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COASSOCIATED PRIMES OF MODULES OVER A COMMUTATIVE RING

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Abstract.

In this paper the concept of the coassociated prime of modules over (not necessarily Noetherian) commutative rings is introduced.

0. Introduction.

The theory of associated primes is very important in the study of modules over commutative Noetherian rings. For modules over non-Noetherian rings however, the classical associated primes do not behave so well, for example there exist non-trivial modules without associated primes. In [1] Bourbaki has introduced the notion of weakly associated primes of M over a (not necessarily Noetherian) commutative ring R: the prime ideal p is weakly associated to M if there exists an element $x \in M$ such that p is a minimal among the prime ideals contaning the annihilator $\operatorname{Ann} x$. In this paper the set of weakly associated primes of M is denoted by $\widetilde{\operatorname{Ass}}\, M$. In [2] Iroz and Rush have studied further the associated primes of modules over a (not necessarily Noetherian) commutative ring R. They have shown that the set $\widetilde{\operatorname{Ass}}\, M$ of weakly associated primes of R-module M is -- in some sense -- the best choice for a notion of associated primes over a (not necessarily Noetherian) commutative ring.

The aim of this paper is to develop a theory dual to that of weakly associated primes.

Let $\mathrm{D}_m(-)$ denote the functor $\operatorname{Hom}(-, \mathrm{E}(R/m))$ for $m \in \operatorname{Max} R$, the maximal spectrum of R, and the injective envelope $\mathrm{E}(R/m)$ of R/m.

In [10] we have introduced the notion of a cocyclic module (that is, a submodule of $\mathrm{D}_m(R)$ for some $m \in \operatorname{Max} R$) and it is used to define the notion of coassociated primes of modules over Noetherian rings (and this is dual to the classical associated primes). In this paper we use the notion of cocyclic modules to introduce the notion of weakly coassociated prime of modules

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(dual notion to that of weakly associated prime). We define the set $\widetilde{\mathrm{Coass}}\, M$ of weakly coassociated primes of an R-module M to be the set of prime ideals p such that there exists a cocyclic homomorphic image L of M with p is a minimal element in $\mathrm{V}(\mathrm{Ann}\, L)$.

We show that $p \in \widetilde{\mathrm{Coass}}\, M$ if and only if there exists a maximal ideal m such that $p \in \widetilde{\mathrm{Ass}}\, \mathrm{D}_m(M)$, and this result will play an important rôle in the rest of the paper.

Also we show that if $p \in \widetilde{\mathrm{Ass}}\, M$, then $p \in \widetilde{\mathrm{Coass}}\, \mathrm{D}_m(M)$ for all ideal $m \in \mathrm{Max}\, R \cap \mathrm{V}(p)$ (but the inverse is not true).

It turns out that weakly coassociated primes have properties similar -- or rather dual -- to those of weakly associated primes. For example, $\widetilde{\mathrm{Coass}}\, M \neq \emptyset$ whenever M is non-trivial, and that the set of coassociated primes of M is a subset of the set of weakly coassociated primes of M. In addition, $\widetilde{\mathrm{Coass}}\, M = \mathrm{Coass}\, M$ when R is Noetherian. Finally, if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of R-modules and R-homomorphisms, then it turns out that there are inclusions:

$$\widetilde{\mathrm{Coass}}\, M'' \subseteq \widetilde{\mathrm{Coass}}\, M \subseteq \widetilde{\mathrm{Coass}}\, M' \cup \widetilde{\mathrm{Coass}}\, M''.$$

In [3] Macdonald defined the set of attached primes of an R-module M. The theory of attached primes is particularly well-behaved when M has a secondary representation (which is the dual notion to primary decomposition). However, in general this theory is not completely satisfactory. In this paper we prove that the set of weakly coassociated primes of M is equal to the set of attached primes of M when M has a secondary representation.

Assume that R is a Noetherian ring and the zero submodule of M has primary decomposition. We find the set of coassociated primes of $\mathrm{Hom}(M, E)$ for any injective module E. This result is a sharpening of [8, 2.6] and [9, 2.1].

We define the *cosupport* of module M over commutative rings (this definition is the same as definition of the cosupport over Noetherian rings, cf. [10]). It follows that the set of weakly coassociated primes is a subset of the cosupport. On the other hand every minimal element of the cosupport belongs to the set of weakly coassociated primes.

In the last section we bring some functorial results. Let $\mathscr{P}$ be the class of all modules M such that the zero submodule has a primary decomposition in M (or $M = 0$), and let $\mathscr{S}$ be the class of all modules having secondary representation (or being zero). If $M \in \mathscr{P}$ then $\mathrm{Hom}(L, M) \in \mathscr{P}$ for any R-module L, $M \otimes F \in \mathscr{P}$ for any flat module F, and $\mathrm{Hom}(M, E) \in \mathscr{S}$ for any injective module E. If $N \in \mathscr{S}$ then $N \otimes L \in \mathscr{S}$, $\mathrm{Hom}(N, L) \in \mathscr{P}$ for any R-module L, and $\mathrm{Hom}(P, N) \in \mathscr{S}$ for any projective module P.

Throughout this paper the ring R is commutative with a non-zero identity element. We write "finite" for "finitely generated". Also, we shall use Max R to denote the set of all maximal ideals of R. For an R-module M its injective envelope is denoted by $\mathrm{E}(M)$.

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1. Weakly associated primes.

Let M be an R-module. The set of weakly associated prime ideals of M is denoted by $\widetilde{\mathrm{Ass}}\, M$ and it is the set of prime ideals p such that there exists $x \in M$ with p a minimal element in $\mathrm{V}(\mathrm{Ann}\, x)$, that is, the set of prime ideals p of R such that there exists a cyclic submodule N of M with p a minimal element in $\mathrm{V}(\mathrm{Ann}\, N)$.

