MATH. SCAND. 80 (1997), 195--248

ON THE COHOMOLOGY RING OF THE FREE LOOP SPACE OF A WEDGE OF SPHERES

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Abstract.

In this paper we consider $H^*(\mathscr{L}X, k)$, the cohomology ring of the free loop space of X, when X is a wedge of spheres of the same dimensions (both even and odd), i.e., $X = S^{2n+1} \bigvee S^{2n+1} \cdots\cdots \bigvee S^{2n+1}$ or $X = S^{2n} \bigvee S^{2n} \cdots\cdots \bigvee S^{2n}$. We prove that in the odd-dimensional case, this ring is an algebra with an infinite number of generators and an infinite number of relations, which however has a very nice algebraic structure. It is the trivial extension of the ring consisting of the elements of degrees $(2n)k, k = 1, 2, \cdots$, by the module consisting of the elements of degrees $(2n+1)k, k = 1, 2, \cdots$. In the even-dimensional case, we prove that this ring is the trivial extension of the ring consisting of the elements of degrees $(2n-1)k, k = 1, 2, \cdots$, by the module consisting of the elements of degrees $(2n)k, k = 1, 2, \cdots$. We prove that $H^*(\mathscr{L}X, k)$(in low dimensions) is a Koszul algebra when $X = \bigvee_{i=1}^{2} S^d (d \geq 3$, odd), but it is not a Koszul algebra When $X = S^4 \bigvee S^4$. However we get strong indications that this algebra satisfies a condition M_3 that has been studied by Löfwall and Roos. We study the torsion of these cohomology rings with coefficients in Z and prove that in odd-dimensional case there is no torsion at all, whereas in even-dimensional case we have torsion. We prove that only 2-torsion is present in this case and determine the number of generators for the 2-torsion part. A general tool that we use is the Eilenberg-Moore spectral sequence

$$E^2_{-p,q} = \mathrm{Tor}_p^{H^*(X\times X,k)}(H^*(X,k), H^*(X;k))^q \Rightarrow H^n(\mathscr{L}X, k).$$

This spectral sequence degenerates if X is a formal space and k is a field of characteristic zero. It reduces our work to the calculation of the Hochschild homology $H_*(\Lambda, \Lambda) = \mathrm{Tor}_*^{\Lambda^e}(\Lambda, \Lambda)$. We also study another Eilenberg-Moore spectral sequence and find that it degenerates for $\mathscr{L}S^5$ but does not for $\mathscr{L}S^4$. This gives a clear indication that the ring structure of $H^*(\mathscr{L}X, k)$, should be more complicated when X is a wedge of even-dimensional spheres.

0. Introduction.

In this paper k is usually a field of characteristic 0, X is a CW–complex with a basepoint, $\mathscr{L}X$ is the free loop space of X, i.e., the space of continuous maps $S^1 \longrightarrow X$ with the compact open topology and ΩX is the loop space of X, basepoint preserving maps, $S^1 \longrightarrow X$. In the literature (cf. [20]) the Betti numbers of the free loop spaces have been studied.

If X is a finite CW-complex, then the "fiber homotopy pull-back diagram" (cf. e.g. [20] page 182):

Received May 5, 1995.

$$\begin{array}{ccc} \mathscr{L}X & \longrightarrow & X^I \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta} & X \times X \end{array},$$

where Δ is the diagonal map, $X^I = \{\phi : I = [0,1] \longrightarrow X\}$ and $X^I \longrightarrow X \times X$ is defined by $\phi \longrightarrow (\phi(0), \phi(1))$, gives rise to an Eilenberg–Moore spectral sequence in the second quadrant.

$$E_2^{-p,q} = \mathrm{Tor}_p^{H^*(X \times X, k)}(H^*(X,k), H^*(X,k))^q \Rightarrow H^n(\mathscr{L}X, k)$$

where

$$H^*(X \times X, k) \cong H^*(X,k) \otimes H^*(X,k).$$

If

$$X = \bigvee_{i=1}^{m} S^{n_i}$$

is a wedge of spheres, then this spectral sequence degenerates and, as a result of the following theorem due to D. Anick, we obtain

$$\coprod_{t \geq 0} \mathrm{Tor}_t^{H^*(X,k) \otimes H^*(X,k)}(H^*(X,k), H^*(X;k)^{t+n} \simeq H^n(\mathscr{L}X, k).$$

THEOREM 0.1. *Let k be a field of characteristic zero and let X be a formal space, then $H^*(\mathscr{L}X, k)$ is naturally bigraded and*

$$\coprod_{t \geq 0} \mathrm{Tor}_t^{H^*(X,k) \otimes H^*(X,k)}(H^*(X,k), H^*(X,k)^{t+n} \simeq H^n(\mathscr{L}X, k) \tag{0.1}$$

PROOF. (cf. [1], page 489).

In fact (0.1) is an isomorphism of rings (on the left hand side we have the Hochschild homology and on the right hand side the cohomoloy ring) when X is a wedge of spheres. See [12] for more information. In [20] the groups

$$\mathrm{Tor}_t^{H^*(X,k) \otimes H^*(X,k)}(H^*(X,k), H^*(X,k)) =$$

$$\frac{(H^+(X,k))^{\otimes t+1}}{\mathrm{Im}\, S_{t+1}} \coprod \mathrm{Ker}((H^+)^{\otimes t} \xrightarrow{S_t} (H^+)^{\otimes t})$$

is calculated when $H^*(X,k) = k + \bar{\Lambda}$ is concentrated in even degrees and where

$$S_n : \bar{\Lambda}^{\otimes n} \to \bar{\Lambda}^{\otimes n} \tag{0.2}$$

is defined by $S_n = 1 - s_n$ and $s_n : \bar{\Lambda}^{\otimes n} \to \bar{\Lambda}^{\otimes n}$ is defined by:

$$\lambda_1 \otimes \lambda_2 \otimes \ldots\ldots \otimes \lambda_n \longrightarrow (-1)^{n-1}\lambda_n \otimes \lambda_1 \otimes \ldots\ldots \otimes \lambda_{n-1}.$$

In this paper we calculate the groups

$$\mathrm{Tor}_t^{\mathrm{H}^*(\mathrm{X},k)\otimes \mathrm{H}^*(\mathrm{X},k)}(\mathrm{H}^*(\mathrm{X},k), \mathrm{H}^*(\mathrm{X},k)) =$$

$$\frac{(\mathrm{H}^+(\mathrm{X},k))^{\otimes t+1}}{\mathrm{Im}\ \mathrm{T}_{t+1}} \coprod \mathrm{Ker}((\mathrm{H}^+)^{\otimes t} \xrightarrow{\mathrm{T}_t} (\mathrm{H}^+)^{\otimes t}),$$

when $\mathrm{H}^*(\mathrm{X},k) = k + \bar{\Lambda}$ is concentrated in odd degrees and where

$$\mathrm{T}_n : \bar{\Lambda}^{\otimes n} \to \bar{\Lambda}^{\otimes n} \tag{0.3}$$

is defined by $\mathrm{T}_n = 1 - t_n$ and $t_n : \bar{\Lambda}^{\otimes n} \to \bar{\Lambda}^{\otimes n}$ is defined by

$$\lambda_1 \otimes \lambda_2 \otimes \ldots\ldots \otimes \lambda_n \longrightarrow \lambda_n \otimes \lambda_1 \otimes \ldots\ldots \otimes \lambda_{n-1}.$$

Using this, we give an explicit formula for $\mathrm{H}^*(\mathscr{L}\mathrm{X},k)$, when $\mathrm{X} = \bigvee_{i=1}^m \mathrm{S}^d (d \geq 3$, odd, and $m \geq 2)$, see Table 2.1.1 below. We prove that the ring $\mathrm{H}^*(\mathscr{L}\mathrm{X},k)$ is the trivial extension of $\mathrm{H}^{2*}(\mathscr{L}\mathrm{X},k)$ by the module

$$\mathrm{H}^{\mathrm{odd}}(\mathscr{L}\mathrm{X},k) = s^{-1}\overline{\mathrm{H}^{2*}(\mathscr{L}\mathrm{X},k)}$$

in the case when $\mathrm{X} = \mathrm{S}^{d_1} \bigvee \mathrm{S}^{d_2} \bigvee \cdots\cdots \mathrm{S}^{d_n} (d_i$, odd) and that the ring $\mathrm{H}^*(\mathscr{L}(\mathrm{S}^4 \bigvee \mathrm{S}^4),k)$ is the trivial extension of $\mathrm{H}^{3*}(\mathscr{L}(\mathrm{S}^4 \bigvee \mathrm{S}^4),k)$ by the module

$$\mathrm{H}^{3*+1}(\mathscr{L}(\mathrm{S}^4 \bigvee \mathrm{S}^4),k) = s^{-1}\overline{\mathrm{H}^{3*}(\mathscr{L}(\mathrm{S}^4 \bigvee \mathrm{S}^4),k)}.$$

Moreover we prove that $\mathrm{H}^*(\mathscr{L}\mathrm{X},k)$ is isomorphic in low dimensions to a Koszul algebra, i.e., $\mathrm{Tor}_{p,q}^R(k,k) = 0$ for $p \neq q$ in the case $\mathrm{X} = \bigvee_{i=1}^2 \mathrm{S}^d (d \geq 3$, odd). Note that the even dimensional subalgebra, i.e., $\mathrm{H}^{2*}(\mathscr{L}\mathrm{X},k)$ is not free (see [12]).

In section 3, we show that the ring $\mathrm{H}^*(\mathscr{L}(\mathrm{S}^4 \bigvee \mathrm{S}^4),k)$ in low dimensions is not a Koszul algebra, but that

$$\mathrm{Ext}_{\mathrm{H}^*(\mathscr{L}(S^4 \bigvee S^4),k)}(k,k)$$

has a very nice form. We also study the torsion of these cohomology rings with coefficients in Z and prove that in the *odd*-dimensional case there is no torsion at all (section 2.1 below), whereas in the *even*-dimensional case we have only 2-torsion (section 3.1 below). We also determine the number of generators for the 2-torsion part.

In section four we consider the degeneration of the Eilenberg-Moore spectral sequence in some special cases as follows.

Let X be a finite simply connected CW–complex with a basepoint x_0, PX the space of paths in X starting in x_0 and $PX \xrightarrow{\pi} X$ the map that to each path associates its endpoint. We have a pull-back diagram

$$\begin{array}{ccc} \Omega X & \longrightarrow & PX \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \{x_0\} & \longrightarrow & X \end{array}$$

This diagram gives rise to another Eilenberg-Moore spectral sequence:

$$\text{(0.4)} \qquad E^2_{-p,q} = \operatorname{Ext}^p_{H^*(X,k)}(k,k)_q \Longrightarrow gr(H_*(\Omega X, k))$$

THEOREM 0.2. ([21] *page* 25). *The Eilenberg–Moore spectral sequence* (0.4) *degenerates if* X *is a finite, simply connected CW–complex with* $\dim X \leq 4$.

Whenever this spectral sequence degenerates

$$\dim_k(H_n(\Omega X, k)) = \sum_{p \geq 0} \dim_k \operatorname{Ext}^p_{H^*(X,k)}(k,k)_{p+n}$$

where the sum is finite.

In section four we replace X in (0.4) by $\mathscr{L}X$ the free loop space of X and prove that it degenerates in the case $X = \mathbf{S}^5$ but it does not if $X = \mathbf{S}^4$.

I wish to thank J.-E. Roos who called my attention to this problem. I am grateful to him for his great help and good advice during the work on this paper that encouraged me to continue. I also wish to thank J. Backelin, R. Fröberg, L. Lambe and C. Löfwall for stimulating discussions.

1. Algebraic preliminaries.

1.1. *The normalized standard free resolution.*

Let $\Lambda = k \oplus \bar{\Lambda}$ be an associative k–algebra with unit (non-graded). Moreover assume that the product is zero in $\bar{\Lambda}$. Then

$$H_*(\Lambda, \Lambda) = \operatorname{Tor}^{\Lambda^e}_*(\Lambda, \Lambda)$$

is the homology of the complex

$$\text{(1.1.1)} \qquad \cdots\cdots \longrightarrow \Lambda \otimes_k \bar{\Lambda}^{\otimes n+1} \longrightarrow \Lambda \otimes_k \bar{\Lambda}^{\otimes n} \xrightarrow{d_n} \Lambda \otimes_k \bar{\Lambda}^{\otimes n-1} \longrightarrow \cdots\cdots$$

where

$$\text{(1.1.2)} \qquad d_n(\lambda \otimes [\lambda_1 \otimes \cdots\cdots \otimes \lambda_n]) = \lambda\lambda_1 \otimes [\lambda_2 \otimes \cdots\cdots \otimes \lambda_n] + (-1)^n \lambda_n \lambda \otimes [\lambda_1 \otimes \cdots\cdots \otimes \lambda_{n-1}].$$

(See [20] page 178).

