_2 the second Chern number. The Hodge theory yields the following identities.

$$b_+ = 2p_g + 1, \qquad b_1 = 2q, \qquad \chi = c_2.$$

Therefore, we rewrite the definition of numerical quadrics.

DEFINITION. A general-type surface X is a *numerical quadric surface* if

$$p_g(X) = 0, \qquad q(X) = 0, \qquad c_2(X) = 4, \qquad \epsilon(X) = 0.$$

By Qin's answer to the Hirzebruch problem in the category of differentiable manifolds [12], no numerical quadric is diffeomorphic to a quadric surface. Whether there are fake quadrics, i.e. general-type surfaces homeomorphic to but not diffeomorphic to a quadric, remains a question. However, when one allows the fundamental group to be non-trivial, there are examples of numerical quadrics constructed by Kuga and Beauville [1] [2]. These examples are discussed in detail by Dolgachev in [6]. Note that in [1], numerical quadrics were called fake quadrics.

It is a direct consequence of Reider and Bombieri's result that the third canonical map on a numerical quadric is an embedding and that the second canonical map is a regular morphism. The aim of this article is to prove the following two theorems.

THEOREM A. *Let X be a numerical quadric. If $\pi_1(X)$ does not have any irreducible* SU(2) *or* SO(3)*-representations, then there exists a divisor F numerically equivalent to the canonical divisor such that the adjunction map of F is a regular bimeromorphism.*

THEOREM B. *Let F be a divisor on a numerical quadric. Suppose that it is not numerically equivalent to the canonical class.*

(1) *If $F - K$ is numerically effective, then the adjunction map of F is a regular bimeromorphism.*

(2) *If $F - K$ is positive, then the adjunction map of F is an embedding.*

Other related results are discussed in the last section of this article.

§2. Preliminary data.

To set up our investigation, we need to describe effective divisors and positive divisors on a numerical quadric. It was already done in Qin's paper, at least when the manifold X is simply-connected [12]. In this section, we follow his techniques to describe divisors which are positive, effective and numerically effective.

When X is a numerical quadric, its second betti number is equal to 2 and its signature is equal to zero. Since the intersection form of X is even, there exists a basis in the torsion-free part of $H^2(X, \mathbf{Z})$ such that the intersection form is the hyperbolic matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.1}$$

LEMMA 2.2. *A numerical quadric does not contain any smooth rational curves. It is a minimal surface and its canonical bundle is ample.*

PROOF. Since the intersection form of a numerical quadric X is even, X is minimal. On a minimal general-type surface, the number of disjoint nonsingular rational curves is bounded by $\frac{2}{3}(c_2 - \frac{1}{3}c_1^2)$ [11]. Since $c_2 = 4$ and the signature is equal to zero, the signature formula shows that $c_1^2 = 8$. Then the number of disjoint rational curves on X is at most $\frac{8}{9}$. In other words, there are no rational curves on X.

The canonical bundle is positive because the only obstruction to the canonical bundle being positive on any general-type surface is the presence of smooth rational curves with self-intersection -2.

On X, two divisors A and B are numerically equivalent if for any divisor D, $A \cdot D = B \cdot D$. By $A \equiv B$, we mean that A and B are numerically equivalent. Consider the induced long exact sequence of the exponential sequence

$$0 \to \mathbf{Z} \to \mathcal{O} \to \mathcal{O}^* \to 1.$$

The vanishing of the geometric genus and irregularity together implies that

$$H^1(X, \mathcal{O}^*) \cong H^2(X, \mathbf{Z}) \cong H^{1,1}(X).$$

The notions of holomorphic line bundles, first Chern class of line bundles and divisor classes are equivalent. For this reason, we shall not use different symbols to distinguish them. Warning: Given a line bundle F or its divisor, we shall use F^2 to represent both the tensor product of line bundles $F \otimes F$

and the self-intersection number of a divisor $F \cdot F$. In contents, it will not be confusing.