1.1. Theorem. *Let M be an R-module. Then the following hold*

(a) $\mathrm{Ass}\, M \subseteq \widetilde{\mathrm{Ass}}\, M \subseteq \mathrm{Supp}\, M$.

(b) $\mathrm{Ass}\, M = \widetilde{\mathrm{Ass}}\, M$ *if R is a Noetherian ring.*

(c) $\widetilde{\mathrm{Ass}}\, M \neq \emptyset$ *if* $M \neq 0$.

(d) *If* $0 \to M' \to M \to M'' \to 0$ *is an exact sequence, then*

$$\widetilde{\mathrm{Ass}}\, M' \subseteq \widetilde{\mathrm{Ass}}\, M \subseteq \widetilde{\mathrm{Ass}}\, M' \cup \widetilde{\mathrm{Ass}}\, M''.$$

(e) $\mathrm{z}(M) = \bigcup_{p \in \widetilde{\mathrm{Ass}}\, M} p$.

(f) *If S is a multiplicative closed system of R, then*

$$\widetilde{\mathrm{Ass}}_{S^{-1}R}\, S^{-1}M = \{pS^{-1}R \mid p \in \widetilde{\mathrm{Ass}}_R\, M \text{ with } p \cap S = \emptyset\}.$$

Proof. See [1, page 289--290] and [2, 6.1--6.2].

1.2. Corollary *Let M be a Noetherian R-module. Then*

$$\widetilde{\mathrm{Ass}}_R\, M = \mathrm{Ass}_R\, M.$$

Proof. Set $R' = R/\mathrm{Ann}\, M$, which is a Noetherian ring, and then use (1.1) (b).

It follows from (1.1) (a) that $\mathrm{Ass}\, M \subseteq \widetilde{\mathrm{Ass}}\, M$ and equality holds when M is a Noetherian by (1.2). The next example shows that equality does not hold in general.

Example. Let k be a field and consider the ring $R = k^{\mathrm{N}}$ (direct product). Set $a = k^{(\mathrm{N})}$ (direct sum) which is an ideal of R. Set $M = R/a$. We claim that

$\operatorname{Ass} M = \emptyset$. Assume that $p \in \operatorname{Ass} M$. Then $a \subseteq p = (a : r)$ for some $r \notin a$. It is easy to find two elements s and t of R such that $sr, tr \notin a$ and $st = 0$. Since $str = 0$ we have $st \in p$. But $s \notin p$ and $t \notin p$ and this is a contradiction.

An R-module $M \neq 0$ is said to be coprimary if for each $a \in R$ the homothety $M \xrightarrow{a\cdot} M$ is either injective or nilpotent. This implies that $\operatorname{nil} M = p$ is a prime ideal of R (note that $\operatorname{nil} M = \sqrt{\operatorname{Ann} M}$), and M is said to be p-coprimary.

If there exist a finite number of primary submodules N_i (in other word M/N_i is coprimary) for $1 \leq i \leq n$ such that $0 = N_1 \cap N_2 \cap \cdots \cap N_n$, then we say that the zero submodule of M has primary decomposition. In this case M is isomorphic to a submodule of $M/N_1 \oplus M/N_2 \oplus \cdots \oplus M/N_n$.

1.3. Lemma. *Let the zero submodule of M have a primary decomposition and let $0 = N_1 \cap N_2 \cap \cdots \cap N_n$ be a minimal primary decomposition of 0 where N_i is p_i-primary submodule of M for $i = 1, 2, \ldots, n$. Then*

$$\widetilde{\operatorname{Ass}}_R M = \{p_1, p_2, \ldots, p_n\}.$$

Proof. "$\subseteq$" First assume that the zero submodule of M is p--primary (in the other words, M is p-coprimary). Let $q \in \widetilde{\operatorname{Ass}} M$. Then there exists a cyclic submodule N of M such that q is a minimal element in $V(\operatorname{Ann} N)$. Since N is also p-coprimary we have $\operatorname{nil} N = p$. Therefore $q = p$.

Now let the zero submodule of M have primary decomposition. Then there exist the exact sequence $0 \to M \to \oplus(M/N_i)$. We have $\widetilde{\operatorname{Ass}} M \subseteq \widetilde{\operatorname{Ass}} (\oplus M/N_i) = \cup \widetilde{\operatorname{Ass}}(M/N_i)$ by (1.1) (d) and $\widetilde{\operatorname{Ass}}(M/N_i) = \{p_i\}$ (as already proved). Now the assertion follows.

"$\supseteq$" Set $K_j = \cap_{i \neq j} N_i$. Then $K_j \neq 0$ by the minimality of the primary decomposition, and $K_j = K_j/(K_j \cap N_j) \cong (K_j + N_j)/N_j \subseteq M/N_j$. Now choose by (1.1) (c) $q \in \widetilde{\operatorname{Ass}} K_j$. Since $\widetilde{\operatorname{Ass}} K_j \subseteq \widetilde{\operatorname{Ass}}(M/N_j) \subseteq \{p_j\}$ by (1.1) (d) and the already established inclusion, we get $p_j = q \in \widetilde{\operatorname{Ass}} K_j \subseteq \widetilde{\operatorname{Ass}} M$ as desired.

