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EFFICIENT PRESENTATIONS OF THE GROUP $\mathrm{PSL}(2, Z_n) \times \mathrm{PSL}(2, Z_m)$, FOR CERTAIN n, m.

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Abstract.

We give deficiency -4 presentations of the groups $\mathrm{PSL}(2, Z_n) \times \mathrm{PSL}(2, Z_m)$, n, m odd numbers and $[(n \equiv 1(\mathrm{mod}6)$ and $m \equiv 1(\mathrm{mod}6))$ or $(n \equiv -1(\mathrm{mod}6)$ and $m \equiv -1(\mathrm{mod}6))$ or $(n \equiv 1(\mathrm{mod}6)$ and $m \equiv -1(\mathrm{mod}6))$ or $(n \equiv -1(\mathrm{mod}6)$ and $m \equiv 1(\mathrm{mod}6))]$. Moreover efficient presentations are given for certain cases of the groups considered.

1. Introduction.

For any commutative ring R with a 1 define $\mathrm{SL}(2, R)$ to be the group of 2×2 matrices with determinant 1 over R. Define $\mathrm{PSL}(2, R) = \mathrm{SL}(2, R)/\{\pm I\}$ where I is the 2×2 identity matrix.

If R is the finite field $\mathrm{GF}(p^n)$, for p a prime , we write $\mathrm{PSL}(2, R) = \mathrm{PSL}(2, p^n)$. The order of $\mathrm{PSL}(2, R)$ is $p^n(p^n - 1)(p^n + 1)/2$. If R is the ring of integers *modulo m*, then we write. $\mathrm{PSL}(2, R) = \mathrm{PSL}(2, Z_m)$. In terms of the prime factorization $m = \Pi p^c$, the order of $\mathrm{PSL}(2, Z_m)$ is see [7], $m^3 \Pi p^c(1 - 1/p^2)/2$.

Given a finite presentation $\langle X \mid Y \rangle$ of a finite group G, the deficiency of the presentation is $\mid X \mid - \mid X \mid \leq 0$. Let $M(G)$ denote the Schur multiplier of G (see Beyl and Tappe [1].). Schur [6] shoved that any presentation G with n generators requires at least $n + \mathrm{rank}(M(G))$ relations. If G has a presentation with n generators and precisely $n + \mathrm{rank}(M(G))$ relations we say that G is efficient.

Questions concerning the efficiency of direct products have been of considerable interest for a number of years. The first questions concerning the efficiency of direct products were posed by Wiegold in [9]. In particular his questions were whether $\mathrm{PSL}(2, 5) \times \mathrm{PSL}(2, 5)$ and $\mathrm{SL}(2, 5) \times \mathrm{SL}(2, 5)$ are efficient. The first of these questions was answered by Kenne in [5]. He showed that $\mathrm{PSL}(2, 5) \times \mathrm{PSL}(2, 5)$ is efficient. The second question was answered by Campbell et al [2]. In [3] C.M.Campbell, E.F.Robertson and P.D.Williams have obtained efficient presentations for certain direct products involving

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fields of the same characteristic. Some work on direct products of groups $\mathrm{PSL}(2, p^{n_i})$ for a fixed prime p and different n_i's is discussed and also some efficient presentations for $\mathrm{PSL}(2, q_1) \times \mathrm{PSL}(2, q_2)$, q_1, q_2 prime powers, is given by Vatansever in [8].

In this paper we consider the problem of efficient presentation for direct product $\mathrm{PSL}(2, \mathrm{Z_n}) \times \mathrm{PSL}(2, \mathrm{Z_m})$.

Given two groups G_1 and G_2 then the Schur-Kunneth formula [4] asserts

$$M(G_1 \times G_2) = M(G_1) \times M(G_2) \times (G_1 \otimes G_2).$$

Thus, when G_1 or G_2 is perfect, $M(G_1 \times G_2) = M(G_1) \times M(G_2)$ so the multiplier of a direct product of simple groups is the direct product of multipliers of the simple groups.

The direct product $\mathrm{PSL}(2, \mathrm{Z_n}) \times \mathrm{PSL}(2, \mathrm{Z_m})$.