Now assume that Λ is a graded algebra which is connected, that is, the unit $\eta : k \longrightarrow \Lambda$ is an isomorphism in degree 0. Denote the cokernel of η by $\bar{\Lambda}$, since Λ is connected we have

$$\bar{\Lambda} = \Lambda^+ = \{\gamma \in \Lambda \mid \deg \gamma > 0\}.$$

Recall (cf. [18] page 228) the bar construction.

$$B^{-n}(\Lambda, N) = \Lambda \otimes_k \bar{\Lambda} \otimes_k \cdots \otimes_k \bar{\Lambda} \otimes_k N.$$

Notice that $B^{-n}(\Lambda, N)$ is a left Λ–module with the extended module action. It is customary to write an element of $B^{-n}(\Lambda, N)$ as $\gamma \otimes [\gamma_1 \otimes \gamma_2 \otimes \ldots\ldots \otimes \gamma_n] \otimes a$ and of $B^0(\Lambda, N)$ as $\gamma \otimes [\quad] \otimes a$. If we write $\bar{\alpha} = (-1)^{1+\deg \alpha}\alpha$ for a homogeneous element, then we can assemble the $B^{-n}(\Lambda, N)$ into a resolution by introducing an external differential

$$\delta : B^{-n}(\Lambda, N) \longrightarrow B^{-n+1}(\Lambda, N)$$

where

$$\begin{aligned}\delta(\gamma[\gamma_1|\gamma_2|\cdots|\gamma_n]a) &= (-1)^{\deg \gamma}(\gamma\gamma_1)[\gamma_2|\cdots|\gamma_n]a \\ &+ \sum_{i=1}^{n-1}(-1)^{\deg \gamma}\gamma[\bar{\gamma}_1|\bar{\gamma}_2|\cdots|\bar{\gamma}_{i-1}|\bar{\gamma}_i \cdot \gamma_{i+1}|\cdots|\gamma_n]a \\ &+ (-1)^{\deg \gamma}\gamma[\bar{\gamma}_1|\cdots|\bar{\gamma}_{n-1}](\gamma_n a)\end{aligned}$$

Now in the case of product zero we obtain:

$$\begin{aligned}(1.1.3)\qquad d_n\lambda \otimes [\lambda_1 \otimes \ldots\ldots \otimes \lambda_n] &= (-1)^{\deg \lambda}\lambda\lambda_1 \otimes [\lambda_2 \otimes \ldots\ldots \otimes \lambda_n] \\ &+ (-1)^{\deg \lambda + \deg \lambda_1 + \cdots\cdots + \deg \lambda_{n-1} + n-1}(-1)^n\lambda_n\lambda \otimes [\lambda_1 \otimes \ldots\ldots \otimes \lambda_{n-1}].\end{aligned}$$

Notice that in (1.1.3) if λ_i's are concentrated in even degrees, then we have the same formula as in the non-graded case. But in this paper we also need to consider the case when λ_i's are concentrated in odd degrees. If this is so, then

$$(-1)^{\deg \lambda + \deg \lambda_1 + \cdots\cdots + \deg \lambda_{n-1} + n-1}(-1)^n = -1, \qquad \forall \qquad n \geq 0.$$

We have assumed λ to be in degree 0, i.e., $\deg \lambda = 0$. Hence (1.1.3) can be written as

$$\begin{aligned}(1.1.4)\qquad d_n[\lambda_1 \otimes \ldots\ldots \otimes \lambda_n] &= \lambda_1 \otimes [\lambda_2 \otimes \ldots\ldots \otimes \lambda_n] \\ &- \lambda_n \otimes [\lambda_1 \otimes \ldots\ldots \otimes \lambda_{n-1}].\end{aligned}$$

In order to calculate

$$\mathrm{Tor}_t^{\mathrm{H}^*(\mathrm{X},k) \otimes \mathrm{H}^*(\mathrm{X},k)}(\mathrm{H}^*(\mathrm{X},k), \mathrm{H}^*(\mathrm{X},k))$$

when $H^*(X,k)/k$ is concentrated in odd degrees, let $H^*(X,k) = \Lambda = k \oplus \bar{\Lambda}$ and use the isomorphism

$$\Lambda \otimes_k \bar{\Lambda}^{\otimes n} \longleftrightarrow \bar{\Lambda}^{\otimes n+1} \oplus 1 \otimes \bar{\Lambda}^{\otimes n}. \tag{1.1.5}$$

Moreover define

$$T_n : \bar{\Lambda}^{\otimes n} \to \bar{\Lambda}^{\otimes n} \tag{1.1.6}$$

by $T_n = 1 - t_n$, where $t_n : \bar{\Lambda}^{\otimes n} \to \bar{\Lambda}^{\otimes n}$ is defined by

$$\lambda_1 \otimes \lambda_2 \otimes \ldots\ldots \otimes \lambda_n \longrightarrow \lambda_n \otimes \lambda_1 \otimes \ldots\ldots \otimes \lambda_{n-1}.$$

Now the resolution (1.1.1) can be written as:

$$\begin{aligned} \ldots\ldots \longrightarrow \bar{\Lambda}^{\otimes n+2} + 1 \otimes \bar{\Lambda}^{\otimes n+1} \; {}_{d_{n+1}} \longrightarrow \bar{\Lambda}^{\otimes n+1} \\ + 1 \otimes \bar{\Lambda}^{\otimes n} \; {}_{d_n} \longrightarrow \bar{\Lambda}^{\otimes n} + 1 \otimes \bar{\Lambda}^{\otimes n-1} \longrightarrow \ldots\ldots \end{aligned} \tag{1.1.7}$$

where the map d_n has the simple form:

$$\begin{aligned} d_n[(v_0 \otimes v_1 \otimes \ldots\ldots \otimes v_n) + (1 \otimes w_1 \otimes \ldots\ldots \otimes w_n)] &= (w_1 \otimes w_2 \otimes \ldots\ldots \otimes w_n) \\ &\quad - (w_n \otimes w_1 \otimes \ldots\ldots \otimes w_{n-1}) \\ &= T_n(w_1 \otimes w_2 \otimes \ldots\ldots \otimes w_n). \end{aligned} \tag{1.1.8}$$

Therefore

$$\operatorname{Ker} d_n = \bar{\Lambda}^{\otimes n+1} + 1 \otimes \operatorname{Ker} T_n$$

and

$$\operatorname{Im} d_{n+1} = \operatorname{Im} T_{n+1}$$

and hence

$$\begin{aligned} &\operatorname{Tor}_t^{H^*(X,k)\otimes H^*(X,k)}(H^*(X,k), H^*(X,k)) \\ &= \frac{(H^+(X,k))^{\otimes t+1}}{\operatorname{Im} T_{t+1}} \coprod \operatorname{Ker}((H^+)^{\otimes t} \xrightarrow{T_t} (H^+)^{\otimes t}). \end{aligned} \tag{1.1.9}$$

DEFINITION 1.1.1. (The shuffle product in the graded case cf. [18]). Let $\gamma_{p,q}$ denote the group of permutations of the set $\{1, 2, \ldots\ldots, p+q\}$. If $\sigma \in \gamma_{p,q}$, then σ is a (p,q) *shuffle* if the following holds

$$\sigma(1) < \sigma(2) < \cdots\cdots < \sigma(p) \quad \text{and}$$
$$\sigma(p+1) < \sigma(p+2) < \cdots\cdots < \sigma(p+q).$$

Define the shuffle product $(*)$ on the level of the standard free resolution (1.1.1) as follows:

Suppose

$$\lambda \otimes [a_1 \otimes a_2 \otimes \ldots\ldots \otimes a_p] \in \Lambda \otimes \bar{\Lambda}^{\otimes p}$$

and

$$\mu \otimes [b_1 \otimes b_2 \otimes \ldots\ldots \otimes b_q] \in \Lambda \otimes \bar{\Lambda}^{\otimes q},$$

then

$$\lambda \otimes [a_1 \otimes a_2 \otimes \ldots\ldots \otimes a_p] * \mu \otimes [b_1 \otimes b_2 \otimes \ldots\ldots \otimes b_q]$$
$$= \sum_{(p,q)-\text{shuffles}(\sigma)} (-1)^{s(\sigma)} \lambda\mu \otimes [c_{\sigma^{-1}(1)} \otimes c_{\sigma^{-1}(2)} \otimes \ldots\ldots \otimes c_{\sigma^{-1}(p+q)}]$$

where $c_{\sigma(i)} = a_{\sigma(i)}$ if $1 \leq \sigma(i) \leq p$, $c_{\sigma(i)} = b_{\sigma(i)-p}$ if $p+1 \leq \sigma(i) \leq p+q$ and

$$s(\sigma) = \sum (\deg c_i + 1)(\deg c_{p+j} + 1)$$

summed over all pairs $(i, p+j)$ with $\sigma(i) > \sigma(p+j)$. This sign reflects the convention that ± 1 is introduced when elements are switched past each other according to their total degrees. Here Λ is a graded algebra and

$$\bar{\Lambda} = \Lambda^+ = \{\gamma \in \Lambda \mid \deg \gamma > 0\}.$$

We study the shuffle product, since this gives the product structure on

$$\mathrm{Tor}_t^{\mathrm{H}^*(\mathrm{X},k) \otimes \mathrm{H}^*(\mathrm{X},k)}(\mathrm{H}^*(\mathrm{X},k), \mathrm{H}^*(\mathrm{X},k)),$$

and hence on $\mathrm{H}^*(\mathscr{L}\mathrm{X}, k)$.

1.2. *Invariant subspaces of Hochschild homology.*

Let k be a commutative field, V a finite dimensional vector space over k and $\Lambda = k \oplus V$ the trivial extension of k by V. In other words Λ consist of the set of pairs $(\kappa, v), \kappa \in k, v \in V$ with pairwise addition and multiplication

$$(k_1, v_1) \cdot (k_2, v_2) = (k_1 k_2, k_1 v_2 + k_2 v_1).$$

Let $\Lambda^e = \Lambda \otimes_k \Lambda^o$, where Λ^o is the oposite ring (here $\Lambda^o = \Lambda$), and let Λ be considered as Λ^e module in the natural way. In [20], the Hochschild homology of trivial ring extension is calculated as follows:

$$\text{(1.2.1)} \qquad \mathrm{H}_n(\Lambda, \Lambda) = \mathrm{Tor}_n^{\Lambda^e}(\Lambda, \Lambda) = \frac{V^{\otimes n+1}}{\mathrm{Im}\, \mathrm{S}_{n+1}} \coprod \mathrm{Ker}(\mathrm{V}^{\otimes n} \xrightarrow{\mathrm{S}_n} \mathrm{V}^{\otimes n}),$$

where

(1.2.2) $$S_n = 1 - s_n \text{ and } s_n(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = (-1)^{n-1} v_n \otimes v_1 \otimes \cdots \otimes v_{n-1}.$$

Now $\operatorname{Ker} S_n$ in (1.2.1) is exactly the vector space $(V^{\otimes n})^{\frac{Z}{nZ}}$ of invariants in $V^{\otimes n}$ for the group Z/nZ acting through its generator $s_n = \bar{1}$ and hence

$$\operatorname{Ker} S_n = (V^{\otimes n})^{\frac{Z}{nZ}} = \frac{1 + s_n + s_n^2 + \cdots + s_n^{n-1}}{n} V^{\otimes n}.$$

In a similar way, $\operatorname{Coker} S_{n+1}$ in (1.2.1) is the space of coinvariants for the analogous action of $Z/(n+1)Z$. But the exact sequence

(1.2.3) $$0 \to \operatorname{Ker} S_n \to V^{\otimes n} \xrightarrow{S_n} V^{\otimes n} \longrightarrow \operatorname{Coker} S_n \to 0 (\text{cf. } [20])$$

shows that $\operatorname{Ker} S_n$ and $\operatorname{Coker} S_n$ always have the same dimension over k. Therefore it follows from (1.2.1) and the preceding discussion that

$$|\operatorname{Tor}_*^{\Lambda^e}(\Lambda, \Lambda)| = |(V^{\otimes n})^{\frac{Z}{nZ}}| + |(V^{\otimes(n+1)})^{\frac{Z}{(n+1)Z}}| \text{ for } n \geq 1.$$

Using the endomorphism

$$\mathrm{N} = \frac{1 + s_n + s_n^2 + \cdots + s_n^{n-1}}{n}$$

of $V^{\otimes n}$ as a projection of $V^{\otimes n}$ onto $(V^{\otimes n})^{\frac{Z}{nZ}}$, J.-E. Roos (cf. [20]) found the dimension of the invariant subspace by the following formula, over a field of characteristic 0:

$$|(V^{\otimes n})^{\frac{Z}{nZ}}| = \operatorname{trace}\mathrm{N} = \frac{1}{n}\sum_{i=0}^{n-1} \operatorname{trace}(s_n)^i$$

where $(s_n)^0$ is the identity on $V^{\otimes n}$.