1.4. Corollary. *If the zero submodule of M has a primary decomposition then $\widetilde{\operatorname{Ass}} M$ is a finite set. In particular, if M in a Noetherian R-module then $\widetilde{\operatorname{Ass}} M$ is a finite set.*

1.5. Theorem. *Let the zero submodule of M have primary decomposition. Then the following hold.*

(a) *The minimal elements of the set* $V(\operatorname{Ann} M)$ *belong to* $\widetilde{\operatorname{Ass}} M$.

(b) $\widetilde{\operatorname{Ass}} (R/\operatorname{Ann} M) \subseteq \widetilde{\operatorname{Ass}} M$.

(c) *The sets* $V(\operatorname{Ann} M)$, $\widetilde{\operatorname{Ass}} (R/\operatorname{Ann} M)$ *and* $\widetilde{\operatorname{Ass}} M$ *have the same minimal elements.*

Proof. Let $0 = N_1 \cap N_2 \cap \cdots \cap N_n$ be a minimal primary decomposition of the zero submodule of M where N_i is a p_i-primary submodule of M for $i = 1, 2, \dots, n$.

(a) There exists an exact sequence $0 \to M \to \oplus_i (M/N_i)$. Thus $\cap_i \mathrm{Ann}\,(M/N_i) = \mathrm{Ann}\, M$ (since the inclusion $\supseteq$ is obvious). Let p be a minimal element of the set $\mathrm{V}(\mathrm{Ann}\, M)$. Then p is a minimal element of the set $\mathrm{V}(\mathrm{Ann}\,(M/N_i))$ for some $1 \leq i \leq n$. Since $\mathrm{nil}(M/N_i) = p_i$ we have $p = p_i$. Now the assertion follows from (1.3).

(b) Let $p \in \widetilde{\mathrm{Ass}}(R/\mathrm{Ann}\, M)$. Then p is a minimal element of the set $\mathrm{V}(\mathrm{Ann}\,(a + \mathrm{Ann}\, M))$ for some $a \in R$. Thus p is a minimal element of $\mathrm{V}(\mathrm{Ann}\, aM)$. Therefore $p \in \widetilde{\mathrm{Ass}}(aM)$. Now the assersion follows from (1.1) (d).

(c) We know that $\mathrm{V}(\mathrm{Ann}\, M)$ and $\widetilde{\mathrm{Ass}}\, M$ have the same minimal elements by (a) and the fact that $\widetilde{\mathrm{Ass}}\, M \subseteq \mathrm{V}(\mathrm{Ann}\, M)$. For any $p \in \mathrm{Ass}(R/\mathrm{Ann}\, M)$ we know that $\mathrm{Ann}\, M \subseteq p$. Now the assertion follows from (b).

1.6. THEOREM. *Let R be a Noetherian ring and let F be a flat R-module. Let p be a prime ideal of R. Then the following are equivalent*

(i) *There exists a p-coprimary module M such that $M \otimes F \neq 0$.*

(ii) *For any p-coprimary module M we have $M \otimes F \neq 0$.*

(iii) *$R/p \otimes F \neq 0$.*

PROOF. Let M be a p-coprimary. Since R is a Noetherian ring and M has primary decomposition we have $\mathrm{Ass}\, M = \{p\}$.

"(i) $\Rightarrow$ (iii)" Since $\mathrm{Ass}\, M = \{p\}$ we have $\mathrm{Ass}(M \otimes F) = \mathrm{Ass}(R/p \otimes F)$ by [1, page 154]. Since $M \otimes F \neq 0$ we have $\mathrm{Ass}(M \otimes F) \neq \emptyset$ and hence $\mathrm{Ass}(R/p \otimes F) \neq \emptyset$. Now the desired implication has been established.

"(iii) $\Rightarrow$ (ii)" The exact sequence $0 \to R/p \to M$ induces the exact sequence $0 \to (R/p) \otimes F \to M \otimes F$. Now the assertion holds.

"(ii) $\Rightarrow$ (i)" This is clear.

1.7. LEMMA. *Let M, N be R-modules. If $\mathrm{Hom}(M, N) \neq 0$ then there exists $p \in \widetilde{\mathrm{Ass}}\, M$ such that $p \subseteq q$ for some $q \in \widetilde{\mathrm{Ass}}\, N$.*

PROOF. Let $\varphi : M \to N$ be a non-zero map. Choose $q \in \widetilde{\mathrm{Ass}}\, \varphi(M)$. Then $q \in \widetilde{\mathrm{Ass}}\, N$. Since $q \in \widetilde{\mathrm{Ass}}\, \varphi(M)$, we can choose $x \in M$ such that q is a minimal element in $\mathrm{V}(\mathrm{Ann}(\varphi(x)))$. Thus $q \in \mathrm{V}(\mathrm{Ann}\, x)$. We can choose p as a minimal element of $\mathrm{V}(\mathrm{Ann}\, x)$ such that $p \subseteq q$. Thus $p \in \widetilde{\mathrm{Ass}}\, M$.

1.8. THEOREM. *Let E be an injective R-module such that $\widetilde{\mathrm{Ass}}\, E = \mathrm{Ass}\, E$. Let p be a prime ideal of R. Then the following are equivalent*

(i) *There exists a p-coprimary module M such that $\mathrm{Hom}(M, E) \neq 0$.*

(ii) *For any p-coprimary module M we have $\mathrm{Hom}(M, E) \neq 0$.*

(iii) $\mathrm{Hom}(R/p, E) \neq 0$.