Let Σ and Θ be elements in $H^2(X, \mathbf{Z})$ such that they span the torsion-free part of $H^2(X, \mathbf{Z})$ and the intersection matrix with respect to this basis is represented by (2.1). Then there are integers a and b such that $K \equiv a\Sigma + b\Theta$. As $K^2 = c_1^2 = 8$, $ab = 4$. Moreover, for any divisor D,

$$\begin{aligned} K \cdot D &= w_2(X) \cdot D, \text{ mod } 2 \\ &= D \cdot D, \text{ mod } 2 \\ &= 0. \text{ mod } 2 \end{aligned}$$

The integers a and b are both even. If follows that either $a = b = 2$ or $a = b = -2$. In the later case, if we choose $\{-\Sigma, -\Theta\}$ as a basis for the torsion-free part of $H^2(X, \mathbf{Z})$, then $K \equiv 2(-\Theta) + 2(-\Sigma)$. With respect to this basis, the intersection form is again represented by the hyperbolic matrix (2.1). To conclude, we have

LEMMA 2.3. *There is a basis $\{\Theta, \Sigma\}$ for the torsion-free part of $H^2(X, \mathbf{Z})$ such that the intersection form is the hyperbolic matrix (2.1) and $K \equiv 2\Theta + 2\Sigma$.*

We choose such a basis for the torsion-free part of $H^2(X, \mathbf{Z})$ once and for all.

LEMMA 2.4. [12] *If a divisor D is effective, then there are non-negative integers a and b such that $D \equiv a\Theta + b\Sigma$ and $a + b > 0$.*

PROOF. It suffices to prove this claim for irreducible effective divisors. As the canonical bundle is positive, $2(a + b) = DK > 0$. Applying the adjunction formula, one sees that if $ab < 0$, then D is a smooth rational curve, contradicting Lemma 2.2.

LEMMA 2.5. *A divisor on X is positive if and only if it is numerically equivalent to $m\Theta + n\Sigma$ such that $m \geq 1$ and $n \geq 1$.*

PROOF. This is a direct consequence of the Nakai's criterion and the above lemma on effective divisors.

COROLLARY 2.6. *A divisor on X is numerically effective if and only if it is numerically equivalent to $m\Theta + n\Sigma$ such that $m \geq 0$ and $n \geq 0$.*

PROOF. For any positive integers h and k, $\Theta + h\Sigma$ and $k\Theta + \Sigma$ are ample. Therefore, if $C \equiv m\Theta + n\Sigma$ is numerically effective, then

$$mh + n = C(\Theta + h\Sigma) \geq 0, \qquad m + nk = C(k\Theta + \Sigma) \geq 0,$$

for all positive integers h and k. It follows that m and n are non-negative.

§3. Linear systems.

We apply the Riemann-Roch formula to find that when $D \equiv m\Theta + n\Sigma$,

$$(3.1) \quad \chi(X, D) = \frac{1}{2}(m\Theta + n\Sigma)((m-2)\Theta + (n-2)\Sigma) + 1 = (m-1)(n-1).$$

By Serre duality $h^2(X, D) = h^0(X, -(D-K))$. Therefore, if $D \neq K$ and $D - K$ is numerically effective, then Lemma 2.4 and Lemma 2.5 together implies that $h^2(X, D)$ is equal to zero. Then (3.1) implies that $h^0(X, D) \geq 2$. When F is a line bundle, we use Φ_{KF} to denote the adjunction map. i.e. the rational map associated to the complete linear system of $|K + F|$.

When F is numerically effective and non-zero, the above discussion shows that the system $|K + F|$ is at least a pencil. The following is a direct consequence of Part (i) of Reider's Theorem 1 [14].

PROPOSITION 3.2. *Suppose that F is a numerically effective divisor on X such that $F \equiv (m\Theta + n\Sigma)$ and $mn \geq 3$. If the system $|K + F|$ has a base point then $m = 1$ or $n = 1$. And if p is a base point, then it is contained in an effective divisor E such that $E \equiv \Theta$ when $F \equiv m\Theta + \Sigma$, and $E \equiv \Sigma$ when $F \equiv \Theta + n\Sigma$.*

To illustrate the involvement of the representations of the fundamental group, we prove the following.