In this part we try to find the dimension of the invariant and coinvariant subspaces over a field of arbitrary characteristic.

THEOREM 1.2.1. *Assume the group* $\frac{Z}{nZ}$ *acts through its generator* $t_n = \bar{1}$ *on* $V^{\otimes n}$ *by*

$$t_n(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_n \otimes v_1 \otimes \cdots \otimes v_{n-1}.$$

Then the dimension of the invariant subspace is independent of the characteristic of the field, moreover

$$|(V^{\otimes n})^{\frac{Z}{nZ}}| = \frac{1}{n}\sum_{i=0}^{n} |V|^{(i,n)}$$

where $(V^{\otimes n})^{\frac{Z}{nZ}}$ *is the invariant subspace under the action of* t_n *and* $||$ *is the dimension as a vector space over* k.

PROOF. Let $|V| = m$, $|V^{\otimes n}| = m^n = l$ and let

$$\{\alpha_1, \alpha_2, \dots\dots, \alpha_l\}$$

be a basis for $V^{\otimes n}$. If the characteristic of k is zero, then the distinct elements of

$$\left\{\frac{1 + t_n + t_n^2 + \cdots + t_n^{n-1}}{n}(\alpha_i) \mid \alpha_i \in V^{\otimes n}\right\} \tag{1.2.4}$$

form a basis for

$$\text{Ker } \mathrm{T}_n = \text{Ker}(1 - t_n) = (V^{\otimes n})^{\frac{Z}{nZ}}.$$

We denote this basis by

$$\begin{aligned} M =& \left\{\frac{1 + t_n + t_n^2 + \cdots + t_n^{n-1}}{n}(\alpha_{i1}),\right. \\ & \frac{1 + t_n + t_n^2 + \cdots + t_n^{n-1}}{n}(\alpha_{i2}), \dots\dots \\ & \left.\dots\dots, \frac{1 + t_n + t_n^2 + \cdots + t_n^{n-1}}{n}(\alpha_{im})\right\} \end{aligned} \tag{1.2.5}$$

Note that

$$\frac{1 + t_n + t_n^2 + \cdots + t_n^{n-1}}{n}(\alpha_i) = \frac{1 + t_n + t_n^2 + \cdots + t_n^{n-1}}{n}(\alpha_j) \tag{1.2.6}$$

if and only if $\alpha_i = t_n^k \alpha_j$ for some $1 \leq k \leq n$. By eliminating the denominator n we get the basis

$$\begin{aligned} M_1 =& \{(1 + t_n + t_n^2 + \cdots + t_n^{n-1})(\alpha_{i1}), (1 + t_n + t_n^2 + \cdots + t_n^{n-1})(\alpha_{i2}), \dots \\ & \dots, (1 + t_n + t_n^2 + \cdots + t_n^{n-1})(\alpha_{im})\}. \end{aligned} \tag{1.2.7}$$

For any $1 \leq j \leq m$let r_j be the least integer $1 \leq r_j \leq n$ such that $t_n^{r_j}(\alpha_{ij}) = \alpha_{ij}$, then

$$(1 + t_n + t_n^2 + \cdots + t_n^{n-1})(\alpha_{ij}) = \frac{n}{r_j}(1 + t_n + t_n^2 + \cdots + t_n^{r_j - 1})(\alpha_{ij}).$$

Now put

$$(1.2.8)\qquad M' = \{\beta_{i1} = (1 + t_n + t_n^2 + \cdots + t_n^{r_1-1})(\alpha_{i1}),$$
$$\beta_{i2} = (1 + t_n + t_n^2 + \cdots + t_n^{r_2-1})(\alpha_{i2}), \cdots$$
$$\ldots\ldots, \beta_{im} = (1 + t_n + t_n^2 + \cdots + t_n^{r_m-1})(\alpha_{im})\}.$$

We prove that M' is a basis for the invariant subspace regardless of the characteristic of k. Clearly M' is a linearly independent set, since the elements of M' are different with coefficients one. To prove that M' generates the invariant subspace, let α be any invariant element in $V^{\otimes n}$. Arrange the basis

$$\{\alpha_1, \alpha_2, ..., \alpha_l\}$$

of $V^{\otimes n}$ as

$$M_2 = \{\alpha_{i1}, t(\alpha_{i1}), t^2(\alpha_{i1}), \ldots\ldots, t^{r_1-1}(\alpha_{i1}), \alpha_{i2}, t(\alpha_{i2}), t^2(\alpha_{i2}), \ldots\ldots,$$
$$t^{r_2-1}(\alpha_{i2}), \ldots\ldots, \alpha_{im}, t(\alpha_{im}), t^2(\alpha_{im}), \ldots\ldots, t^{r_m-1}(\alpha_{im})\}$$

In M_2 we replaced t_n by t for simplicity. As a linear combination of elements of M_2, let

$$(1.2.9)\quad \alpha = b_{i1}^0\alpha_{i1} + b_{i1}^1 t(\alpha_{i1}) + b_{i1}^2 t^2(\alpha_{i1}) + \cdots\cdots + b_{i1}^{r_1-1} t^{r_1-1}(\alpha_{i1})$$
$$+ b_{i2}^0\alpha_{i2} + b_{i2}^1 t(\alpha_{i2}) + b_{i2}^2 t^2(\alpha_{i2}) + \cdots\cdots + b_{i2}^{r_2-1} t^{r_2-1}(\alpha_{i2}) + \cdots\cdots$$
$$+ b_{im}^0\alpha_{im} + b_{im}^1 t(\alpha_{im}) + b_{im}^2 t^2(\alpha_{im}) + \cdots\cdots + b_{im}^{r_m-1} t^{r_m-1}(\alpha_{im})$$

Now applying t to α, we obtain:

$$(1.2.10)\, t(\alpha) = b_{i1}^0 t(\alpha_{i1}) + b_{i1}^1 t^2(\alpha_{i1}) + \cdots\cdots + b_{i1}^{r_1-2} t^{r_1-1}(\alpha_{i1}) + b_{i1}^{r_1-1}(\alpha_{i1})$$
$$+ b_{i2}^0 t(\alpha_{i2}) + b_{i2}^1 t^2(\alpha_{i2}) + \cdots\cdots + b_{i2}^{r_2-2} t^{r_2-1}(\alpha_{i2}) + b_{i2}^{r_2-1}(\alpha_{i2}) + \cdots$$
$$+ b_{im}^0 t(\alpha_{im}) + b_{im}^1 t^2(\alpha_{im}) + \cdots\cdots + b_{im}^{r_m-2} t^{r_m-1}(\alpha_{im}) + b_{im}^{r_m-1}(\alpha_{im})$$

An element $\alpha \in V^{\otimes n}$ is invariant if and only if

$$(1.2.11)\qquad t(\alpha) = \alpha.$$

It follows from substituting (1.2.9) and (1.2.10) into (1.2.11) that

$$b_{i1}^0 = b_{i1}^1 = b_{i1}^2 = \cdots\cdots = b_{i1}^{r_1-2} = b_{i1}^{r_1-1} = b_{i1}^0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$b_{ij}^0 = b_{ij}^1 = b_{ij}^2 = \cdots\cdots = b_{ij}^{r_j-2} = b_{ij}^{r_j-1} = b_{ij}^0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$b_{im}^0 = b_{im}^1 = b_{im}^2 = \cdots\cdots = b_{im}^{r_m-2} = b_{im}^{r_m-1} = b_{im}^0$$

and hence

$$\alpha = b_{i1}^0 \beta_{i1} + b_{i2}^0 \beta_{i2} + \cdots\cdots + b_{im}^0 \beta_{im}.$$

This shows that α is a linear combination of elements of M'. This proves the first part of the theorem.

For the second part, later in section 2 in Theorem 2.1.1, we prove that

$$|(V^{\otimes n})^{\frac{Z}{nZ}}| = \frac{1}{n}\sum_{i=0}^{n} |V|^{(i,n)} \tag{1.2.12}$$

when the characteristic of k is zero. Now the result follows by the first part of the theorem.

THEOREM 1.2.2. *Assume the group* $\frac{Z}{nZ}$ *acts through its generator* $s_n = \bar{1}$ *on* $V^{\otimes n}$ *by*

$$s_n(v_1 \otimes v_2 \otimes \cdots\cdots \otimes v_n) = (-1)^{n-1} v_n \otimes v_1 \otimes \cdots\cdots \otimes v_{n-1}.$$

Then the dimension of the invariant subspace is independent of the characteristic of the field if the characteristic is different from 2. If this is the case, then

$$|(V^{\otimes n})^{\frac{Z}{nZ}}| = \frac{1}{n}\sum_{i=0}^{n} (-1)^{i(n-1)} |V|^{(i,n)}$$

where $(V^{\otimes n})^{\frac{Z}{nZ}}$ *is the invariant subspace under the action of* s_n. *In characteristic 2, the dimension of the invariant subspace can be calculated as in Theorem 1.2.1, i.e.,*

$$|(V^{\otimes n})^{\frac{Z}{nZ}}| = \frac{1}{n}\sum_{i=0}^{n} |V|^{(i,n)}.$$

PROOF. If characteristic of k is 2, then $-1 = +1$ and hence:

$$\begin{aligned} s_n(v_1 \otimes v_2 \otimes \cdots\cdots \otimes v_n) &= (-1)^{n-1}(v_n \otimes v_1 \otimes \cdots\cdots \otimes v_{n-1}) \\ &= (v_n \otimes v_1 \otimes \cdots\cdots \otimes v_{n-1}) = t_n(v_1 \otimes v_2 \otimes \cdots\cdots \otimes v_n). \end{aligned}$$

This proves the second part of the theorem. The proof of the first part is almost the same as in Theorem 1.2.1. If n is odd then:

$$s_n(v_1 \otimes v_2 \otimes \cdots\cdots \otimes v_n) = t_n(v_1 \otimes v_2 \otimes \cdots\cdots \otimes v_n)$$

and hence the dimension of the invariant subspace is independent of the characteristic of the field as we proved in Theorem 1.2.1. Consider the case that n is even, i.e.,

$$s_n(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = -(v_n \otimes v_1 \otimes \cdots \otimes v_{n-1})$$

As in Theorem 1.2.1, let $|V| = m, |V^{\otimes n}| = m^n = l$ and let

$$\{\alpha_1, \alpha_2, \ldots\ldots, \alpha_l\}$$

be a basis for $V^{\otimes n}$ over a field k of characteristic zero. Consider the set

$$G = \{(1 + s + s^2 + \cdots + s^{n-1})(\alpha_i) | \alpha_i \quad \text{is a basis element of} \quad V^{\otimes n}\} \tag{1.2.13}$$

(we have replaced s_n by s for simplicity). For any $1 \leq i \leq l$ let k_i be the least integer $1 \leq k_i \leq n$such that $s^{k_i}(\alpha_i) = \pm\alpha_i$. If $s^{k_i}(\alpha_i) = -\alpha_i$, i.e., if k_i is odd, then

$$(1 + s + s^2 + \cdots + s^{n-1})(\alpha_i) = 0$$

and if $s^{k_i}(\alpha_i) = \alpha_i$, i.e., if k_i is even, then

$$(1 + s + s^2 + \cdots + s^{n-1})(\alpha_i) = \frac{n}{k_i}(1 + s + s^2 + \cdots + s^{k_i-1})(\alpha_i).$$

Now let G' be the set of all distinct non-zero elements of G. Assume

$$\begin{aligned} G' =\{&(1 + s + s^2 + \cdots + s^{n-1})(\alpha_{i1}), (1 + s + s^2 + \cdots + s^{n-1})(\alpha_{i2}), \\ &\ldots\ldots, (1 + s + s^2 + \cdots + s^{n-1})(\alpha_{iq})\}. \end{aligned} \tag{1.2.14}$$

To prove that G' is a basis for the invariant subspace is easy and is left to the reader. For any $1 \leq j \leq q$ let u_j be the least integer $1 \leq u_j \leq n$ such that $s^{u_j}(\alpha_{ij}) = \alpha_{ij}$, then

$$(1 + s + s^2 + \cdots + s^{n-1})(\alpha_{ij}) = \frac{n}{u_j}(1 + s + s^2 + \cdots + s^{u_j-1})(\alpha_{ij}).$$

Put

$$\begin{aligned} G'' =\{&\beta_{i1} = (1 + s + s^2 + \cdots + s^{u_1-1})(\alpha_{i1}), \\ &\beta_{i2} = (1 + s + s^2 + \cdots + s^{u_2-1})(\alpha_{i2}), \\ &\ldots\ldots, \beta_{iq} = (1 + s + s^2 + \cdots + s^{u_q-1})(\alpha_{iq})\}. \end{aligned} \tag{1.2.15}$$

We prove that G'' is a basis for the invariant subspace over a field k_p of characteristic $p(p \neq 2)$ and hence the dimension of the invariant subspace is independent of the characteristic if the characteristic is not 2. Clearly G'' is linearly independent over k_p, since it contains distinct elements with coefficient 1. It remains to prove that it generates the invariant subspace. First let us construct an special basis of $V^{\otimes n}$. Consider the basis $\{\alpha_1, \alpha_2, \ldots\ldots, \alpha_l\}$ of $V^{\otimes n}$ and put

(1.2.16) $H = \{(1 + s + s^2 + \cdots + s^{k_i - 1})(\alpha_i) | \alpha_i \text{ is a basis element of } V^{\otimes n}\}$.