PROOF. "(iii) $\Rightarrow$ (ii)" Since $p \in \widetilde{\mathrm{Ass}}\, M$ we have $x \in M$ such that p is minimal in $\mathrm{V}(\mathrm{Ann}(Rx))$. The sequence $Rx \to R/p \to 0$ is exact and induces the exact sequence $0 \to \mathrm{Hom}(R/p, E) \to \mathrm{Hom}(Rx, E)$. Thus $\mathrm{Hom}(Rx, E) \neq 0$. In addition, the exact sequence $0 \to Rx \to M$ induces the exact sequence $\mathrm{Hom}(M, E) \to \mathrm{Hom}(Rx, E) \to 0$. Therefore $\mathrm{Hom}(M, E) \neq 0$.

"(ii) $\Rightarrow$ (i)" This is clear.

"(i) $\Rightarrow$ (iii)" Since $\widetilde{\mathrm{Ass}}\, M = \{p\}$ by (1.3), and $\mathrm{Hom}(M, E) \neq 0$ we have $p \subseteq q$ for some $q \in \widetilde{\mathrm{Ass}}\, E = \mathrm{Ass}\, E$ by (1.7). Thus $\mathrm{Hom}(R/q, E) \neq 0$. Now the exact sequence $R/p \to R/q \to 0$ induce the exact sequence $0 \to \mathrm{Hom}(R/q, E) \to \mathrm{Hom}(R/p, E)$. Thus $\mathrm{Hom}(R/p, E) \neq 0$.

1.9. REMARK. If R is a Noetherian ring then (i), (ii) and (iii) are equivalent for *any* injective R-module E.

Let R be a Noetherian ring and let N be a finite R-module. It is well-known that for any R-module M we have $\mathrm{Hom}(N, M) \neq 0$ if and only if $p \supseteq \mathrm{Ann}\, N$ for some $p \in \mathrm{Ass}\, M$, cf. [1, page 267]. In the next example we show that this result does not hold for $\widetilde{\mathrm{Ass}}\, M$ when R is not Noetherian.

EXAMPLE. Let R and M be the same as in the example after (1.2). Assume that $m \in \widetilde{\mathrm{Ass}}\, M$ so $m \in \mathrm{Max}\, R$ since that ring R is von Neumann regular. Since $m \notin \mathrm{Ass}\, M$ we have $\mathrm{Hom}(R/m, M) = 0$.

2. Weakly coassociated primes.

In this section we introduce the notion of weakly coassociated primes (dual notion to that of weakly associated prime). First we bring some definitions and results of [10].

In [10] we have introduced the notion of cocyclic module, and it is used to define the coassociated prime ideals and cosupport of modules over Noetherian rings. Now note that the next definitions are over (not necessarily Noetherian) commutative rings. For any maximal ideal m of R the functor $\mathrm{Hom}(-, \mathrm{E}(R/m))$ is denoted by $\mathrm{D}_m(-)$.

2.1. DEFINITION. An R-module L is said to be *cocyclic* if L is a submodule of $\mathrm{D}_m(R)$ for some $m \in \mathrm{Max}\, R$.

2.2. REMARK. The module M is cocyclic if there exists an ideal a of R such that $M \cong \mathrm{D}_m(R/a)$. This follows from the fact that $\mathrm{D}_m(R/a) \subseteq \mathrm{D}_m(R)$.

2.3. DEFINITION. Let M be an R-module. A prime ideal p of R is said to be a *coassociated* prime of M if there exists a cocyclic homomorphic image L of M such that $p = \mathrm{Ann}\, L$. The set of coassociated prime ideals of M is denoted by $\mathrm{Coass}\, M$.

The next theorem was proved in [10, 1.7] with the Noetherian condition on R, but the proof in the non-Noetherian case is the same.

2.4. THEOREM. *Let M be an R-module. The following are equivalent*
(i) $p \in \mathrm{Coass}\, M$
(ii) *There exists* $m \in \mathrm{Max}\, R \cap \mathrm{V}(p)$ *such that* $p \in \mathrm{Ass}\, \mathrm{D}_m(M)$.

2.5. DEFINITION. For an R-module M the subset $\mathrm{w}(M)$ of R is defined by

$$\mathrm{w}(M) = \{a \in R \mid M \xrightarrow{a\cdot} M \text{ is not surjective}\}.$$

2.6. DEFINITION. Let M be an R-module. The cosupport of M, written $\mathrm{Cosupp}\, M$, is the set of prime ideals p such that there exists a cocyclic homomorphic image L of M with $p \in \mathrm{V}(\mathrm{Ann}\, L)$.

2.7. DEFINITION. Let M be an R-module. A prime ideal p of R is said to be a *weakly coassociated* prime of M if there exists a *cocyclic* homomorphic image L of M such that p is a minimal element in $\mathrm{V}(\mathrm{Ann}\, L)$. The set of weakly coassociated prime ideals of M is denoted by $\widetilde{\mathrm{Coass}}\, M$.

2.8. LEMMA *For any R-module M,* $\widetilde{\mathrm{Coass}}\, M \subseteq \mathrm{Cosupp}\, M$ *and every minimal element of the set* $\mathrm{Cosupp}\, M$ *belongs to* $\widetilde{\mathrm{Coass}}\, M$.

PROOF. It follows directly from (2.6) and (2.7).

2.9. THEOREM. *Let M be an R-module. The following are equivalent*
(i) $p \in \widetilde{\mathrm{Coass}}\, M$
(ii) *There exists* $m \in \mathrm{Max}\, R \cap \mathrm{V}(p)$ *such that* $p \in \widetilde{\mathrm{Ass}}\, \mathrm{D}_m(M)$.