PROPOSITION 3.3. *Suppose that the fundamental group of X does not have any irreducible* SO(3)*-representations. Let F be a numerically effective divisor on X such that $F \equiv (m\Theta + n\Sigma)$ and $mn = 2$. If p is a base point, then it is contained in an effective divisor E such that $E \equiv \Theta$ when $F \equiv 2\Theta + \Sigma$, and $E \equiv \Sigma$ when $F \equiv \Theta + 2\Sigma$.*

PROOF. Let p be a base point of the system $|K + F|$. Let $\mathscr{I}_Z$ be the ideal sheaf of p. By a construction of Serre [5], there is a rank-2 holomorphic bundle $\mathscr{E}$ defined by the extension

$$(3.4) \qquad 0 \to \mathcal{O} \to \mathscr{E} \to \mathscr{I}_Z(F) \to 0$$

As $c_1(\mathscr{E}) = F$ and $c_2(\mathscr{E}) = 1$, the discriminant of $\mathscr{E}$ is given by

$$c_1^2(\mathscr{E}) - 4c_2(\mathscr{E}) = 2mn - 4 = 2(mn-2) = 0.$$

When $mn = 2$, $w_2(F) \neq 0$. Due to a generalization of Donaldson's Stable Bundle Theorem [7] [10], the stability of the bundle $\mathscr{E}$ is due to the existence of an anti-self-dual connection on the SO(3)-bundle $\wp := \mathrm{Ad}(\mathscr{E})$ with Pontryagin class

$$p_1(\wp) = c_1(\mathscr{E})^2 - 4c_2(\mathscr{E}) = 0.$$

As the connection is also anti-self-dual, Chern-Weil theory implies that the connection on $\wp$ is flat. Therefore, the holonomy representation of $\wp$ produces an irreducible SO(3)-representation of $\pi_1(X)$.

With the absence of such a representation, the bundle $\mathscr{E}$ is unstable with respect to any polarization. Let M be a maximally destabilizing sheaf with respect to the bundle F. Then there is a 0-cycle A and a line bundle E such that $\mathscr{E}$ is contained in the following extension.

$$(3.5) \qquad 0 \to \mathscr{O}(M) \to \mathscr{E} \to \mathscr{I}_A(E) \to 0.$$

As M is destabilizing with respect to F,

$$(3.6) \qquad MF \geq \frac{1}{2} c_1(\mathscr{E}) F = \frac{1}{2} F^2 > 0.$$

Therefore, $H^0(X, \mathscr{O}(M^{-1}))$ vanishes. When (3.5) is twisted by M^{-1} to

$$(3.7) \qquad 0 \to \mathscr{O} \to \mathscr{E} \otimes M^{-1} \to \mathscr{I}_A(EM^{-1}) \to 0,$$

we see that the bundle $\mathscr{E} \otimes M^{-1}$ has non-trivial sections. When (3.4) is twisted, one has the exact sequence

$$(3.8) \qquad 0 \to \mathscr{O}(M^{-1}) \to \mathscr{E} \otimes M^{-1} \to \mathscr{I}_Z(FM^{-1}) \to 0.$$

Its induced long exact sequence and the vanishing of $H^0(X, \mathscr{O}(M^{-1}))$ together shows that p is contained in an effective divisor E such that $E = F - M$. Note that (3.6) is equivalent to $(M - E)F = (2M - F)F \geq 0$. We have $M^2 \geq E^2$. By Lemma 2.4, $M^2 \geq E^2 \geq 0$.

If $M^2 = E^2 = 0$, then $ME > 0$ because $F^2 = (M + E)^2 > 0$. If $M^2 > 0$, then by Lemma 2.5, either M is positive or $-M$ is positive. By (3.6), M is positive. As E is effective, then $ME > 0$. In conclusion, $ME > 0$ for all cases.

From (3.4) and (3.5), the second Chern class of $\mathscr{E}$ is equal to $1 = c_2(\mathscr{E}) = ME + \deg A$. Therefore, $ME = 1$ and $\deg A = 0$. Given $ME = 1$ and $F = M + E$, with Lemma 2.4 and Corollary 2.6, one deduces that either $E \equiv \Theta$ and $M \equiv (m-1)\Theta + \Sigma$, or $E \equiv \Sigma$ and $M \equiv \Theta + (n-1)\Sigma$.

The following is a direct consequence of part (ii) of Reider's Theorem 1 [14]. We give some details to the proof as it helps to understand other related observations.