Recall that for any $1 \leq i \leq l, k_i$ is the least integer, $1 \leq k_i \leq n$ such that $s^{k_i}(\alpha_i) = \pm\alpha_i$. If k_i is odd then

$$(1 + s + s^2 + \cdots + s^{n-1})(\alpha_i) = 0$$

and hence the element $(1 + s + s^2 + \cdots + s^{k_i - 1})(\alpha_i)$ of H in (1.2.16) does not appear in G'' in (1.2.15). But if k_i is even then the element $(1 + s + s^2 + \cdots + s^{k_i - 1})(\alpha_i)$ lies in both H in (1.2.16) and G'' in (1.2.15). So G'' is a subset of H. Rewrite the set H by arranging the elements as follows:

(1.2.17)
$$\begin{aligned} H = \{&\beta_{i1} = (1 + s + s^2 + \cdots + s^{u_1 - 1})(\alpha_{i1}),\\ &\beta_{i2} = (1 + s + s^2 + \cdots + s^{u_2 - 1})(\alpha_{i2}), \ldots\ldots,\\ &\beta_{iq} = (1 + s + s^2 + \cdots + s^{u_q - 1})(\alpha_{iq}),\\ &\beta_{i(q+1)} = (1 + s + s^2 + \cdots + s^{u_{(q+1)} - 1})(\alpha_{i(q+1)}),\\ &\beta_{i(q+2)} = (1 + s + s^2 + \cdots + s^{u_{(q+2)} - 1})(\alpha_{i(q+2)}), \ldots,\\ &\beta_{i(q+w)} = (1 + s + s^2 + \cdots + s^{u_{(q+w)} - 1})(\alpha_{i(q+w)})\} \end{aligned}$$

where $u_1, u_2, \ldots\ldots, u_q$ are even and $u_{q+1}, u_{q+2}, \ldots\ldots, u_{q+w}$ are odd. Now an special basis for $V^{\otimes n}$ is:

(1.2.18)
$$\begin{aligned} G_1 = \{&\alpha_{i1}, -s(\alpha_{i1}), s^2(\alpha_{i1}), \ldots\ldots, s^{u_1 - 2}(\alpha_{i1}), -s^{u_1 - 1}(\alpha_{i1}),\\ &\alpha_{i2}, -s(\alpha_{i2}), s^2(\alpha_{i2}), \ldots\ldots, s^{u_2 - 2}(\alpha_{i2}), -s^{u_2 - 1}(\alpha_{i2}), \ldots\ldots,\\ &\alpha_{iq}, -s(\alpha_{iq}), s^2(\alpha_{iq}), \ldots\ldots, s^{u_q - 2}(\alpha_{iq}), -s^{u_q - 1}(\alpha_{iq}),\\ &\alpha_{i(q+1)}, -s(\alpha_{i(q+1)}), s^2(\alpha_{i(q+1)}), \ldots\ldots, -s^{u_{(q+1)} - 2}(\alpha_{i(q+1)}),\\ &s^{u_{(q+1)} - 1}(\alpha_{i(q+1)}),\\ &\alpha_{i(q+2)}, -s(\alpha_{i(q+2)}), s^2(\alpha_{i(q+2)}), \ldots\ldots, -s^{u_{(q+2)} - 2}(\alpha_{i(q+2)}),\\ &s^{u_{(q+2)} - 1}(\alpha_{i(q+2)}), \ldots\ldots,\\ &\alpha_{i(q+w)}, -s(\alpha_{i(q+w)}), s^2(\alpha_{i(q+w)}), \ldots\ldots, -s^{u_{(q+w)} - 2}(\alpha_{i(q+w)}),\\ &s^{u_{(q+w)} - 1}(\alpha_{i(q+w)})\}. \end{aligned}$$

Now let α be an invariant element. As an element of $V^{\otimes n}, \alpha$ is a linear combination of elements of G_1 in (1.2.18). Let

$$
\begin{aligned}
(1.2.19)\qquad \alpha = {} & c_{i1}^0\alpha_{i1} - c_{i1}^1 s(\alpha_{i1}) + c_{i1}^2 s^2(\alpha_{i1}) + \cdots\cdots + c_{i1}^{u_1-2} s^{u_1-2}(\alpha_{i1}) \\
& - c_{i1}^{u_1-1} s^{u_1-1}(\alpha_{i1}) \\
& + c_{i2}^0\alpha_{i2} - c_{i2}^1 s(\alpha_{i2}) + c_{i2}^2 s^2(\alpha_{i2}) - \cdots\cdots + c_{i2}^{u_2-2} s^{u_2-2}(\alpha_{i2}) \\
& - c_{i2}^{u_2-1} s^{u_2-1}(\alpha_{i2}) + \cdots\cdots \\
& + c_{iq}^0\alpha_{iq} - c_{iq}^1 s(\alpha_{iq}) + c_{iq}^2 s^2(\alpha_{iq}) - \cdots\cdots + c_{iq}^{u_q-2} s^{u_q-2}(\alpha_{iq}) \\
& - c_{iq}^{u_q-1} s^{u_q-1}(\alpha_{iq}) \\
& + c_{i(q+1)}^0\alpha_{i(q+1)} - c_{i(q+1)}^1 s(\alpha_{i(q+1)}) + c_{i(q+1)}^2 s^2(\alpha_{i(q+1)}) + \cdots\cdots \\
& - c_{i(q+1)}^{u_{(q+1)}-2} s^{u_{(q+1)}-2}(\alpha_{i(q+1)}) + c_{i(q+1)}^{u_{(q+1)}-1} s^{u_{(q+1)}-1}(\alpha_{i(q+1)}) \\
& + c_{i(q+2)}^0\alpha_{i(q+2)} - c_{i(q+2)}^1 s(\alpha_{i(q+2)}) + c_{i(q+2)}^2 s^2(\alpha_{i(q+2)}) - \cdots\cdots \\
& - c_{i(q+2)}^{u_{(q+2)}-2} s^{u_{(q+2)}-2}(\alpha_{i(q+2)}) + c_{i(q+2)}^{u_{(q+2)}-1} s^{u_{(q+2)}-1}(\alpha_{i(q+2)}) + \cdots\cdots \\
& + c_{i(q+w)}^0\alpha_{i(q+w)} - c_{i(q+w)}^1 s(\alpha_{i(q+w)}) + c_{i(q+w)}^2 s^2(\alpha_{i(q+w)}) - \cdots\cdots \\
& - c_{i(q+w)}^{u_{(q+w)}-2} s^{u_{(q+w)}-2}(\alpha_{i(q+w)}) + c_{i(q+w)}^{u_{(q+w)}-1} s^{u_{(q+w)}-1}(\alpha_{i(q+w)}).
\end{aligned}
$$

Now our aim is to prove that α is a linear combination of elements of G'' to conclude that G'' is really a basis for the invariant subspace. To do this we prove that all coefficients of $\alpha_{ij} (j \geq q+1)$ in (1.2.19) must vanish. As a sample we prove that the coefficients of $\alpha_{i(q+1)}$ are zero. Rewrite α as:

$$
\begin{aligned}
(1.2.20)\qquad \alpha = {} & c_{i(q+1)}^0\alpha_{i(q+1)} - c_{i(q+1)}^1 s(\alpha_{i(q+1)}) + c_{i(q+1)}^2 s^2(\alpha_{i(q+1)}) + \cdots\cdots \\
& - c_{i(q+1)}^{u_{(q+1)}-2} s^{u_{(q+1)}-2}(\alpha_{i(q+1)}) + c_{i(q+1)}^{u_{(q+1)}-1} s^{u_{(q+1)}-1}(\alpha_{i(q+1)}) + \beta
\end{aligned}
$$

Applying s to α, we obtain

$$
\begin{aligned}
(1.2.21)\qquad s(\alpha) = {} & c_{i(q+1)}^0 s(\alpha_{i(q+1)}) - c_{i(q+1)}^1 s^2(\alpha_{i(q+1)}) + c_{i(q+1)}^2 s^3(\alpha_{i(q+1)}) + \cdots\cdots \\
& - c_{i(q+1)}^{u_{(q+1)}-2} s^{u_{(q+1)}-1}(\alpha_{i(q+1)}) - c_{i(q+1)}^{u_{(q+1)}-1}(\alpha_{i(q+1)}) + s(\beta).
\end{aligned}
$$

Equating (1.2.20) and (1.2.21), since α is an invariant element, we get

$$c_{i(q+1)}^0 = -c_{i(q+1)}^1 = c_{i(q+1)}^2 = -c_{i(q+1)}^3 = \cdots\cdots = -c^{u_{(q+1)}-2} = c^{u_{(q+1)}-1} = -c_{i(q+1)}^0$$

and this implies

$$2c_{i(q+1)}^0 = 0.$$

Since the caracteristic of k is not 2, $c_{i(q+1)}^0 = 0$ and hence

$$c_{i(q+1)}^0 = c_{i(q+1)}^1 = c_{i(q+1)}^2 = c_{i(q+1)}^3 = \cdots\cdots = c^{u_{(q+1)}-2} = c^{u_{(q+1)}-1} = 0.$$

2. The free loop space on a wedge of odd spheres.

2.1. *Graded vector space structure and torsion.*

THEOREM 2.1.1. *Let* $\mathrm{X} = \mathrm{S}^d \bigvee \cdots \bigvee \mathrm{S}^d$ *be the wedge of* $m, d-$*spheres* $(d \geq 3,$*odd*$)$. *We have the following explicit formula for* $\mathrm{H}^*(\mathscr{L}\mathrm{X}, k)$, *where* $\bar{\Lambda} = \mathrm{H}^{>0}(\mathscr{L}\mathrm{X}, k)$.

(2.1.1)

deg	0	$d-1$	d	$2(d-1)$	$2(d-1)+1$	$\cdots\cdots$
$\mathrm{H}^*(\mathscr{L}\mathrm{X}, k)$	k	$1 \otimes \bar{\Lambda}$	$\bar{\Lambda}$	$1 \otimes \mathrm{Ker}\,\mathrm{T}_2$	$\dfrac{\bar{\Lambda}^{\otimes 2}}{\mathrm{Im}\,\mathrm{T}_2}$	$\cdots\cdots$

(T_n is defined in (1.1.6)), *and moreover*

$$\left|\mathrm{H}^{s(d-1)}(\mathscr{L}\mathrm{X}, k)\right| = \left|1 \otimes \mathrm{Ker}\,\mathrm{T}_s\right| = \left|\frac{\bar{\Lambda}^{\otimes s}}{\mathrm{Im}\,\mathrm{T}_s}\right| = \left|\mathrm{H}^{s(d-1)+1}(\mathscr{L}\mathrm{X}, k)\right| = \frac{1}{s}\sum_{i=1}^{s} m^{(i,s)}$$

In the above formula (i, s) means the greatest common divisor of i and s and $||$ is the dimension as a vector space over k.

REMARK 2.1.2. The proof of this theorem is essentially identical to that in the even case given by Roos in [20].

PROOF.

$$\mathrm{H}^N(\mathscr{L}\mathrm{X}, k) = \coprod_{n \geq 0} \mathrm{Tor}_n^{\mathrm{H}^*(\mathrm{X},k)\otimes \mathrm{H}^*(\mathrm{X},k)}(\mathrm{H}^*(\mathrm{X}, k), \mathrm{H}^*(\mathrm{X}, k))^{N+n} \tag{2.1.2}$$

(cf. the introduction). For given N, the Tor_n in (2.1.2) can according to (1.1.9) only occur if n satisfies either $n + N = dn$ or $n + N = d(n+1)$, i.e., only if $n = N/(d-1)$ or $n = (N-d)/(d-1)$, which requires N or $N-1$ to be divisible by $d-1$. In the first case the contribution to $\mathrm{H}^{\mathrm{N}}(\mathscr{L}\mathrm{X}, k)$ is $1 \otimes \mathrm{Ker}\mathrm{T}_n$ and in the second case the contribution to $\mathrm{H}^{\mathrm{N}}(\mathscr{L}\mathrm{X}, k)$ is $\bar{\Lambda}^{\otimes n+1}/\mathrm{Im}\mathrm{T}_{n+1}$ but the first case occurs when $N = k(d-1)$, for $k = 1, 2, 3, \cdots\cdots$ and the second case occurs when $N = k(d-1)+1$,for $k = 1, 2, 3, \cdots\cdots$. This proves the first part of the theorem.