PROOF. (i) $\Rightarrow$ (ii). If $p \in \widetilde{\mathrm{Coass}}\, M$, then there exists a cocyclic homomorphic image L of M such that p is a minimal element in $\mathrm{V}(\mathrm{Ann}\, L)$. Let $\varphi : M \to L$ be the surjective homomorphism. Thus $\mathrm{Ann}\, \varphi = \mathrm{Ann}\, L$. Therefore p is a minimal element in $\mathrm{V}(\mathrm{Ann}\, \varphi)$ and hence $p \in \widetilde{\mathrm{Ass}}\, \mathrm{Hom}(M, L)$. Since L is a submodule of $\mathrm{D}_m(R)$ for some $m \in \mathrm{Max}\, R$, we have $p \in \widetilde{\mathrm{Ass}}\, \mathrm{D}_m(M)$ for the same $m \in \mathrm{Max}\, R$ by (1.1) (d).

(ii) $\Rightarrow$ (i). If $p \in \widetilde{\mathrm{Ass}}\, \mathrm{D}_m(M)$, then there exists $\varphi \in \mathrm{D}_m(M)$ such that p is a minimal element in $\mathrm{V}(\mathrm{Ann}\, \varphi)$. Let $L = \varphi(M)$ (submodule of $\mathrm{D}_m(R)$). Thus L is a cocyclic and $\mathrm{Ann}\, L = \mathrm{Ann}\, \varphi$. Hence we have $p \in \widetilde{\mathrm{Coass}}(M)$.

2.10. THEOREM. *For any R-module M the following hold.*
(a) $\mathrm{Coass}\, M \subseteq \widetilde{\mathrm{Coass}}\, M$.
(b) *If R is a Noetherian ring then* $\mathrm{Coass}\, M = \widetilde{\mathrm{Coass}}\, M$.
(c) *If M is an Artinian R-module then* $\mathrm{Coass}\, M = \widetilde{\mathrm{Coass}}\, M$.

PROOF. (a) It is clear.

(b) Let $p \in \widetilde{\text{Coass}}\, M$. There exists $m \in \text{Max}\, R$ with $p \in \widetilde{\text{Ass}}\, \text{D}_m(M)$ and hence $p \in \text{Ass}\, \text{D}_m(M)$ by (1.1) (b). Therefore we have $p \in \text{Coass}\, M$ by (2.4).

(c) Let $p \in \widetilde{\text{Coass}}\, M$. Then $p \in \widetilde{\text{Ass}}\, \text{D}_m(M)$ for some ideal $m \in \text{Max}\, R \cap \text{V}(p)$. We have $\widetilde{\text{Ass}}\, (\text{D}_m(M)) = \text{Ass}\, (\text{D}_m(M))$, since M is an Artinian R-module, cf. [7, 2.9]. Therefore $p \in \text{Ass}\, \text{D}_m(M)$ and hence $p \in \text{Coass}\, M$ by (2.4). Now the assertion follows from (a).

2.11. THEOREM. *Let M be an R-module. Then* $\widetilde{\text{Coass}}\, M \neq \emptyset$ *if* $M \neq 0$.

PROOF. Assume $M \neq 0$. Since $\text{D}_m(M) \cong \text{Hom}_{R_m}(M_m, \text{E}(R/m))$ we have then $\text{D}_m(M) \neq 0$ for some $m \in \text{Max}\, R$ (namely for $m \in \text{Supp}\, M \cap \text{Max}\, R$). Thus $\widetilde{\text{Ass}}\, \text{D}_m(M) \neq \emptyset$. Hence $\widetilde{\text{Coass}}\, M \neq \emptyset$ by (2.9).

2.12. THEOREM. *If* $0 \to M' \to M \to M'' \to 0$ *is an exact sequence, then* $\widetilde{\text{Coass}}\, M'' \subseteq \widetilde{\text{Coass}}\, M \subseteq \widetilde{\text{Coass}}\, M' \cup \widetilde{\text{Coass}}\, M''$.

PROOF. If $p \in \widetilde{\text{Coass}}\, M''$, then there exists $m \in \text{Max}\, R \cap \text{V}(p)$ such that $p \in \widetilde{\text{Ass}}\, \text{D}_m(M'')$ by (2.9). We have an exact sequence

$$(*) \qquad 0 \to \text{D}_m(M'') \to \text{D}_m(M) \to \text{D}_m(M') \to 0.$$

Thus $p \in \widetilde{\text{Ass}}\, \text{D}_m(M)$. Hence $p \in \widetilde{\text{Coass}}\, M$ by (2.9).

If now $p \in \widetilde{\text{Coass}}\, M$ then there exists $m \in \text{Max}\, R \cap \text{V}(p)$ such that $p \in \widetilde{\text{Ass}}\, \text{D}_m(M)$ by (2.9). Consider once more the exact sequence $(*)$. It follows that $p \in \widetilde{\text{Ass}}\, \text{D}_m(M'')$ or $p \in \widetilde{\text{Ass}}\, \text{D}_m(M')$, and hence $p \in \widetilde{\text{Coass}}\, M''$ or $p \in \widetilde{\text{Coass}}\, M'$ by (2.9).

2.13. COROLLARY. *For R-modules* $M_1, \ldots, M_n$ *we have*

$$\widetilde{\text{Coass}}(M_1 \oplus \cdots \oplus M_n) = \widetilde{\text{Coass}}(M_1) \cup \cdots \cup \widetilde{\text{Coass}}(M_n).$$

PROOF. "$\subseteq$" It is clear from (2.12) by induction.

"$\supseteq$" It follows from the fact that M_i is a homomorphic image of $M_1 \oplus \cdots \oplus M_n$ for all i.

2.14. LEMMA. *Let M be an R-module. If* $p \in \widetilde{\text{Ass}}\, M$, *then* $p \in \widetilde{\text{Coass}}\, \text{D}_m(M)$ *for all* $m \in \text{Max}\, R \cap \text{V}(p)$.