PROPOSITION 3.9. *Suppose that F is a numerically effective divisor on X such that $F \equiv (m\Theta + n\Sigma)$ and $mn \geq 5$. If Z is a 0-cycle of degree two not separated by sections of the bundle KF, then $m = 1$ or $n = 1$. And the support of Z is contained in an effective divisor E such that*

(1) *$E \equiv \Theta$ or 2Θ when $F \equiv m\Theta + \Sigma$, $E \equiv \Sigma$ or 2Σ when $F \equiv \Theta + n\Sigma$; or*

(2) *$E \equiv \Theta$ when $F \equiv m\Theta + 2\Sigma$, $E \equiv \Sigma$ when $F \equiv 2\Theta + n\Sigma$.*

PROOF. Given Z, there is a rank-2 holomorphic bundle $\mathcal{E}$ defined by the extension

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{I}_Z(F) \to 0 \tag{3.10}$$

As $c_1(\mathcal{E}) = F$ and $c_2(\mathcal{E}) = \deg Z = 2$, the discriminant of $\mathcal{E}$ is equal to

$$c_1^2(\mathcal{E}) - 4c_2(\mathcal{E}) = 2mn - 8 = 2(mn - 4). \tag{3.11}$$

With the assumption that $mn \geq 5$, the bundle $\mathcal{E}$ is Bogomolov-unstable [3] [13]. Let M be a maximally destabilizing sheaf with respect to the bundle F. Then there is a 0-cycle A and a line bundle E such that $\mathcal{E}$ is contained in the following extension.

$$0 \to \mathcal{O}(M) \to \mathcal{E} \to \mathcal{I}_A(E) \to 0.$$

As M is destabilizing with respect to F, $MF \geq \frac{1}{2}c_1(\mathcal{E})F = \frac{1}{2}F^2 > 0$. In particular, $H^0(X, \mathcal{O}(M^{-1}))$ vanishes. Applying the argument in the second half of the proof of Proposition 3.3, one finds an effective divisor E such that $E = F - M$ and support of Z is contained in E and $ME > 0$.

The second Chern class of $\mathcal{E}$ is equal to $2 = c_2(\mathcal{E}) = ME + \deg A$. Since $ME > 0$, either $ME = 1$ or $ME = 2$. With Lemma 2.4 and Corollary 2.6, one deduces the following. When $ME = 1$, then $E \equiv \Theta$, $M \equiv (m-1)\Theta + \Sigma$; or $E \equiv \Sigma$, $M \equiv \Theta + (n-1)\Sigma$. When $ME = 2$, then $E \equiv \Theta$, $M \equiv (m-1)\Theta + 2\Sigma$; or $E \equiv \Sigma$, $M \equiv 2\Theta + (n-1)\Sigma$; or $E \equiv 2\Theta$, $M \equiv (m-2)\Theta + \Sigma$; or $E \equiv 2\Sigma$, $M \equiv \Theta + (n-2)\Sigma$.

It is convenient to introduce the following definitions.

DEFINITION 3.12. *Let F be a numerically effective divisor. Let Z be any 0-cycle with finite degree. When a rank-2 vector bundle is obtained in the extension*

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{I}_Z(F) \to 0,$$

it is called an extension bundle of the divisor F with respect to Z.

Suppose that $\mathcal{E}$ is an extension bundle. If $\mathcal{E}$ is unstable with respect to a polarization, then there is a maximally destabilizing line bundle M and a 0-cycle A of finite degree such that $\mathcal{E}$ is contained in the following destabilizing exact sequence.

$$0 \to \mathcal{O}(M) \to \mathcal{E} \to \mathcal{I}_A(FM^{-1}) \to 0.$$

DEFINITION 3.13. *A divisor E such that $E = F - M$ is called a degeneracy divisor.*

Reading the proof of Proposition 3.9 again. Equality (3.11) shows that one cannot apply Bogomolov's theorem if $mn = 4$. Suppose that the extension bundle $\mathscr{E}$ is stable. If F has a square root $F^{1/2}$ in the Picard group, then $\mathscr{F} = \mathscr{E} \otimes F^{-1/2}$ is a stable bundle with vanishing Chern classes. Then Donaldson's Hermitian-Einstein metric on $\mathscr{F}$ is a flat SU(2)-metric. If F does not have square root, then Ad($\mathscr{E}$) is a flat SO(3)-bundle. Therefore, the holonomy representations of such bundles determine irreducible SU(2) and SO(3)-representations of $\pi_1(X)$ respectively. Then from the proof of Proposition 3.3, we have the following.