For the second part, note that it has been shown in [20] that

$$|\mathrm{H}^{s(d-1)}(\mathscr{L}\mathrm{X}, k)| = |\mathrm{Ker}\,\mathrm{T}_s| = \left|\frac{\bar{\Lambda}^{\otimes s}}{\mathrm{Im}\,\mathrm{T}_s}\right| = \frac{1}{s}\sum_{i=1}^{s}(-1)^{i(s-1)} m^{(i,s)}$$

when $\mathrm{T}_s : \bar{\Lambda}^{\otimes s} \to \bar{\Lambda}^{\otimes s}$ is defined by $\mathrm{T}_s = 1 - t_s$, and where $t_s : \bar{\Lambda}^{\otimes s} \to \bar{\Lambda}^{\otimes s}$ is defined by

$$\lambda_1 \otimes \lambda_2 \otimes \ldots\ldots \otimes \lambda_s \longrightarrow (-1)^{s-1}\lambda_s \otimes \lambda_1 \otimes \ldots\ldots \otimes \lambda_{s-1}.$$

The only thing we need to do is to eliminate the sign coefficient $(-1)^{i(s-1)}$ (see the proof in [20]) because now $t_s : \bar{\Lambda}^{\otimes s} \to \bar{\Lambda}^{\otimes s}$ is defined by $\lambda_1 \otimes \lambda_2 \otimes \ldots\ldots \otimes \lambda_s \longrightarrow \lambda_s \otimes \lambda_1 \otimes \ldots\ldots \otimes \lambda_{s-1}$ and has no sign.

Put $\mathrm{N} = \dfrac{1 + t + t^2 + \cdots\cdots + t^{n-1}}{n}$, where t is defined in (1.1.6), and define the map

$$\mathrm{B} : \bar{\Lambda}^{\otimes n} \longrightarrow \bar{\Lambda}^{\otimes(n+1)}$$

by

$$\mathrm{B}(v_1 \otimes v_2 \otimes \cdots\cdots \otimes v_n) = \sum_{i=1}^{n} 1 \otimes v_i \otimes v_{i+1} \otimes \cdots\cdots \otimes v_n \otimes v_1 \otimes \cdots\cdots \otimes v_{i-1}.$$

It is clear that

$$1 \otimes \operatorname{Ker} \mathrm{T_n} = 1 \otimes \mathrm{N}(\bar{\Lambda}^{\otimes n}) = \mathrm{B}(\bar{\Lambda}^{\otimes n}).$$

Moreover as the following lemma shows, B induces an isomorphism:

$$\mathrm{H}^{(\mathrm{d}-1)\mathrm{n}+1}(\mathscr{L}\mathrm{X}, k) = \frac{\bar{\Lambda}^{\otimes n}}{\operatorname{Im} \mathrm{T}_n} \subseteq \operatorname{Tor}_{n-1}^{\mathrm{H}^*(\mathrm{X},k)\otimes \mathrm{H}^*(\mathrm{X},k)}(\mathrm{H}^*(\mathrm{X}, k), \mathrm{H}^*(\mathrm{X}, k))$$

$$= \frac{\bar{\Lambda}^{\otimes n}}{\operatorname{Im} \mathrm{T}_n} \coprod 1 \otimes \operatorname{Ker} \mathrm{T}_{n-1}$$

$$\big\downarrow \mathrm{B}$$

$$\mathrm{H}^{(d-1)n}(\mathscr{L}\mathrm{X}, k) = 1 \otimes \operatorname{Ker} \mathrm{T}_n \subseteq \operatorname{Tor}_{n}^{\mathrm{H}^*(\mathrm{X},k)\otimes \mathrm{H}^*(\mathrm{X},k)}(\mathrm{H}^*(\mathrm{X}, k), \mathrm{H}^*(\mathrm{X}, k))$$

$$= \frac{\bar{\Lambda}^{\otimes n+1}}{\operatorname{Im} \mathrm{T}_{n+1}} \coprod 1 \otimes \operatorname{Ker} \mathrm{T}_n$$

where $\mathrm{X} = \mathbf{S}^{\mathrm{d}} \bigvee \mathbf{S}^{d}$.

LEMMA 2.1.3. *Let* $\{\mathrm{B}(\alpha_1), \mathrm{B}(\alpha_2), \cdots, \mathrm{B}(\alpha_l)\}$ *be a basis for*

$$\mathrm{H}^{(d-1)n}(\mathscr{L}\mathrm{X}, k) = 1 \otimes \operatorname{Ker} \mathrm{T_n}$$

as a vector space over k, *then* $\{[\alpha_1], [\alpha_2], \cdots, [\alpha_l]\}$ *can be chosen as a basis for*

$$\mathrm{H}^{(d-1)n+1}(\mathscr{L}\mathrm{X}, k) = \frac{\bar{\Lambda}^{\otimes n}}{\mathrm{T_n}}$$

as a vector space over k.

PROOF. It is enough to prove that $[\alpha_1], [\alpha_2], \ldots, [\alpha_l]$ are linearly independent. If

$$c_1[\alpha_1] + c_2[\alpha_2] + \cdots + c_l[\alpha_l] = 0,$$

then

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_l\alpha_l \in \operatorname{Im} \mathrm{T}_n$$

and hence

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_l\alpha_l = (1-t)\alpha \qquad \text{for some } \alpha.$$

Thus

$$c_1\mathrm{B}\alpha_1 + c_2\mathrm{B}\alpha_2 + \cdots + c_l\mathrm{B}\alpha_l = \mathrm{B}(1-t)\alpha = 1 \otimes \mathrm{N}((1-t)\alpha) = 0$$

which implies that

$$c_1 = c_2 = \cdots = c_l = 0.$$

LEMMA 2.1.4.

$$\mathrm{B}(\alpha) = 0 \Rightarrow [\alpha] = 0 \text{in} \frac{\bar{\Lambda}^{\otimes n}}{\mathrm{T}_n}.$$

PROOF. We have

$$\mathrm{B}(\alpha) = 1 \otimes \mathrm{N}(\alpha) = 0 \Rightarrow \mathrm{N}(\alpha) = 0$$

but

$$\bar{\Lambda}^{\otimes n} = \mathrm{N}(\bar{\Lambda}^{\otimes n}) \oplus (1-t)(\bar{\Lambda}^{\otimes n}),$$

hence

$$\mathrm{N}(\alpha) = 0 \Rightarrow \alpha \in (1-t)(\bar{\Lambda}^{\otimes n}).$$

LEMMA 2.1.5. *The spectral sequence:*

$$\mathrm{E}_2^{-p,q} = \mathrm{Tor}_p^{\mathrm{H}^*(\mathrm{X}\times\mathrm{X},k)}(\mathrm{H}^*(\mathrm{X},k), \mathrm{H}^*(\mathrm{X};k))^q \Rightarrow \mathrm{H}^n(\mathscr{L}\mathrm{X},k)$$

degenerates in the case

$$\mathrm{X} = \mathrm{S}^d \bigvee \mathrm{S}^d \cdots \bigvee \mathrm{S}^d \quad (d, \text{odd})$$

over a field k of arbitrary characteristic.

PROOF. The elements of $\mathrm{H}^+(\mathrm{X},k)$ are concentrated in degree d and hence by (1.1.9), the elements of

$$\mathrm{Tor}_p^{H^*(X,k)\otimes H^*(X,k)}(H^*(X,k), H^*(X,k))$$

are concentrated in degrees dp and $dp+d$. This shows that only the terms

$$E_2^{-p,dp} = \mathrm{Tor}_p^{H^*(X\times X,k)}(H^*(X,k), H^*(X;k))^{dp}$$

and

$$E_2^{-p,dp+d} = \mathrm{Tor}_p^{H^*(X\times X,k)}(H^*(X,k), H^*(X;k))^{dp+d}$$

are non zero. Note that the spectral sequence is (E_r, d_r) where d_r has bidegree $(-r, r-1)$. Consider the two complexes

(2.1.3) $$0 = E_r^{-p+r,dp-r+1} \xrightarrow{d_r} E_r^{-p,dp} \xrightarrow{d_r} E_r^{-p-r,dp+r-1} = 0$$

and

(2.1.4) $$0 = E_r^{-p+r,dp+d-r+1} \xrightarrow{d_r} E_r^{-p,dp+d} \xrightarrow{d_r} E_r^{-p-r,dp+d+r-1} = 0$$

Since (2.1.3) and (2.1.4) hold for all $r \geq 2$, we have

$$E_2^{-p,q} = E_\infty^{-p,q} \mathrm{for all} p, q$$

and hence the spectral sequence degenerates.

Lemma 2.1.5, together with Theorem 1.2.1, which asserted that the dimension of the invariant subspace under the action of

$$t(v_1 \otimes v_2 \otimes \ldots\ldots \otimes v_n) = v_n \otimes v_1 \otimes \ldots\ldots \otimes v_{n-1}$$

is independent of the characteristic of the field, together with (1.1.9) imply that

$$H^*(\mathscr{L}(S^d \bigvee S^d), Z)$$

has no torsion part at all.

By the same argument as above it can easily be seen that

$$H^*(\mathscr{L}(S^g \bigvee S^g \cdots\cdots \bigvee S^g), k) \quad (g, \mathrm{odd})$$

has no torsion part at all.

2.2. *The ring structure of* $H^*(\mathscr{L}(S^d \bigvee S^d), k)(d \geq 3, odd)$.

2.2.1. Global observations.

LEMMA 2.2.1. *Let* $X = S^d \bigvee \cdots\cdots \bigvee S^d$ *be a wedge of* m, d*−spheres* ($d \geq 3$, *odd) and let* $\Lambda = H^*(X, k)$. *Then the shuffle product on the level of the standard free resolution* (1.1.1) *is given by*

$$\lambda[a_1 \otimes a_2 \otimes \ldots\ldots \otimes a_p] * \mu[a_{p+1} \otimes a_{p+2} \otimes \ldots\ldots \otimes a_{p+q}]$$

$$= \sum_{(p,q)-\text{shuffles}(\sigma)} \lambda\mu[a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \ldots\ldots \otimes a_{\sigma^{-1}(p+q)}]$$

PROOF. All elements in $\bar{\Lambda} = \text{coker}(k \longrightarrow \Lambda)$ are concentrated in degree d and d is odd, so

$$s(\sigma) = \sum (\deg c_i + 1)(\deg c_{p+j} + 1) = \sum (d+1)(d+1).$$

This implies $(-1)^{s(\sigma)} = 1$.