In this section we shall investigate the direct products $\mathrm{PSL}(2, Z_n) \times \mathrm{PSL}(2, Z_m), n, m$ odd numbers and

(i) $n \equiv 1(\mathrm{mod}6)$ and $m \equiv 1(\mathrm{mod}6)$
(ii) $n \equiv -1(\mathrm{mod}6)$ and $m \equiv -1(\mathrm{mod}6)$
(iii) $n \equiv 1(\mathrm{mod}6)$ and $m \equiv -1(\mathrm{mod}6)$
(iv) $n \equiv -1(\mathrm{mod}6)$and $m \equiv 1(\mathrm{mod}6)$

in an attempt to prove that these groups are efficient. For a particular n, m we shall give efficient presentations which were not previously known to be efficient.We consider in details case (iii). The other cases can be deduced from case (iii).

Let $G = \mathrm{PSL}(2, Z_n) \times \mathrm{PSL}(2, Z_m)$. Then, using a presentation for $\mathrm{PSL}(2, Z_p)$ given in [7]. $G = \langle a, b, c, d \mid a^2 = b^n = (ab)^3 = (ab^4ab^{(n+1)/2})^2 = 1,$ $c^2 = d^m = (cd)^3 = (cd^4cd^{(m+1)/2})^2 = 1,\ [a, c] = [a, d] = [b, c] = [b, d] = 1\rangle$. Put $x = bcd,\ y = abd$. Then let $n = 6k + 1,\ m = 6t - 1$. We have $x^3 = b^3 \Rightarrow$ $x^{n-1} = b^{-1}$ so $b = x^{1-n}$. Similarly $y^3 = d^3 \Rightarrow y^{m+1} = d^{m+1} = d$ so $d = y^{m+1}$. Since $x = bcd \Rightarrow c = x^n y^{-m-1}$. Also since $y = abd \Rightarrow a = y^{-m}x^{n-1}$. We have proved:

LEMMA 1. *If $x = bcd$, $y = abd$ then $a = y^{-m}x^{n-1}$, $b = x^{1-n}$, $c = x^n y^{-m-1}$, $d = y^{m+1}$.*

We write down the 12 relations of G written in terms of x and y in the order they appear in the presentation above

(1) $(y^{-m}x^{n-1})^2 = 1$
(2) $(x^{1-n})^n = 1$
(3) $y^{-3m} = 1$

(4) $(y^{-m}x^{-3n+3}y^{-m}x^{r})^2 = 1$, where $r = -(n-1)^2/2$
(5) $(x^n y^{-m-1})^2 = 1$
(6) $(y^{m+1})^m = 1$
(7) $x^{3n} = 1$
(8) $(x^n y^{3m+3} x^n y^s)^2 = 1$, where $s = (m^2 - 1)/2$
(9) $[y^{-m}x^{n-1}, x^n y^{-m-1}] = 1$
(10) $[y^{-m}x^{n-1}, y^{m+1}] = 1$
(11) $[x^{1-n}, x^n y^{-m-1}] = 1$
(12) $[x^{1-n}, y^{m+1}] = 1$

LEMMA 2. *In Lemma* 1 *the relations* (2), (6), (10), (11) *are redundant.*

PROOF. Since $3|(1-n)$ then x^{1-n} is a power of x^3 so, since $x^{3n} = 1$ we have $(x^{1-n})^2 = 1$. Since $3|(m+1)$ then y^{m+1} is a power of y^3 so, since $y^{3m} = 1$ we have $(y^{m+1})^m = 1$. Also (10) and (11) are immediate consequences of (12).

We now tidy up a little. Since $[x^{1-n}, y^{m+1}] = 1$, cubing these two elements we get $[x^3, y^3] = 1$. Also using (7) we can replace r in (4) by $(n-1)/2$. Using (3) we can replace s in (8) by $(-m-1)/2$. We now have the presentation for G as follows.