PROPOSITION 3.14. *Suppose that the fundamental group of X does not have any irreducible* SO(3)*-representations. Suppose that F is a divisor such that $F \equiv (4\Theta + \Sigma)$ or $(\Theta + 4\Sigma)$. If Z is a* 0*-cycle of degree two not separated by sections of the bundle KF, then Z is contained in an effective divisor E such that $E \equiv \Theta$ or $E \equiv 2\Theta$ when $F \equiv 4\Theta + \Sigma$; and $E \equiv \Sigma$ or $E \equiv 2\Sigma$ when $F \equiv \Theta + 4\Sigma$.*

PROPOSITION 3.15. *Suppose that the fundamental group of X does not have any irreducible* SO(3) *or* SU(2)*-representations. Suppose that F is a divisor such that $F \equiv K$. If Z is a* 0*-cycle of degree two not separated by sections of the bundle KF, then Z is contained in an effective divisor E such that $E \equiv \Theta$, Σ, or $\Theta + \Sigma$.*

§4. Proof of The Main Theorems.

All propositions in the last section focus our attention on *effective* divisors E numerically equivalent to Σ, Θ or $\Theta + \Sigma$. In order to prove Theorem A and Theorem B in the introduction, we investigate the properties of such divisors.

LEMMA 4.1. *Suppose that E is a divisor numerically equivalent to Θ or Σ. Then* $\dim|E| \leq 0$.

PROOF. We prove this claim when $E \equiv \Sigma$. Suppose that $\dim|E| \geq 1$. Due to Lemma 2.4, every element in this system $|E|$ is irreducible. As $EE = \Sigma\Sigma = 0$, any two elements in this system are mutually disjoint. In particular, this system is free of base points. By Bertini's theorem, a generic element in this system is a non singular irreducible curve. Let D be such an element. Then one has the exact sequence

$$0 \to \mathscr{O} \to \mathscr{O}(E) \to \mathscr{O}_D(E) \to 0.$$

Since $EE = 0$ and $\dim|E| \geq 1$, $\mathscr{O}_D(E) = \mathscr{O}_D$. Therefore, $q = 0$ implies that $|E|$ is a pencil.

Let D be any nonsingular irreducible element in the pencil $|E|$. By the adjunction formula, the canonical divisor K_D of the curve D is the restriction of $K+E$ and the genus of D is equal to 2. It follows that

$$h^0(D, \mathcal{O}_D(K_D)) = 2, \qquad h^1(D, \mathcal{O}_D(K_D)) = h^0(D, \mathcal{O}_D) = 1.$$

As $p_g = 0$ and $q = 0$, we deduce an isomorphism

$$H^0(X, \mathcal{O}(KE)) \cong H^0(D, \mathcal{O}_D(K_D)) \cong \mathrm{C}^2$$

from the following exact sequence.

$$0 \to \mathcal{O}(K) \to \mathcal{O}(KE) \to \mathcal{O}_D(K_D) \to 0.$$

It means that the system $|K+E|$ is a pencil. As $(K+E)^2 = 12$, this pencil has base points. Let p be a base point. Let G be the unique element in the pencil $|E|$ through p. If G is singular at p, let q be any point of G different from p. As the system $|K+E|$ is a pencil, there is an element C in this system passing through q. The intersection multiplicity of C and G is at least one at q and two at p. Yet, $(K+E)\cdot G = 2$. Since G is irreducible, G is a component of C. Therefore, $K = C - G$ is effective. This is a contradiction to the vanishing of the geometric genus of X. So G is not singular at p. Similarly, let q be any singular point of G. This point is different from p. Let C be the unique element in the pencil $|K+E|$ through q, then G and C intersect at least once at p and twice at q. Again, we deduce that G is a component of C. Then $K = C - G$ is effective. This contradiction to the vanishing of p_g shows that G is non-singular. When G is non-singular, the conclusion of the last paragraph gives the following isomorphism.

$$H^0(X, \mathcal{O}(KE)) \cong H^0(G, \mathcal{O}_G(K_G)) \cong C^2.$$

As p is a base point of the system $|K+E|$ and is contained in the curve G, the above isomorphism obtained by restriction implies that p is a base point of the canonical system of the curve G. But G is a non-singular curve with genus 2, its canonical systemis base-point-free. This contradiction shows that $|E|$ cannot be a pencil.