PROOF. Let $p \in \widetilde{\text{Ass}}\, M$ and let $m \in \text{Max}\, R \cap \text{V}(p)$. Then there exists $x \in M$ such that p is a minimal element in $\text{V}(\text{Ann}\, x)$. The exact sequence $Rx \to R/p \to 0$ induced the exact sequence $0 \to \text{D}_m(R/p) \to \text{D}_m(Rx)$. Since $\text{Ann}\, (\text{D}_m(R/p)) = p$ and $\text{Ann}\, x \subseteq \text{Ann}\, (\text{D}_m(Rx))$, we have p is a minimal element in $\text{V}(\text{Ann}(\text{D}_m(Rx)))$. On the other hand, Since $Rx \cong R/\text{Ann}\, x$, we have that $\text{D}_m(Rx)$ is a cocyclic by (2.2). Thus $p \in \widetilde{\text{Coass}}\, \text{D}_m(Rx)$ and hence $p \in \widetilde{\text{Coass}}\, \text{D}_m(M)$ by (2.12).

2.15. THEOREM. *For any R-module M there is an equality*

$$\mathrm{w}(M) = \bigcup_{p \in \widetilde{\mathrm{Coass}}\, M} p.$$

PROOF. If $a \in \bigcup_{p\in\widetilde{\mathrm{Coass}}\, M} p$, then there exists $p \in \widetilde{\mathrm{Coass}}\, M$ with $a \in p$. By (2.9) there exists $m \in \mathrm{Max}\, R$ with $p \in \widetilde{\mathrm{Ass}}\, \mathrm{D}_m(M)$ and hence $a \in \mathrm{z}(\mathrm{D}_m(M))$. Thus $aM \neq M$ and hence $a \in \mathrm{w}(M)$.

On the other hand let $a \in (M)$. Since $M/aM \neq 0$, we have $\widetilde{\mathrm{Coass}}(M/aM) \neq \emptyset$ by (2.11). Let $p \in \widetilde{\mathrm{Coass}}(M/aM)$. Then $a \in p$ and $p \in \widetilde{\mathrm{Coass}}\, M$ by (2.12). Therefore $a \in \bigcup_{p\in\widetilde{\mathrm{Coass}}\, M} p$.

3. Attached primes.

In [3] I. G. Macdonald has developed the theory of attached prime ideals and secondary representation of a module, which is (in a certain sense) a dual to the theory of associated prime ideals and primary decompositions. Now we want to prove that if M has a secondary representation then $\mathrm{Att}(M) = \widetilde{\mathrm{Coass}}(M)$. We use the notation of [3]. An R-module $M \neq 0$ is called secondary if for each $a \in R$ multiplication by a on M is either surjective or nilpotent. Then $\mathrm{nil}\, M = p$ is a prime ideal and M is called p-secondary. We say that M has a secondary representation if there is a finite number of secondary submodules $M_1, M_2, \ldots, M_n$ such that $M = M_1 + M_2 + \cdots + M_n$. One may assume that the prime ideals $\mathrm{nil}\, M_i = p_i$, $i = 1, 2, \ldots, n$, are all distinct and, by omitting redundant summands, that the representation is minimal. Then the set of prime ideals $\{p_1, \ldots, p_n\}$ does not depend on the representation, and it is called the set of attached prime ideals and denoted by $\mathrm{Att}(M)$. In [3] Macdonald showed that Artinian modules have secondary representation.

3.1. THEOREM. *If M have a secondary representation and $M = S_1 + S_2 + \cdots + S_n$ is a minimal secondary representation of M where S_i is a p_i-secondary submodule of M for $i = 1, 2, \ldots, n$, then*

$$\widetilde{\mathrm{Coass}}\, M = \{p_1, p_2, \ldots, p_n\} = \mathrm{Att}\, M.$$

PROOF. First assume that M is a p-secondary. Then $\mathrm{Att}\, M = \{p\}$. Let $q \in \widetilde{\mathrm{Coass}}\, M$. There exists then a cocyclic homomorphic image L of M such that q is a minimal element in $\mathrm{V}(\mathrm{Ann}\, L)$. Since L is also p-secondary we have $\mathrm{nil}\, L = p$ and hence $q = p$.

Now let M have secondary representation. Then there exists an exact sequence $\oplus S_i \to M \to 0$. Since $\widetilde{\mathrm{Coass}}\, M \subseteq \cup_i \widetilde{\mathrm{Coass}}\, S_i$ by (2.12) and (2.13), we have $\widetilde{\mathrm{Coass}}\, M \subseteq \mathrm{Att}\, M$ (since it has already been proved that $\widetilde{\mathrm{Coass}}\, S_i = \mathrm{Att}\, S_i$ for any $i = 1, 2, \ldots, n$).

Set $K_j = \sum_{i \neq j} N_i$. Then $M/K_j \neq 0$ by the minimality of the representation, and $M/K_j = (K_j + S_j)/K_j \cong S_j/(K_j \cap S_j)$ is a homomorphic image of S_j. Choose by (2.11) $q \in \widetilde{\text{Coass}}(M/K_j)$. Since $\widetilde{\text{Coass}}(M/K_j) \subseteq \widetilde{\text{Coass}}\, S_j \subseteq \{p_j\}$ by (2.12) and the already established equality, we get $p_j = q \in \widetilde{\text{Coass}}(M/K_j) \subseteq \widetilde{\text{Coass}}\, M$ as desired.