THEOREM 2.2.2. *Let Q^* denote the indecomposables in $\mathrm{H}^*(\mathscr{L}(\mathrm{S}^d \bigvee \mathrm{S}^d), k)$, $d \geq 3$, odd. The elements $Y_1, Y_2, \ldots\ldots, Y_s$ are representatives of a basis of $Q^{(d-1)n+1}$ if and only if the elements $X_1 = \mathrm{B}(Y_1), X_2 = \mathrm{B}(Y_2), \cdots, X_s = \mathrm{B}(Y_s)$ are representatives of a basis of $Q^{(d-1)n}$.*

PROOF. We prove this theorem by induction. We have two generators X_1, X_2 in degree $(d-1)$ and two generators Y_1, Y_2 in degree d that satisfy

$$X_1 = \mathrm{B}(Y_1) \qquad \text{and} \qquad X_2 = \mathrm{B}(Y_2).$$

Assume that the claim is true for $n \leq p$. We prove that

$$X = \mathrm{B}(Y) \text{ represents an indecomposable in } \mathrm{H}^{(\mathrm{d}-1)(\mathrm{p}+1)}(\mathscr{L}(\mathrm{S}^{\mathrm{d}} \bigvee \mathrm{S}^{\mathrm{d}}), k)$$

if and only if

$$Y \text{ represents an indecomposable in } \mathrm{H}^{(\mathrm{d}-1)(\mathrm{p}+1)+1}(\mathscr{L}(\mathrm{S}^{\mathrm{d}} \bigvee \mathrm{S}^{\mathrm{d}}), k).$$

To prove this, assume that $X = \mathrm{B}(Y)$ represents an indecomposable in degree $(d-1)(p+1)$ but Y does not represent an indecomposable in degree $(d-1)(p+1)+1$, i.e.,

$$Y = \sum a_{k_1 k_2 \cdots k_t} X_1^{k_1} X_2^{k_2} \cdots X_t^{k_t} Y_j$$

where

$$\deg X_i \leq (d-1)p, \ \deg Y_j \leq (d-1)p + 1 \quad \text{and}$$
$$X_i = \mathrm{B}(Y_i) \quad \text{for all} \quad i = 1, 2, 3, \ldots\ldots t.$$

Since

$$\mathrm{B}(Y') * \mathrm{B}(Y) = \mathrm{B}(Y' * \mathrm{B}(Y)) = \mathrm{B}(\mathrm{B}(Y') * Y) \text{ (cf. [13])},$$

we obtain that

$$X = \mathrm{B}(Y) = \sum a_{k_1k_2\cdots k_t}\mathrm{B}(X_1^{k_1}X_2^{k_2}\cdots X_t^{k_t}Y_j) = \sum a_{k_1k_2\cdots k_t}X_1^{k_1}X_2^{k_2}\cdots X_t^{k_t}\mathrm{B}(Y_j)$$

$$= \sum a_{k_1k_2\cdots k_t}X_1^{k_1}X_2^{k_2}\cdots X_t^{k_t}X_j.$$

But this is a contradiction to the fact that X was an indecomposable element. To prove the converse, assume that Y appears as anindecomposable in degree $(d-1)(p+1)+1$ but $X = \mathrm{B}(Y)$ does not appear as an indecomposable in degree $(d-1)(p+1)$, i.e.,

$$X = \sum a_{k_1k_2\cdots k_t}X_1^{k_1}X_2^{k_2}\cdots X_t^{k_t}$$

where

$$\deg X_i \leq (d-1)p \text{ and} X_i = \mathrm{B}(Y_i) \text{for all}\, i = 1, 2, 3, \ldots\ldots, t.$$

This implies that

$$\mathrm{B}(Y) = \sum a_{k_1k_2\cdots k_t}\mathrm{B}(Y_1)^{k_1}\mathrm{B}(Y_2)^{k_2}\cdots\mathrm{B}(Y_t)^{k_t}$$

$$= \sum a_{k_1k_2\cdots k_t}\mathrm{B}[Y_1\mathrm{B}(Y_1)^{k_1-1}\mathrm{B}(Y_2)^{k_2}\cdots\mathrm{B}(Y_t)^{k_t}]$$

and hence

$$Y = \sum a_{k_1k_2\cdots k_t}Y_1X_1^{k_1-1}X_2^{k_2}\cdots X_t^{k_t}.$$

which is a contradiction.

We have used the following

LEMMA 2.2.3. *Let* $\alpha \in V^{\otimes n}$ *and* $\gamma \in V^{\otimes m}$, *then*

$$\alpha * \mathrm{B}(\gamma) = \mathrm{B}(\alpha) * \gamma$$

PROOF. (cf. [13]). This lemma has been proved in the non graded case, i.e., when $t : V^{\otimes n} \longrightarrow V^{\otimes n}$ is defined by

$$t(v_1 \otimes v_2 \otimes \cdots\cdots \otimes v_n) = (-1)^{n-1}v_n \otimes v_1 \otimes \cdots\cdots \otimes v_{n-1}$$

and

$$\mathrm{B} : V^{\otimes n} \longrightarrow V^{\otimes(n+1)}$$

by

$$\mathrm{B}(v_1 \otimes v_2 \otimes \cdots\cdots \otimes v_n) = \sum_{i=1}^{n} (-1)^{in} 1 \otimes v_i \otimes v_{i+1} \otimes \cdots\cdots \otimes v_n \otimes v_1 \otimes \cdots\cdots \otimes v_{i-1}.$$

Here we have a similar proof with no sign coefficient.

Denote the ring structure of $\mathrm{H}^{2*}(\mathscr{L}(\mathrm{S}^{\mathrm{d}} \bigvee \mathrm{S}^{\mathrm{d}}), k)$ by R and denote $\mathrm{H}^{\mathrm{odd}}(\mathscr{L}(\mathrm{S}^{\mathrm{d}} \bigvee \mathrm{S}^{\mathrm{d}}), k)$ by M. Then $M = S^{-1}\bar{R}$ (where $\bar{R} = R/k$ and $(S^{-1}\bar{R})^p = (\bar{R})^{p-1}$) is an R module with shuffle product and we have

THEOREM 2.2.4. *Let* $\mathrm{X} = \mathbf{S}^d \bigvee \mathbf{S}^d$ *be a wedge of* 2, *d*−*spheres* ($d \geq 3$,*odd*), *then the ring structure of* $\mathrm{H}^*(\mathscr{L}\mathrm{X}, k)$ *is* $R \oplus s^{-1}\bar{R}$, *the trivial extension of R by* $s^{-1}\bar{R}$ *(see Definition* 2.2.11 *below).*

PROOF. Both $R = \mathrm{H}^{2*}(\mathscr{L}(\mathrm{S}^{\mathrm{d}} \bigvee \mathrm{S}^{\mathrm{d}}), k)$ and $s^{-1}\bar{R} = \mathrm{H}^{\mathrm{odd}}(\mathscr{L}(\mathrm{S}^{\mathrm{d}} \bigvee \mathrm{S}^{\mathrm{d}}), k)$ are R modules with respect to the suffle product. The relation

$$\mathrm{B}(\mathrm{B}(Y) * Y') = \mathrm{B}(Y) * \mathrm{B}(Y')$$

for $\mathrm{B}(Y) \in R$ and $Y' \in M$ shows that the map:

$$\mathrm{H}^{\mathrm{odd}}(\mathscr{L}(\mathrm{S}^{\mathrm{d}} \bigvee \mathrm{S}^{\mathrm{d}}), k) \xrightarrow{\mathrm{B}} \mathrm{H}^{2*}(\mathscr{L}(\mathrm{S}^{\mathrm{d}} \bigvee \mathrm{S}^{\mathrm{d}}), k)$$

is an R module isomorphism.

REMARK 2.2.5. Theorem 2.2.4 is true even for $\mathrm{X} = \mathbf{S}^{d_1} \bigvee \mathbf{S}^{d_2} \bigvee \cdots\cdots \bigvee \mathbf{S}^{d_n}$ (d_i, odd).

Although we are not able to compute the entire ring structure of $R \oplus s^{-1}\bar{R}$ we prove the following lemma.

LEMMA 2.2.6. *All relations involving elements of odd degree, except commutators of elements of odd degree, are induced by* B^{-1} *from relations involving only elements of even degree.*

PROOF. Any relation not of the form $Y_i Y_j$ in odd degrees must be of the form

$$(*) \qquad \sum a_{k_1 k_2 \cdots k_t} X_1^{k_1} X_2^{k_2} \cdots X_t^{k_t} Y_j = 0$$

If we set $X_j = \mathrm{B}(Y_j)$, then the relation $(*)$ is induced by the relation

$$\sum a_{k_1 k_2 \cdots k_t} X_1^{k_1} X_2^{k_2} \cdots X_t^{k_t} X_j = 0$$

THEOREM 2.2.7. *If* $\mathrm{H}^{2*}(\mathscr{L}\mathrm{X}, k) \cong \mathrm{T}(W)/I$, *then* $\mathrm{H}^*(\mathscr{L}\mathrm{X}, k)$ *is isomorphic as a ring to*

$$\frac{\mathrm{T}(W \oplus W')}{(I + I' + W'^2 + C)},$$

where $W' = \mathrm{B}^{-1}(W), I' = 1 \otimes \mathrm{B}^{-1}(I)$ *and* $C = \{XY - YX | X \in W, Y \in W'\}$.

PROOF. The map $\mathrm{B} : W' \longrightarrow W$ induces the following diagram:

$$\begin{array}{ccc} I' & \xrightarrow{i'} & \overline{\mathrm{T}(W) \otimes W'} \\ \downarrow & & \downarrow 1\otimes\mathrm{B} \\ I & \xrightarrow{i} & \mathrm{T}(W) \end{array}$$

Here i and i' are inclusion maps and $1 \otimes \mathrm{B} : W^{\otimes n} \otimes W' \longrightarrow W^{\otimes n} \otimes W$. Now if the element

$$\sum a_{k_1 k_2 \cdots k_t} X_1^{k_1} X_2^{k_2} \cdots X_t^{k_t} X_j$$

lies in I, then the above diagram shows that the coresponding element, i.e.,

$$\sum a_{k_1 k_2 \cdots k_t} X_1^{k_1} X_2^{k_2} \cdots X_t^{k_t} Y_j$$

lies in I'

2.2.2. *Explicit low-dimensional calculations.*

THEOREM 2.2.8. *Let* $\mathrm{X} = \mathrm{S}^d \bigvee \mathrm{S}^d$ *be a wedge of* 2, *d*−*spheres* ($d \geq 3$,*odd*), *then the ring structure of* $\mathrm{H}^*(\mathscr{L}\mathrm{X}, k)$ *in low dimensions (up to degree* $12(d-1) - 1$) *is of the form*

$$R = \frac{k[X_1, X_2, \ldots\ldots, X_{76}, Y_1, Y_2, \ldots\ldots, Y_{76}]}{(I)} \tag{2.2.1}$$

where 2, 1, 4, 9, 8, 20, 32 *of* X_i*'s are respectively in degrees*

$$(d-1), 4(d-1), 6(d-1), 8(d-1), 9(d-1), 10(d-1) \text{ and } 11(d-1)$$

and 2, 1, 4, 9, 8, 20, 32 *of* Y_i*'s are respectively in degrees*

$$(d-1)+1, 4(d-1)+1, 6(d-1)+1, 8(d-1)+1,$$
$$9(d-1)+1, 10(d-1)+1 \text{ and } 11(d-1)+1$$

and where

$$I = \{Y_i Y_j (i \leq j), (X_i Y_j - X_j Y_i)(i < j)\}.$$

In other words, I is generated by the relations of the form $Y_i Y_j$ *and the relations that make the following matrix symmetric:*

$$A = \begin{pmatrix} X_1 Y_1 & X_1 Y_2 & \dots & X_1 Y_n \\ X_2 Y_1 & X_2 Y_2 & \dots & X_2 Y_n \\ \vdots & \vdots & \ddots & \vdots \\ X_n Y_1 & X_n Y_2 & \dots & X_n Y_n \end{pmatrix}.$$

REMARK 2.2.9. In Theorem 2.2.8 we have chosen the algebra generators X_i's in degree $(d-1)n$ and the algebra generators Y_i's in degree $(d-1)n+1$ such that $X_i = \mathrm{B}(Y_i)$ which is possible by Theorem 2.2.2.

PROOF OF THEOREM 2.2.8. Put $\mathrm{H}(\mathrm{S}^d \bigvee \mathrm{S}^d), k) = k + \bar{\Lambda}$, then $\bar{\Lambda} = < x, y >$ is a vector space of dimension 2 and both x and y are in degree d. By Theorem 2.1.1, we obtain

(2.2.2)

$\mathrm{H}^*(\mathscr{L}\mathrm{X}, k)$	k	$1 \otimes \bar{\Lambda}$	$\bar{\Lambda}$	$1 \otimes \mathrm{Ker}\,\mathrm{T}_2$	$\frac{\bar{\Lambda}^{\otimes 2}}{\mathrm{Im}\,\mathrm{T}_2}$	$1 \otimes \mathrm{Ker}\,\mathrm{T}_3$	$\frac{\bar{\Lambda}^{\otimes 3}}{\mathrm{Im}\,\mathrm{T}_3}$	$\cdots$
deg	0	$d-1$	d	$2(d-1)$	$2(d-1)+1$	$3(d-1)$	$3(d-1)+1$	$\cdots$
dim	1	2	2	3	3	4	4	$\cdots$

Recall that $\mathrm{Ker}\,\mathrm{T}_n$ in (2.2.2) is exactly the vector space $(\bar{\Lambda}^{\otimes n})^{\frac{Z}{nZ}}$ of invariants in $\bar{\Lambda}^{\otimes n}$ for the group $\frac{Z}{nZ}$ acting through its generator $t_n = \bar{1}$ by

$$t_n(v_1 \otimes v_2 \otimes \cdots \cdot \otimes v_n) = v_n \otimes v_1 \otimes \cdots \cdots \otimes v_{n-1}.$$

Hence

$$\mathrm{Ker}\,\mathrm{T}_n = (\bar{\Lambda}^{\otimes n})^{\frac{Z}{nZ}} = \frac{1 + t_n + t_n^2 + \cdots\cdots + t_n^{n-1}}{n} \bar{\Lambda}^{\otimes n}.$$

Consider $\mathrm{H}^*(\mathscr{L}(\mathrm{S}^d \bigvee \mathrm{S}^d), k)$ in even degrees and denote it by $\mathrm{H}^{2*}(\mathscr{L}\mathrm{X}, k)$. In low dimensions we have:

(2.2.3)

$\mathrm{H}^{2*}(\mathscr{L}\mathrm{X}, k)$	k	$l1 \otimes \bar{\Lambda}$	$1 \otimes \mathrm{Ker}\,\mathrm{T}_2$	$1 \otimes \mathrm{Ker}\,\mathrm{T}_3$	$1 \otimes \mathrm{Ker}\,\mathrm{T}_4$	$1 \otimes \mathrm{Ker}\,\mathrm{T}_5$	$1 \otimes \mathrm{Ker}\,\mathrm{T}_6$	$\cdots$
deg	0	$d-1$	$2(d-1)$	$3(d-1)$	$4(d-1)$	$5(d-1)$	$6(d-1)$	$\cdots$
dim	1	2	3	4	6	8	14	$\cdots$

Start with $R_{d-1} = k[X_1, X_2]$, where X_1, X_2 are the two basis elements of $\mathrm{H}^{d-1}(\mathscr{L}(\mathrm{S}^d \bigvee \mathrm{S}^d), k)$ that commute in R_{d-1} and contain no other relations. We have chosen the notation R_{d-1} because this ring contains all generators up to degree $d-1$. Denote the Hilbert series of R_{d-1} by $\mathrm{H}_{R_{d-1}}(t)$. We have

(2.2.4) $$\mathrm{H}_{R_{d-1}}(t) = \frac{1}{(1-t^4)^2} = 1 + 2t^4 + 3t^8 + 4t^{12} + 5t^{16} + \cdots\cdots$$

Comparing (2.2.3) and (2.2.4) shows that a generator of degree $4(d-1)$ is needed. Call this new generator X_3 and add it to R_{d-1}. Denote the new ring by $R_{4(d-1)}$ then

$$R_{4(d-1)} = k[X_1, X_2, X_3],$$

and

$$\text{(2.2.5)} \qquad \mathrm{H}_{R_{4(d-1)}}(t) = \frac{1}{(1-t^4)^2(1-t^{16})} = 1 + 2t^4 + 3t^8 + 4t^{12} + 6t^{16} + 8t^{20} + 10t^{24} + \cdots\cdots$$

Comparing (2.2.3) and (2.2.5), we realize that four new generators are needed in degree $6(d-1)$. Continuing this way, we can find the number of generators in each degree (even degrees). This ring, i.e., $\mathrm{H}^{2*}(\mathscr{L}\mathrm{X}, k)$. is a subring of a polynomial ring. If it were a polynomial ring it would have: $2, 1, 4, 9, 8, 20, 32, 68, \cdots\cdots$ generators respectively in degrees:

$$(d-1), 4(d-1), 6(d-1), 8(d-1), 9(d-1),$$
$$10(d-1), 11(d-1), 12(d-1), \cdots\cdots.$$

However this is only true up to degree $11(d-1)$ since C. Löfwall and L. Lambe have proved the existence of four relations in degree $12(d-1)$. Moreover they have proved the non-existence of any relations in previous degrees, i.e., degrees less than $12(d-1)$. (see [12]). It remains an open problem to find out what this ring looks like in all dimensions.

By continuing, we find the ring

$$R_{11(d-1)} = k[X_1, X_2, X_3, \ldots\ldots, X_{76}]$$

isomorphic to $\mathrm{H}^{2*}(\mathscr{L}\mathrm{X}, k)$ up to degree $12(d-1) - 1$.

In order to prove that

$$R = \frac{k[X_1, X_2, X_3, \ldots\ldots, X_{76}, Y_1, Y_2, Y_3, \ldots\ldots, Y_{76}]}{(Y_iY_j[i \leq j], (X_iY_j - X_jY_i)[i < j])} \quad \text{and } \mathrm{H}^*(\mathscr{L}\mathrm{X}, k)$$

are isomorphic up to degree $12(d-1)$, first we prove that there is an onto ring map $R \longrightarrow \mathrm{H}^*(\mathscr{L}\mathrm{X}, k)$ and then that R and $\mathrm{H}^*(\mathscr{L}\mathrm{X}, k)$ have the same Hilbert series. It is clear that $\mathrm{H}^*(\mathscr{L}\mathrm{X}, k)$ contains all relations $Y_iY_j[i \leq j]$, by definition of the shuffle product in $\mathrm{H}^*(\mathscr{L}\mathrm{X}, k)$ and also contains the relations of the form $X_iY_j - X_jY_i$, (see Lemma 2.2.3).

To prove that R and $\mathrm{H}^*(\mathscr{L}\mathrm{X}, k)$ have the same Hilbert series up to degree $12(d-1)$, let

$$R'_n = K[X_1, X_2, \cdots\cdots, X_n],$$

$$R_n = \frac{k[X_1, X_2, X_3, \cdots\cdots, X_n, Y_1, Y_2, Y_3, \cdots\cdots, Y_n]}{(Y_i Y_j[i \leq j], (X_i Y_j - X_j Y_i)[i < j])},$$

$$A_n = \frac{k[X_1, X_2, X_3, \cdots\cdots, X_n, Y_1, Y_2, Y_3, \cdots\cdots, Y_n]}{(Y_i Y_j[i \leq j], X_i Y_j[i < j])}.$$

Considering all X_i's and Y_j's as of degree 1 we get

$$\mathrm{Hilb}_{R'_n}(t) = \sum h_{k,n} t^k = \frac{1}{(1-t)^n}, \quad h_{k,n} = \binom{n+k-1}{n-1}.$$

It can easily be seen that $(Y_i Y_j[i \leq j], (X_i Y_j - X_j Y_i)[i < j])$ is a Groebner basis, so

$$\mathrm{Hilb}_{R_n}(t) = \mathrm{Hilb}_{A_n}(t) = \sum h_k t^k, \; h_0 = 1.$$

The monomials of degree k in A_n are of the following types:

M_0 = monomials of degree k in $\{X_1, X_2, ..., X_n\}$ $\dim \mathrm{M}_0 = h_{k,n}$
M_1 = $Y_1\cdot$ monomials of degree $k-1$ in $\{X_1, X_2, ..., X_n\}$ $\dim \mathrm{M}_1 = h_{k-1,n}$
M_2 = $Y_2\cdot$ monomials of degree $k-1$ in $\{X_2, X_3, ..., X_n\}$ $\dim \mathrm{M}_2 = h_{k-1,n-1}$
M_3 = $Y_3\cdot$ monomials of degree $k-1$ in $\{X_3, X_4, ..., X_n\}$ $\dim \mathrm{M}_3 = h_{k-1,n-2}$
...
...
M_n = $Y_n\cdot$ monomials of degree $k-1$ in $\{X_n\}$ $\dim \mathrm{M}_n = h_{k-1,1} = 1$

and hence

(2.2.6) $$h_k = h_{k,n} + h_{k-1,1} + h_{k-1,2} + \cdots\cdot + h_{k-1,n}$$

It is also easy to see that

(2.2.7) $$h_{k-1,1} + h_{k-1,2} + \cdots\cdot + h_{k-1,n} = h_{k,n}$$

By (2.2.6) and (2.2.7) we get $h_k = 2h_{k,n}$ for $(k \geq 1)$ and hence

$$\mathrm{Hilb}_{R_n}(t) = \mathrm{Hilb}_{A_n}(t) = \sum h_k t^k = \frac{2}{(1-t)^n} - 1.$$

This shows that R_n and $\mathrm{H}^*(\mathscr{L}\mathrm{X}, k)$ have the same Hilbert series, up to some degree. The above argument works, however, for any $R_n (n \geq 1)$, so

$$R = \frac{k[X_1, X_2, X_3, \cdots\cdots, X_{76}, Y_1, Y_2, Y_3, \cdots\cdots, Y_{76}]}{(Y_i Y_j[i \leq j], (X_i Y_j - X_j Y_i)[i < j])} \quad \text{and} \quad \mathrm{H}^*(\mathscr{L}\mathrm{X}, k)$$

have the same Hilbert series up to degree $12(d-1) - 1$.

In this part we are going to show that the ring

$$R = \frac{k[X_1, X_2, X_3, \cdots\cdots, X_{76}, Y_1, Y_2, Y_3, \cdots\cdots, Y_{76}]}{(Y_i Y_j[i \leq j], (X_i Y_j - X_j Y_i)[i < j])},$$

i.e., $H^*(\mathscr{L}X, k)$ up to degree $12(d-1)-1$. is a homogeneous Koszul algebra (see Definition 2.2.10 below). In order to do this we need the following.

DEFINITION 2.2.10. Let R be an algebra over k. We call R a *homogeneous Koszul algebra,* if the ring $\mathrm{Ext}^*_R(k,k)$ is generated by $\mathrm{Ext}^{1,*}_R(k,k)$.

DEFINITION 2.2.11. Let R be a local ring and let M be an R–module. The ring $R \oplus M$ whose elements consist of pairs (r, m) with addition componentwise and multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$$

is called *the trivial extension of R by M*.

DEFINITION 2.2.12. We say that the bigraded ring R is a *Koszul algebra* (with respect to the first degree) up to degree $\leq n$ (with respect to the second degree), if there is a ring K (K a Koszul algebra) (with respect to the first degree) and a ring homomorphism $K \overset{\phi}{\longrightarrow} R$ that is isomorphism up to degree $\leq n$ (with respect to the second degree).

THEOREM 2.2.13. *If the finitely generated ring R is a homogeneous Koszul-algebra then the trivial extension ring $R \oplus \bar{R}$ is also a homogeneous Koszul-algebra.*

REMARK 2.2.14. We have considered the generators X_i*'s* and Y_i*'s* as bigraded elements with bidegree $(1, \deg X_i\text{'s})$ and $(1, \deg Y_i\text{'s})$ respectively. Now if we only consider this new degree 1 (the first degree), then $H^*(\mathscr{L}(S^d \bigvee S^d), k)$ is nothing but $R \oplus \bar{R}$. This is why we consider the ring extension $R \oplus \bar{R}$ in Theorem 2.2.13.

PROOF OF THEOREM 2.2.13. According to the Fröberg formula, [8], It is enough to show that

$$P_{R\oplus\bar{R}}(x, y) \cdot H_{R\oplus\bar{R}}(-xy) = 1$$

where P denotes the Poincaré-betti series and where H denotes the Hilbert series. It is clear that

$$H_{R\oplus\bar{R}}(t) = H_R(t) + (H_R(t) - 1). \tag{2.2.8}$$

Next we prove that

$$P_{R\oplus\bar{R}}(x, y) = P_R(x, y)/(1 - (P_R(x, y) - 1)) \tag{2.2.9}$$

In [11] Herzog has found

$$P_{R\oplus\bar{R}}(z) = P_R(z)/(1 - zP_R^{\bar{R}}(z)) \tag{2.2.10}$$

but the exact sequence

$$0 \longrightarrow \bar{R} \longrightarrow R \longrightarrow k \longrightarrow 0$$

gives

$$\mathrm{Ext}_R^*(\bar{R}, k) = \overline{\mathrm{Ext}}_R^{*+1}(k, k),$$

i.e,

$$P_R^{\bar{R}}(z) = 1/z(P_R(z) - 1).$$

This gives the proof of (2.2.9). Now using (2.2.8) and (2.2.9) we get

$$\begin{aligned}(2.2.11)\qquad H_{R\oplus\bar{R}}(-xy)\cdot P_{R\oplus\bar{R}}(x,y) &= (H_R(-xy) + H_R(-xy) - 1)\\ &\quad\cdot P_R(x,y)/(1 - (P_R(x,y) - 1))\\ &= (2 - P_R(x,y))/(1 - (P_R(x,y) - 1)) = 1.\end{aligned}$$

THEOREM 2.2.15. *The ring*:

$$\frac{k[X_1, X_2, X_3, \ldots\ldots, X_{76}, Y_1, Y_2, Y_3, \ldots\ldots, Y_{76}]}{(Y_iY_j[i \le j], (X_iY_j - X_jY_i)[i < j])}$$

which is isomorphic to $\mathrm{H}^*(\mathscr{L}(\mathrm{S^d} \bigvee \mathrm{S^d}), k)$ *in degrees less than* $12(d-1)-1$ *is a Koszul-algebra, where all generators* X_i *and* Y_i *are considered to be in degree* 1.