Lemma 4.2. *Suppose that $E_1 \equiv \Theta + \Sigma$. Suppose that the map $\Phi_{KE_1^2}$ is not an embedding. Let E_2 be an effective degeneracy divisor of the system $|K+2E_1|$. Then $2E_1 \neq 2E_2$. In particular, $E_1 \neq E_2$.*

Proof. We only need to investigate the case when $E_2 \equiv \Theta + \Sigma$. In this case, the maximally destabilizing exact sequence is

$$0 \to \mathcal{O}(E_1^2 E_2^{-1}) \to \mathcal{E} \to \mathcal{O}(E_2) \to 0.$$

Therefore, the bundle $\mathcal{E}$ is determined by a non-trivial element in $H^1(X, E_1^2 E_2^{-2})$. As $q = 0$, the bundle $E_1^2 E_2^{-2}$ cannot be trivial.

PROOF OF THEOREM A. Let F be *any* divisor such that $F \equiv K$. Proposition 3.2 shows that the map Φ_{KF} is regular. Suppose that Φ_{KF} fails to be an embedding. Let Z be a degree-2 0-cycle not separated by sections of the bundle KF. Given the assumption on the fundamental group, the extension bundle of F with respect to Z is unstable. Proposition 3.15 shows that the support of Z is contained in an effective degeneracy divisor.

If all effective degeneracy divisors are numerically equivalent to either Σ or Θ, then Φ_{KF} is a bimeromorphism because the regularity of X implies that there are only finitely many such divisor classes. By Lemma 4.1, there is at most one effective element in the complete linear system of such a divisor class. Therefore, Φ_{KF} fails to be a bimeromorphism only when there is a degeneracy divisor E_1 such that $E_1 \equiv \Theta + \Sigma$ and $\dim|E_1| = 1$.

Given such E_1, define F_1 by E_1^2. Consider the adjunction map of F_1. Suppose that this map is not a bimeromorphism. Applying the conclusion of the last paragraph to F_1, one finds a degeneracy divisor E_2 such that $E_2 \equiv \Theta + \Sigma$ and $\dim|E_2| = 1$.

Due to Lemma 4.2, $E_1^2 \neq E_2^2$. Define $F_2 := E_2^2$. Φ_{KF_2} must be a bimeromorphism. For otherwise, there is a degeneracy divisor E_3 such that $E_3 \equiv \Theta + \Sigma$ and $\dim|E_3| = 1$. By Lemma 4.2, $E_1 \neq E_2$ and $E_2 \neq E_3$. If $E_1 = E_3$, then

$$H^0(X, E_1E_2E_3) = H^0(X, E_1^2E_2) \supseteq S^2H^0(X, E_1) \otimes H^0(X, E_2) \equiv C^6.$$

If $E_1 \neq E_3$, then

$$H^0(X, E_1E_2E_3) \supseteq H^0(X, E_1) \otimes H^0(X, E_2) \otimes H^0(X, E_3) \equiv C^8.$$

Since $p_g = 0$ and $h^0(X, E_1E_2E_3) \geq 6$, $h^2(X, E_1E_2E_3) = 0$. The divisors of the bundle $K(E_1E_2E_3)^{-1}$ is numerically equivalent to $-(\Theta + \Sigma)$. By Lemma 2.5, it is a negative line bundle. By Kodaira's vanishing theorem and Serre duality, $h^1(X, E_1E_2E_3) = 0$. By (3.1), $\chi(X, E_1E_2E_3) = 4$. It follows that $h^0(X, E_1E_2E_3) = 4$. This is a contradiction to $h^0(X, E_1E_2E_3) \geq 6$. Therefore if Φ_{KF_1} is not a bimeromorphism, then Φ_{KF_2} is a bimeromorphism.

The proof of Theorem A is completed.

PROOF OF THEOREM B. When $F - K$ is ample, then $F \equiv m\Theta + n\Sigma$ such that $m \geq 3$ and $n \geq 3$. Therefore, Φ_{KF} is regular because of Proposition 3.2 and is an embedding because of Proposition 3.9.