3.2. COROLLARY. *If M have a secondary representation then* $\widetilde{\text{Coass}}\, M$ *is a finite set. In particular, if M is an Artinian R-module then* $\widetilde{\text{Coass}}\, M$ *is a finite set.*

3.3. THEOREM. *Let M have a secondary representation. Then the following hold*

(a) *The minimal elements of the set* $\text{V}(\text{Ann}\, M)$ *belong to* $\text{Coass}\, M$.

(b) $\widetilde{\text{Ass}}(R/\text{Ann}\, M) \subseteq \widetilde{\text{Coass}}\, M$.

(c) *The sets* $\text{V}(\text{Ann}\, M)$, $\widetilde{\text{Ass}}(R/\text{Ann}\, M)$ *and* $\widetilde{\text{Coass}}\, M$ *have the same minimal elements.*

PROOF. (a) Let $M = S_1 + S_2 + \cdots + S_n$ be a minimal secondary representation of M where S_i is a p_i-secondary for $i = 1, 2, \ldots, n$. We have the sequence $\oplus S_i \to M \to 0$ is exact. Thus $\cap \text{Ann}\, S_i = \text{Ann}\, M$. Let p be a minimal element of the set $\text{V}(\text{Ann}\, M)$. Then p is a minimal element of the set $\text{V}(\text{Ann}\, S_i)$ for some $1 \leq i \leq n$. Since $\text{nil}\, S_i = p_i$ we have $p = p_i$ and hence $p \in \text{Att}\, S_i$. Now the assertion follows from (3.1).

(b) Since $\text{Ann}\, M = \cap \text{Ann}\, S_i$ and $\text{Ann}\, S_i$ is a p_i primary ideal we have $R/\text{Ann}\, M$ has primary decomposition and $\widetilde{\text{Ass}}(R/\text{Ann}\, M) \subseteq \text{Att}\, M$.

(c) We know that $\widetilde{\text{Coass}}_R\, M = \{p_1, p_2, \ldots, p_n\}$ by (3.1), and that $\text{Ann}\, M \subseteq \text{Ann}\, S_i$ for any $1 \leq i \leq n$. Therefore $\text{V}(\text{Ann}\, M)$ and $\widetilde{\text{Coass}}\, M$ have the same minimal elements by (a). Since $\text{Ann}\, M \subseteq p$ for any $p \in \widetilde{\text{Ass}}(R/\text{Ann}\, M)$ the assertion follows from (b).

3.4. THEOREM. *Let M be a Noetherian R-module. For an injective R-module E we have*

$$\widetilde{\text{Coass}}\, \text{Hom}(M, E) = \{p \in \widetilde{\text{Ass}}\, M \mid p \subseteq q \text{ for some } q \in \widetilde{\text{Ass}}\, E\}.$$

PROOF. It follows from [5, Lemma 4] and (3.1).

3.5. REMARK. Let M be an R-module and let E be an injective R-module. Then

$$\{p \in \widetilde{\text{Ass}}\, M \mid p \subseteq q \text{ for some } q \in \widetilde{\text{Ass}}\, E\} \subseteq \widetilde{\text{Coass}}\, \text{Hom}(M, E),$$

cf. [10, Remark after (1.17)] and (2.10) (a). But equality does not hold in general, cf. [10, Example after (1.8)].

Assume R is a Noetherian ring. In [8] Sharp has found the set of attached primes of injective modules E and in [9] Toroghy and Sharp have found the set of attached primes of $\mathrm{Hom}(M,E)$ for any finite module M and injective module E. These results are particular cases of the following.

3.6. THEOREM. *Assume R is a Noetherian ring. Let the zero submodule of R-module M have a primary decomposition and let $0 = N_1 \cap N_2 \cap \cdots \cap N_n$ be a minimal primary decomposition of the zero submodule of M where N_i is a p_i-primary submodule of M for $i = 1, 2, \ldots, n$. For an injective R-module E,*

$$\mathrm{Coass}\,\mathrm{Hom}(M,E) = \{p \in \widetilde{\mathrm{Ass}}\, M \mid p \subseteq q \text{ for some } q \in \widetilde{\mathrm{Ass}}\, E\}$$

PROOF. First note that $\widetilde{\mathrm{Ass}}\, M = \{p_1, p_2, \ldots, p_n\} = \widetilde{\mathrm{Ass}}\, M$ by (1.2) and (1.3). There exists an exact sequence $0 \to M \to \oplus M/N_i$. Thus the sequence $\oplus\mathrm{Hom}(M/N_i, E) \to \mathrm{Hom}(M,E) \to 0$ is exact. Therefore $\mathrm{Coass}\,\mathrm{Hom}(M,E) \subseteq \cup\,\mathrm{Coass}\,\mathrm{Hom}(M/N_i, E)$. Since for any $a \in R$ the homothety $M/N_i \xrightarrow{a\cdot} M/N_i$ is either injective or nilpotent we have $\mathrm{Hom}(M/N_i, E) \xrightarrow{a\cdot} \mathrm{Hom}(M/N_i, E)$ is either surjective or nilpotent for $1 \le i \le n$. Thus $\mathrm{Hom}(M/N_i, E)$ is either p_i-secondary or zero. On the other hand if $\mathrm{Hom}(M/N_i, E) \ne 0$ then $\mathrm{Hom}(R/p_i, E) \ne 0$ by (1.8), and hence $p_i \subseteq q$ for some $q \in \mathrm{Ass}\, E$ by (1.7). Thus $\mathrm{Coass}\,\mathrm{Hom}(M,E) \subseteq \{p \in \mathrm{Ass}\, M \mid p \subseteq q \text{ for some } q \in \mathrm{Ass}\, E\}$. Now the assertion follows from (3.5).