LEMMA 3. *G is generated by x and y subject to the relations*

(i)	$(y^{-m}x^{n-1})^2 = 1$	(v)	$[x^n, y^m] = 1$
(ii)	$(x^n y^{-m-1})^2 = 1$	(vi)	$[x^3, y^3] = 1$
(iii)	$(y^{-m}x^3 y^{-m}x^{(n-1)/2})^2 = 1$	(vii)	$x^{3n} = 1$
(iv)	$(x^n y^3 x^n y^{-(m+1)/2})^2 = 1$	(viii)	$y^{3m} = 1$

Consider (i). We have $y^{-m}x^{n-1}y^{-m}x^{n-1} = 1$ and using (v) and (vii) this gives $y^{-m}x^{-1}y^{-m}x^{-n-1} = 1$. Hence $x^{-n-1} = y^m x y^m$ replaces (i). Similarly consider (ii). We have $x^n y^{-m-1} x^n y^{-m-1} = 1$ and using (v) and (viii) this gives $x^n y^{-1} x^n y^{m-1} = 1$. Hence $y^{-m+1} = x^n y^{-1} x^n$ replaces (ii). But consider again $y^{-m}x^{n-1}y^{-m}x^{n-1} = 1$ and this time use (*vi*) in the form $[x^{n-1}, y^{m+1}] = 1$. We have $y^{-m}x^{n-1}y^{-m-1}yx^{n-1} = 1 \Rightarrow y^{-2m-1}x^{n-1}yx^{n-1} = 1 \Rightarrow y^{m-1}x^{n-1}yx^{n-1} = 1$ and substituting $y^{m-1} = x^{1-n}y^{-1}x^{1-n}$ into this relation gives $x^n y^{-1} x^n y^{m-1} = x^n y^{-1} x^n x^{1-n} y^{-1} x^{1-n} = xy^{-1}xy^{-1} = 1$ so $(xy^{-1})^2 = 1$. This now replaces $y^{-m+1} = x^n y^{-1} x^n$. Use this to replace $x^{-n-1} = y^m x y^m$ by $x^{-n-1} = y^{m+1}x^{-1}y^{m+1}$. We have new relations (i)* and (ii)* to replace respectively (i) and (ii). They are

(i)* $x^{-n-1} = y^{m+1}x^{-1}y^{m+1}$
(ii)* $(xy^{-1})^2 = 1$.

LEMMA 4. *In Lemma* 3 $[x^n, y^m] = 1$ *and* $[x^3, y^3] = 1$ *are redundant.*

PROOF. Using the (ii)* we can rewrite (i)* as $x^{-n} = y^m x y^m x$ so $[y^m x, x^n] = 1 \Rightarrow [y^m, x^n] = 1$. Consider (i)* i.e. $x^{-n-1} = y^{m+1}x^{-1}y^{m+1} \Rightarrow x^{-n-2} = (y^{m+1}x^{-1})^2 \Rightarrow [y^{m+1}, x^{n+2}] = 1$. Cubing the first term in $[y^{m+1}, x^{n+2}] = 1$and using (viii) we have $[y^3, x^{n+2}] = 1$. Cubing the second term in $[y^3, x^{n+2}] = 1$ and using (vii) we have $[y^3, x^6] = 1$. Now considering $[y^3, x^{n-1+3}] = 1$ and using the fact that $6|(n-1)$ and using $[y^3, x^6] = 1$ it can be seen that $[y^3, x^3] = 1$.

Next we simplify (iii) and (iv). Notice that we can still use (v) and (vi) which are consequences of (i) and (ii). Write (iii) as

$$y^{-m}x^3y^{-m}x^{(n-1)/2}y^{-m}x^3y^{-m}x^{(n-1)/2} = 1$$
$$\Rightarrow x^3yx^{(n-1)/2}yx^3yx^{(n-1)/2}y^{-m-3} = 1 \text{ since } 3|(m+1)$$
$$(x^3yx^{(n-1)/2}y)^2 = y^{m+4}$$

Write (iv) as

$$x^ny^3x^ny^{-(m+1)/2}x^ny^3x^ny^{-(m+1)/2} = 1$$
$$\Rightarrow x^{-2}y^3xy^{-(m+1)/2}xy^3xy^{-(m+1)/2}x^{n-1} = 1 \text{ since } 3|(n-1)$$
$$(y^3xy^{-(m+1)/2}x)^2 = x^{4-n}.$$

We now write the relations of G as:

THEOREM 1. *G is generated by x and y subject to the relations*

(.1.)	$x^{3n} = 1$	(.4.)	$(x^3yx^{(n-1)/2}y)^2 = y^{m+4}$
(.2.)	$y^{3m} = 1$	(.5.)	$(y^3xy^{-(m+1)/2}x)^2 = x^{4-n}$
(.3.)	$(xy^{-1})^2 = 1$	(.6.)	$x^{-n} = (y^m x)^2$

LEMMA 5. *In G we have $[x^n, yx^3y^{-1}] = [x^n, y^{-1}x^3y] = 1$, $[y^m, xy^3x^{-1}] = [y^m, x^{-1}y^3x] = 1$.*

PROOF. Since $y^m = x^{1-n}y^{-1}x^{1-n}y = yx^{1-n}y^{-1}x^{1-n}$ we have $[x^n, yx^{1-n}y^{-1}] = [x^n, y^{-1}x^{1-n}y] =1$ and cubing the second term in the commutators gives the result.

From (i)* we can deduce $x^n = y^{-m-1}xy^{-m-1}x^{-1} = x^{-1}y^{-m-1}xy^{-m-1}$. We have $[y^m, xy^{-m-1}x^{-1}] = [y^m, x^{-1}y^{-m-1}x] = 1$ and cubing the second term in the commutators gives the result.

LEMMA 6. *Relations (.4.) and (.5.) in Theorem* 1 *can be replaced by*

(.4.)* $(yx^{(n-1)/2}y^{-1}x^{-4})^2 = x^n$

(.5.)* $(xy^{(m+1)/2}x^{-1}y^4)^2 = y^m$

PROOF. To obtain the new relation to replace (.4.) start from (iii)

$$(y^{-m}x^3y^{-m}x^{(n-1)/2})^2 = 1$$
$$(y^{-m}x^3y^{-m-1}yx^{(n-1)/2})^2 = 1$$

$(y^{-2m-1}x^3yx^{(n-1)/2})^2 = 1$ since $3|(-m-1)$
Use $y^{-2m-1} = x^{1-n}y^{-1}x^{1-n}$ to get $(y^{-1}x^{4-n}yx^{(-n+1)/2})^2 = 1$ so
$yx^{(n-1)/2}y^{-1}x^{n-4}yx^{(n-1)/2}y^{-1}x^{n-4} = 1$. But $3|(n-1)/2$,
so using Lemma 5 we have $yx^{(n-1)/2}y^{-1}x^{-4}yx^{(n-1)/2}y^{-1}x^{-4-n} = 1$
giving $(yx^{(n-1)/2}y^{-1}x^{-4})^2 = x^n$.

To obtain the new relation to replace (.5.) start from (iv)
$(x^ny^3x^ny^{-(m+1)/2})^2 = 1$
$(x^{2n-1}y^3xy^{-(m+1)/2})^2 = 1$
$(x^{-n-1}y^3xy^{-(m+1)/2})^2 = 1$
Use $x^{-n-1} = y^{m+1}x^{-1}y^{m+1}$ to get $(x^{-1}y^{m+4}xy^{(m+1)/2})^2 = 1$ so
$xy^{(m+1)/2}x^{-1}y^{m+4}xy^{(m+1)/2}x^{-1}y^{m+4} = 1$.
But $3|(m+1)/2$, so using Lemma 5 we have $(xy^{(m+1)/2}x^{-1}y^4) = y^m$.

Hence replacing the relations (.4.) and (.5.) respectively by (.4.)* and (.5.)* in Theorem 1 the presentation for G will be as in the following corollary.

COROLLARY.

(I)	$x^{3n} = 1$	(IV)	$(yx^{(n-1)/2}y^{-1}x^{-4})^2 = x^n$
(II)	$y^{3m} = 1$	(V)	$(xy^{(m+1)/2}x^{-1}y^4)^2 = y^m$
(III)	$(xy^{-1})^2 = 1$	(VI)	$x^{-n} = (y^mx)^2$

The presentation given in Corollary is not efficient since $M(\mathrm{PSL}(2, Z_n) \times \mathrm{PSL}(2, Z_m)) = C_2 \times C_2$. The presentation given in Corollary has deficiency -4. However we conjecture:

CONJECTURE. For $n = 6k+1$ and $m = 6t-1$, $\mathrm{PSL}(2, Z_n) \times \mathrm{PSL}(2, Z_m)$ has the efficient presentation
$G = \langle x, y \mid x^{3n} = 1, (xy^{-1})^2(yx^{(n-1)/2}y^{-1}x^{-4})^{-2} = x^{-n},$
$(xy^{(m+1)/2}x^{-1}y^4)^2 = y^m, x^{-n} = (y^mx)^2\rangle$

(i) If $n \equiv 1(\mathrm{mod}6)$ and $m \equiv 1(\mathrm{mod}6)$ then replace m by $-m$ in the above presentation.

(ii) If $n \equiv -1(\mathrm{mod}6)$ and $m \equiv -1(\mathrm{mod}6)$ then replace n by $-n$ in the above presentation.

(iii) If $n \equiv -1(\mathrm{mod}6)$ and $m \equiv 1(\mathrm{mod}6)$ then replace n by $-n$ and m by $-m$ in the above presentation.

We have verified the conjecture for
(a) $n = 7, 13, 19, 25, 31, 37, 43, 49, 55$, and $m = 5$
(b) $n = 49$ and $m = 7$
(c) $n = 5, 11, 23, 29, 35, 41, 47, 65$ and $m = 5$
which for the cases $n = 25$, $m = 5$; $n = 55$, $m = 5$; $n = 49$, $n = 7$; $n = 35$, $m = 5$; $n = 65$, $m = 5$; the efficiency of G was previously not known.

(a); Here we will verify the conjecture for case ($n = 25$, $m = 5$) the other cases can be verified by using the same method therefore they are omitted. *Case* $n = 25$, $m = 5$: Since $25 \equiv 1(\text{mod}6)$ and $5 \equiv -1(\text{mod}6)$, in Corollary we have to replace n by 25 and m by 5. Using TC(A machine implementation of Todd-Coxeter.) on subgroup $\langle x \rangle$ it can be seen that the relation (II) in Corollary is redundant and combining relations (III) and (IV) as in the conjecture and again using TC on subgroup $\langle x \rangle$ it can be seen that the index of subgroup is 6000. So the presentation for G is efficient. We only need to verify that x has order 75. Adding respectively , $x^3 = 1$, $x^5 = 1$, $x^{15} = 1$, $x^{25} = 1$we get respectively indexes 20, 12, 240, 300 for subgroup $\langle x \rangle$. So the order of x is 75.

(b); *Case* $n = 49$, $m = 7$: Since $49 \equiv 1(\text{mod}6)$ and $7 \equiv 1(\text{mod}6)$, in Corollary we have to replace n by 49 and m by -7. Using TC on subgroup $\langle x \rangle$ it can be seen that the relation (II) in Corollary is redundant and combining relations (I) and (III) and again using TC on subgroup $\langle x \rangle$ it can be seen that index of subgroup is 65856. So the presentation for G is efficient. We only need to verify that x has order 147. Adding respectively $a^3 = 1$, $a^7 = 1$, $a^{21} = 1$, $a^{49} = 1$, $a^{147} = 1$, we get respectively indexes 56, 24, 1344, 1176, 65856 for subgroup $\langle x \rangle$. So the order of x is 147.

(c); Here we will verify the conjecture for case ($n = 35$, $m = 5$) the other cases can be verified by using the same method therefore they are omitted. *Case* $n = 35, m = 5$: Since $35 \equiv -1(\text{mod}6)$ and $5 \equiv -1(\text{mod}6)$, in Corollary we have to replace n by -35 and m by 5. Using TC on subgroup $\langle x \rangle$ it can be seen that the relation (II) in Corollary is redundant and combining relations (III) and (IV) as in the conjecture and again using TC on subgroup $\langle x \rangle$ it can be seen that the index of subgroup is 11520. So the presentation for G is efficient. We only need to verify that x has order 105. Adding respectively $x^3 = 1$, $x^5 = 1$, $x^7 = 1$, $x^{15} = 1$, $x^{21} = 1$, $x^{35} = 1$, we get respectively indexes 20, 12, 24, 240, 480, 576 for subgroup $\langle x \rangle$. So the order of x is 105.

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