Suppose that F is not numerically equivalent to K and $F - K$ is numerically effective. If $F - K$ is not ample, then $F \equiv m\Theta + 2\Sigma$ such that $m \geq 3$ or $F \equiv 2\Theta + n\Sigma$ such that $n \geq 3$. Therefore, Φ_{KF} is regular because of Proposition 3.2. Due to part (2) of Proposition 3.9, Lemma 4.1 and the finiteness of the first integral homology group, one concludes that Φ_{KF} is a bimeromorphism.

The proof of Theorem B is completed.

§5. Other results.

The adjunction maps of F when $F \equiv m\Theta + \Sigma$ and $F \equiv \Theta + n\Sigma$ are yet to be dealt with. We study these maps with an additional topological assumption that the first integral homology group $H_1(X, \mathbf{Z})$ vanishes. Under this condition, two divisors are linearly equivalent if and only if there are numerically equivalent.

PROPOSITION 5.1. *Suppose that X is a numerical quadric such that $H_1(X, \mathbf{Z})$ vanishes and $\pi_1(X)$ does not have any irreducible $SO(3)$-representations. Let F be $2\Theta + \Sigma$ or $\Theta + 2\Sigma$. Then the system $|K + F|$ has at most one base point.*

PROOF. Assume that $F = 2\Theta + \Sigma$. If p is a base point of the system $|K + F|$. Let $\mathscr{E}$ be the extension bundle of F with respect to p. By Proposition 3.3, Θ is an effective divisor and the maximally destabilizing sequence (3.5) is equivalent to

$$0 \to \Theta + \Sigma \to \mathscr{E} \to \Theta \to 0. \tag{5.2}$$

Therefore, $\mathscr{E}$ is defined by a non-trivial element in $H^1(X, \Sigma)$.

As $p_g = 0$, $h^0(X, \Theta + \Sigma) = 0$. Then Lemma 4.1 and exact sequence (5.2) together shows that $h^0(X, \mathscr{E}) \leq 1$. By definition, $\mathscr{E}$ is obtained by the following extension.

$$0 \to \mathscr{O} \to \mathscr{E} \to \mathscr{I}_p(2\Theta + \Sigma) \to 0.$$

As $q = 0$ and $h^0(X, \mathscr{E}) \leq 1$, the above exact sequence shows that $h^0(X, \mathscr{E}) = 1$ and $h^0(X, \mathscr{I}_p(2\Theta + \Sigma) = 0$. Therefore, $h^0(X, 2\Theta + \Sigma) = 1$.

Since Θ is effective and the geometric genus of X is equal to zero, $h^0(X, \Sigma) = 0$. Then by (3.1), $h^2(X, \Sigma) = h^1(X, \Sigma)$. By Serre duality, we have $h^0(X, 2\Theta + \Sigma) = h^1(X, \Sigma)$. With the conclusion of the last paragraph, we have $h^1(X, \Sigma) = 1$.

As $h^0(X, \mathscr{E}) = 1$, the zeroes of any non-trivial sections of $\mathscr{E}$ is the point p. As $\mathscr{E}$ is the only non-trivial extension given by $H^1(X, \Sigma)$, when there were any other base points q, its extension bundle would have been the given $\mathscr{E}$. Moreover, $\mathscr{E}$ would have had a section vanishing at q. Yet $h^0(X, \mathscr{E}) = 1$ and $c_2(\mathscr{E}) = 1$. It shows that $p = q$.

PROPOSITION 5.3. *Suppose that X is a numerical quadric such that $H_1(X, \mathbf{Z})$ vanishes. Let F be $3\Theta + \Sigma$or $\Theta + 3\Sigma$. Then the adjunction map Φ_{KF} is a regular morphism.*

PROOF. Suppose that $F = 3\Theta + \Sigma$. By Proposition 3.2, when p is a base point of the system $|K + F|$, (3.5) is equivalent to

$$0 \to \mathcal{O}(2\Theta + \Sigma) \to \mathcal{E} \to \mathcal{O}(\Theta) \to 0.$$

Therefore, $\mathcal{E}$ is determined by a non-trivial element in $H^1(X, \Theta + \Sigma)$.

As the geometric genus of X vanishes, $h^0(X, \Theta + \Sigma) = 0$ and $h^2(X, \Theta + \Sigma) = 0$. Then by the Riemann-Roch formula, $-h^1(X, \Theta + \Sigma) = \chi(X, \Theta + \Sigma) = 0$. It shows that the extension bundle $\mathcal{E}$ does not exist.