4. Conilradical.

In this section we introduce the notion of conilradical (dual notion to that of nilradical).

4.1. DEFINITION. For an R-module M we denote the *conilradical* of M by conil (M) defined as the set of all elements $a \in R$ such that for each cocyclic homomorphic image L of M there exists $n \in \mathbb{N}$ with $a^n L = 0$.

4.2. THEOREM. *For any R-module M we have*

$$\mathrm{conil}(M) = \bigcap_{p \in \widetilde{\mathrm{Coass}}\, M} p\,.$$

PROOF. Let $a \in \mathrm{conil}(M)$ and let $p \in \widetilde{\mathrm{Coass}}\, M$. It follows that there exists a cocyclic homomorphic image L of M such that p is a minimal element in $\mathrm{V}(\mathrm{Ann}\, L)$. Thus there exists $t \in \mathbb{N}$ such that $a^t L = 0$ by the definition. Since $\mathrm{Ann}\, L \subseteq p$, we have $a^t \in p$, and hence $a \in p$.

Now let $a \in \bigcap_{p \in \widetilde{\mathrm{Coass}}\, M} p$ and let L be a cocyclic homomorphic image of M. Assume that $a^t \notin \mathrm{Ann}\, L$ for all $t \in \mathbb{N}$. Thus $a \notin \mathrm{nil}\, L$. It follows that there exists $q \in \mathrm{Spec}(R)$ such that $\mathrm{Ann}\, L \subseteq q$ and $a \notin q$. Therefore $q \in \mathrm{Cosupp}\, M$ and $a \notin q$. Now the contradiction follows from (3.7).

4.3. COROLLARY. *Let M be an R-module. Then the following areequivalent*
(i) $\widetilde{\text{Coass}}\, M$ *has exactly one element.*
(ii) $\text{conil}(M) = \text{w}(M)$.

PROOF. "(i) $\Rightarrow$ (ii)" It follows from (4.2) and (2.15).
"(ii) $\Rightarrow$ (i)" For any $p \in \widetilde{\text{Coass}}\, M$ we have $(M) \subseteq p \subseteq \text{w}(M)$ by (4.2) and (2.15). Now the assertion follows.

5. Functorial results.

Let $\mathcal{P}$ be the class of all R-modules M such that the zero submodule has primary decomposition (or $M = 0$), and let $\mathcal{S}$ be the class of all R-modules having secondary representation (or being zero). In this section we show some functorial relation between these two classes.

5.1. THEOREM. *Let* F *be a linear functor over the category of R-modules. Then the following hold*
(a) *If* F *is left exact and covariant and if $M \in \mathcal{P}$ then* $\text{F}(M) \in \mathcal{P}$. *In particular, If F is a flat R-module and $M \in \mathcal{P}$ then $M \otimes F \in \mathcal{P}$, and if $M \in \mathcal{P}$ then* $\text{Hom}(N, M) \in \mathcal{P}$ *for any R-module N.*
(b) *If* F *is right exact and contravariant and if $M \in \mathcal{P}$ then* $\text{F}(M) \in \mathcal{S}$. *In particular, If E is an injective R-module and $M \in \mathcal{P}$ then* $\text{Hom}(M, E) \in \mathcal{S}$.
(c) *If* F *is right exact and covariant and if $M \in \mathcal{S}$ then* $\text{F}(M) \in \mathcal{S}$. *In particular, If $M \in \mathcal{S}$ then $M \otimes N \in \mathcal{S}$. In addition, If P is a projective R-module and $M \in \mathcal{S}$ then* $\text{Hom}(P, M) \in \mathcal{S}$.
(d) *If* F *is left exact and contravariant and if $M \in \mathcal{S}$ then* $\text{F}(M) \in \mathcal{P}$. *In particular, If $M \in \mathcal{S}$ then* $\text{Hom}(M, N) \in \mathcal{P}$ *for any N.*

PROOF. First let $M \in \mathcal{P}$ and let $0 = N_1 \cap N_2 \cap \cdots \cap N_n$ be a minimal primary decomposition of the zero submodule of M where N_i is a p_i-coprimary submodule of M for $i = 1, 2, \ldots, n$.
(a) Since for any $a \in R$ the homothety $M/N_i \xrightarrow{a\cdot} M/N_i$ is either injective or nilpotent we have $\text{F}(M/N_i) \xrightarrow{a\cdot} \text{F}(M/N_i)$ is either injective or nilpotent for $1 \leq i \leq n$. Thus $\text{F}(M/N_i)$ is either p_i-coprimary or zero. On the other hand the exact sequence $0 \to M \to \oplus_i(M/N_i)$ induce the exact sequence $0 \to \text{F}(M) \to \oplus_i \text{F}(M/N_i)$.
(b) Since for any $a \in R$ the homothety $M/N_i \xrightarrow{a\cdot} M/N_i$ is either injective or nilpotent we have $\text{F}(M/N_i) \xrightarrow{a\cdot} \text{F}(M/N_i)$ is either surjective or nilpotent for $1 \leq i \leq n$. Thus $\text{F}(M/N_i)$ is either p_i-secondary or zero. On the other hand the exact sequence $0 \to M \to \oplus_i(M/N_i)$ induce the exact sequence $\oplus_i \text{F}(M/N_i) \to \text{F}(M) \to 0$.
Now let M have a secondary representation and $M = S_1 + S_2 + \cdots + S_n$ is

a minimal secondary representation of M where S_i is a p_i-secondary submodule of M for $i = 1, 2, \ldots, n$.

The proof of (c) is similar to the proof (a), and the proof of (d) is similar to the proof (b) using the exact sequence $\oplus S_i \to M \to 0$.

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