PROPOSITION 5.4. *Suppose that X is a numerical quadric such that $H_1(X, \mathbf{Z})$ vanishes and $\pi_1(X)$ does not have any* SU(2)*-representations. Let F be $4\Theta + \Sigma$ or $\Theta + 4\Sigma$. Then the adjunction map Φ_{KF} is a regular morphism.*

PROOF. Suppose that $F = 4\Theta + \Sigma$. If $|K + F|$ has a base point p, the extension bundle is contained in

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{I}_p(4\Theta + \Sigma) \to 0. \tag{5.5}$$

The maximally destabilizing sequence is

$$0 \to \mathcal{O}(3\Theta + \Sigma) \to \mathcal{E} \to \mathcal{O}(\Theta) \to 0. \tag{5.6}$$

The last exact sequence shows that $\mathcal{E}$ is determined by a non-trivial class in $H^1(X, 2\Theta + \Sigma)$.

As Θ is effective and $p_g = 0$, $h^0(X, \Sigma) = 0$. Then by Serre duality, $h^2(X, 2\Theta + \Sigma) = 0$. By (3.1), $\chi(X, 2\Theta + \Sigma) = 0$. Therefore, $h^0(X, 2\Theta + \Sigma) = h^1(X, 2\Theta + \Sigma) \neq 0$. Let s be a non-trivial element in $H^0(X, 2\Theta + \Sigma)$. Then s^3 is a section of the bundle KF. In particular, s vanishes at p, a base point of the system $|K + F|$.

Twisting the exact sequences (5.5) and (5.6) by -2Θ, we have

$$0 \to \mathcal{O}(-2\Theta) \to \mathcal{E}(-2\Theta) \to \mathcal{I}_p(2\Theta + \Sigma) \to 0. \tag{5.7}$$

and

$$0 \to \mathcal{O}(\Theta + \Sigma) \to \mathcal{E}(-2\Theta) \to \mathcal{O}(-\Theta) \to 0.$$

The last exact sequence yields the vanishing of $H^0(X, \mathcal{E}(-2\Theta))$. As s is an element in $H^0(X, \mathcal{I}_p(2\Theta + \Sigma))$, then exact sequence (5.7) implies that $H^1(X, -2\Theta) \neq 0$. It follows that there is a non-trivial extension.

$$0 \to \mathcal{O} \to \mathcal{F} \to 2\Theta \to 0. \tag{5.8}$$

The discriminant of the bundle $\mathcal{F}$ is equal to zero and the determinant bundle of $\mathcal{F}$ is Θ^2. When $\pi_1(X)$ does not have any irreducible SU(2)-representations, the bundle $\mathcal{F}$ is unstable with respect to any polarization. Let M be a maximally destabilizing bundle with respect to $\Theta + \Sigma$. One has the following exact sequence.

$$0 \to M \to \mathscr{F} \to \mathscr{I}_A(2\Theta - M) \to 0, \tag{5.9}$$

with

$$M(\Theta + \Sigma) \geq \frac{1}{2} c_1(\mathscr{F})(\Theta + \Sigma) = \frac{1}{2}. \tag{5.10}$$

Therefore, the bundle M^{-1} does not have any non-trivial sections. Due to (5.8) and (5.9), $2\Theta - M$ is effective. Let a and b be non-negative integers such that $2\Theta - M = a\Theta + b\Sigma$. (5.10) implies that $2 - a - b \geq 1$. As $a\Theta + b\Sigma$ is effective, Lemma 2.3 implies that $a + b = 1$. As $c_2(\mathscr{F}) = 0$, $M(2\Theta - M) +$ $\deg A = 0$. Therefore, $2b(1 - a) + \deg A = 0$. Since $a + b = 1$, we have $b = 0$ and $\deg A = 0$. So we re-write (5.9) as

$$0 \to \Theta \to \mathscr{F} \to \Theta \to 0.$$

Then $\mathscr{F}$ is also defined by a non-trivial element in $H^1(X, \mathcal{O})$. As the surface is regular, such $\mathscr{F}$ cannot exist.

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DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF CALIFORNIA AT RIVERSIDE,
RIVERSIDE CA92521,
U.S.A.
E-mail address: ypoon@ucrmath.ucr.edu