To prove Theorem 2.2.15 we need the following:

DEFINITION 2.2.16. Let $M = \oplus_{i,j\ge 0} M^{i,j}$ be biagraded vector space. We define the *bigraded vector space* $s^{p,q}M$ by putting

$$(s^{p,q}M)^{i,j} = M^{i+p,j+q}.$$

Now if we consider the generators X_i*'s* of $\mathrm{H}^{2*}(\mathscr{L}(\mathrm{S^d} \bigvee \mathrm{S^d}), k)$ and the generators Y_i*'s* of $\mathrm{H}^{2*+1}(\mathscr{L}(\mathrm{S^d} \bigvee \mathrm{S^d}), k)$ as bigraded elements with bidegrees $(1, \deg X_i)$ and $(1, \deg Y_i)$ respectively, then using the notation of Definition 2.2.16, we have

$$\mathrm{H}^*(\mathscr{L}(\mathrm{S^d} \bigvee \mathrm{S^d}), k) = \mathrm{H}^{2*}(\mathscr{L}(\mathrm{S^d} \bigvee \mathrm{S^d}), k) \oplus s^{0,-1}\overline{\mathrm{H}^{2*}(\mathscr{L}(\mathrm{S^d} \bigvee \mathrm{S^d}), k)}.$$

In another words the ring structure of $\mathrm{H}^*(\mathscr{L}(\mathrm{S^d} \bigvee \mathrm{S^d}), k)$ is the trivial extension of $\mathrm{H}^{2*}(\mathscr{L}(\mathrm{S^d} \bigvee \mathrm{S^d}), k)$ by $s^{0,-1}\mathrm{H}^{2*}(\mathscr{L}(\mathrm{S^d} \bigvee \mathrm{S^d}), k)$.

PROOF OF THEOREM 2.2.15. The ring $R = \mathrm{H}^{2*}(\mathscr{L}(\mathrm{S}^{\mathrm{d}} \bigvee \mathrm{S}^{\mathrm{d}}), k)$ is a polynomial ring in degrees less than $12(d-1)-1$ and hence a Koszul-algebra, if we consider the generators to be in degree 1. Now the result follows from Theorem 2.2.13.

We devote the last part of this section to finding an explicit formula for $\mathrm{Ext}_R(k,k)$, when

$$R = \frac{k[X_1, X_2, X_3, \ldots\ldots, X_n, \ldots\ldots, Y_1, Y_2, Y_3, \ldots\ldots, Y_n, \ldots\ldots]}{(Y_iY_j[i \le j], (X_iY_j - X_jY_i)[i < j])} \tag{2.2.12}$$

DEFINITION 2.2.17. If a and b are bigraded elements with bigrade (s_1, t_1) and (s_2, t_2), then the *graded commutator* $[a, b]$ is defined by $ab - (-1)^{s_1s_2+t_1t_2}ba$ for $a \neq b$ and $[a, a] = a^2$ or zero according to whether $s_1 + t_1$ is odd or even.

The following lemma is due to Löfwall ([14]).

LEMMA 2.2.18. *Let $k[X_1, X_2, X_3, \ldots\ldots, X_n]$ denote the free graded strictly commutative algebra on the variables $X_1, X_2, X_3, \ldots\ldots, X_n$ of nonnegative degree. Let*

$$f_i = \sum_{j \le k} b_{ijk} X_j X_k \qquad b_{ijk} \in k, i = 1, 2, 3, \ldots\ldots, r$$

be homogeneous elements. Put

$$R = \frac{k[X_1, X_2, X_3, \ldots\ldots, X_n]}{(f_1, f_2, \ldots\ldots, f_r)},$$

then

$$A = [\mathrm{Ext}^1_R(k,k)] = \frac{k < T_1, T_2, \ldots\ldots, T_n >}{(\phi_1, \phi_2, \ldots\ldots, \phi_s)}.$$

As bigraded algebras, T_i has bidegree $(1, \deg X_i)$, and

$$\phi_i = \sum_{j \le k} c_{ijk} [T_j, T_k] \qquad c_{ijk} \in k, i = 1, 2, 3, \ldots\ldots, s.$$

where $[T_j, T_k]$ is graded commutator, and $(c_{ijk})_{jk}, i = 1, 2, 3, \ldots\ldots, s$ is a basis to the solutions of the linear system

$$\sum_{j \le k} b_{ijk} X_{jk} = 0 \qquad i = 1, 2, 3, \ldots\ldots, r$$

$\left(\text{hence } s = \dfrac{n(n+1)}{2} - r\right)$

PROOF. (cf. [19] and [14]).

Using lemma 2.2.18, we easily compute $[\mathrm{Ext}^1_R(k,k)]$, where R is the ring in (2.2.12) as the following

$$\begin{aligned}\mathrm{Ext}_R(k,k) &= [\mathrm{Ext}^1_R(k,k)] \\ &= \frac{k < T_1, T_2, T_3, \ldots\ldots, T_n, \ldots\ldots, S_1, S_2, S_3, \ldots\ldots, S_n, \ldots\ldots >}{(F)},\end{aligned}$$

where F is generated by all graded commutators $[T_i, T_j]$ $i \leq j$ and those relations that make the following matrix skew symmetric, i.e., $A = -A^T$.

$$A = \begin{pmatrix} T_1S_1 & T_1S_2 & \ldots & T_1S_n & \ldots \\ T_2S_1 & T_2S_2 & \ldots & T_2S_n & \ldots \\ \vdots & \vdots & \ddots & \vdots & \\ T_nS_1 & T_nS_2 & \ldots & T_nS_n & \ldots \\ \vdots & \vdots & \ddots & \vdots & \end{pmatrix}$$

In other words, F is generated by all relations of the form $[T_i, S_i]$ $i = 1, 2, 3, \ldots\ldots, n$ and $[T_i, S_j] + [T_j, S_i]$ $i < j$

3. The free loop space on a wedge of even spheres.

3.1. *Graded vector space structure and torsion.*

The following theorem is due to J. E. Roos.

THEOREM 3.1.1. *Let* $\mathrm{X} = \mathbf{S}^4 \bigvee \mathbf{S}^4$ *and put* $\mathrm{H}(\mathbf{S}^4 \bigvee \mathbf{S}^4, k) = k + V$, *where* $V = < x, y >$ *is a vector space of dimension 2 and both x and y are in degree 4, then we have the following explicit formula for* $\mathrm{H}^*(\mathscr{L}\mathrm{X}, k)$:

(3.1.1)

deg	0	$4-1$	4	$2(4-1)$	$2(4-1)+1$	$3(4-1)$	$3(4-1)+1$	$\cdots\cdots$
$\mathrm{H}^*(\mathscr{L}\mathrm{X}, k)$	k	$1 \otimes V$	V	$1 \otimes \mathrm{Ker}\,\mathrm{S}_2$	$\dfrac{V^{\otimes 2}}{\mathrm{Im}\,\mathrm{S}_2}$	$1 \otimes \mathrm{Ker}\,\mathrm{S}_3$	$\dfrac{V^{\otimes 3}}{\mathrm{Im}\,\mathrm{S}_3}$	$\cdots\cdots$

where S_n *is defined in* (1.2.2). *Moreover we have*

$$(3.1.2)\qquad |\, \mathrm{H}^{s(4-1)}(\mathscr{L}\mathrm{X},k)\,| = |\, 1\otimes \mathrm{Ker}\, \mathrm{S}_s \,| = \left| \frac{V^{\otimes s}}{\mathrm{Im}\, \mathrm{S}_s} \right|$$

$$= |\, \mathrm{H}^{s(4-1)+1}(\mathscr{L}\mathrm{X},k)\,| = \frac{1}{s}\sum_{i=1}^{s}(-1)^{i(s-1)}2^{(i,s)}$$

In the above formula (i,s) means the greatest common divisor of i and s and $|\,|$ is the dimension as a vector space over k.

PROOF. (cf. [20]).

Definition 1.1.1 implies that the shuffle product in

$$\mathrm{Tor}_t^{\mathrm{H}^*(\mathrm{X},k)\otimes \mathrm{H}^*(\mathrm{X},k)}(\mathrm{H}^*(\mathrm{X},k),\mathrm{H}^*(\mathrm{X},k))$$

is given by

$$\lambda[a_1\otimes a_2\otimes \ldots\ldots \otimes a_p] * \mu[a_{p+1}\otimes a_{p+2}\otimes \ldots\ldots\otimes a_{p+q}]$$
$$= \sum_{(p,q)-\text{shuffles}(\sigma)} \mathrm{sign}(\sigma)\lambda\mu[a_{\sigma^{-1}(1)}\otimes a_{\sigma^{-1}(2)}\otimes\ldots\ldots\otimes a_{\sigma^{-1}(p+q)}],$$

when $\mathrm{X} = \mathrm{H}^*(\mathscr{L}(\mathrm{S}^4 \bigvee \mathrm{S}^4),\mathrm{k})$, since all elements of $\mathrm{H}^*(\mathrm{S}^4\bigvee \mathrm{S}^4)$ are concentrated in even degrees.

If we put:

$$\mathrm{N} = \frac{1+s+s^2+\cdots+s^{n-1}}{n}$$

and define the map:

$$\mathrm{B}: V^{\otimes n} \longrightarrow V^{\otimes n+1}$$

by

$$\mathrm{B}(v_1\otimes v_2\otimes\cdots\otimes v_n) = \sum_{i=1}^{n}(-1)^{in}1\otimes v_i\otimes v_{i+1}\otimes\cdots\otimes v_n\otimes v_1\otimes\cdots\otimes v_{i-1}.$$

Then it is clear that

$$1\otimes \mathrm{Ker}\,\mathrm{S}_n = 1\otimes \mathrm{N}(V^{\otimes n}) = \mathrm{B}(V^{\otimes n}).$$

Moreover we have

$$H^{3n+1}(\mathscr{L}(X,k)) = \frac{V^{\otimes n}}{\mathrm{ImS}_n} \subseteq \mathrm{Tor}_{n-1}^{H^*(X,k)\otimes H^*(X,k)}(H^*(X,k),H^*(X,k))$$

$$= \frac{V^{\otimes n}}{\mathrm{ImS}_n}\coprod 1\otimes \mathrm{Ker}(S_{n-1})$$

$$\big\downarrow B$$

$$H^{3n}(\mathscr{L}(X,k)) = 1\otimes \mathrm{Ker}(S_n) \subseteq \mathrm{Tor}_{n}^{H^*(X,k)\otimes H^*(X,k)}(H^*(X,k),H^*(X,k))$$

$$= \frac{V^{\otimes n+1}}{\mathrm{ImS}_{t+1}}\coprod 1\otimes \mathrm{KerS}_n$$

where $X = \mathbf{S}^4 \bigvee \mathbf{S}^4$.

The proofs of the following two lemmas are essentially identical to those of Lemmas 2.1.3 and 2.1.4.

LEMMA 3.1.2. *If*

$$\{B(\alpha_1), B(\alpha_2), \ldots\ldots, B(\alpha_s)\}$$

is a k–basis of

$$H^{3n}(\mathscr{L}(S^4 \bigvee S^4), k) = 1\otimes \mathrm{KerS}_n,$$

then

$$\{[\alpha_1], [\alpha_2], \ldots\ldots, [\alpha_s]\}$$

is a k–basis of

$$H^{3n+1}(\mathscr{L}(S^4 \bigvee S^4), k) = \frac{V^{\otimes n}}{S_n}.$$

LEMMA 3.1.3.

$$B(\alpha) = 0 \Rightarrow [\alpha] = 0 \mathrm{in} \frac{V^{\otimes n}}{S_n}.$$

In the last part of this section we try to compute the torsion part in

$$H^*(\mathscr{L}(S^4 \bigvee S^4), Z)$$

LEMMA 3.1.4. *The spectral sequence*

$$E_2^{-p,q} = \mathrm{Tor}_p^{H^*(X\times X,k)}(H^*(X,k), H^*(X;k))^q \Rightarrow H^n(\mathscr{L}X,k)$$

degenerates in the case

$$X = S^d \bigvee S^d \cdots \bigvee S^d \quad (d, \text{ even})$$

over a field k of arbitrary characteristic.

PROOF. See Lemma 2.1.5.

Theorem 1.2.2, which asserted that the dimension of invariant under the action of

$$s(v_1 \otimes v_2 \otimes \ldots\ldots \otimes v_n) = (-1)^{n-1} v_n \otimes v_1 \otimes \ldots\ldots \otimes v_{n-1}$$

is independent of the characteristic of the field if the characteristic is not 2, together with Lemma 3.1.4 imply that

$$H^*(\mathscr{L}(S^4 \bigvee S^4), Z)$$

has only 2-torsion part. Moreover the series of this 2-torsion can be calculated by considering the equality of formal power series, provided by the universal coefficient theorem:

$$\sum_{i \geq 0} v_2(H_i(\mathscr{L}X, Z))t^i = (1+t)^{-1}(H^{Z/2Z}_{\mathscr{L}X}(t) - H^{Q}_{\mathscr{L}X}(t)), \tag{3.1.3}$$

where $v_p(Ab)$ denotes the minimal number of generators of the p−torsion subgroup of an abelian group Ab.

Theorem 3.1.1 implies that over a field of characteristic zero the Hilbert series of

$$H^*(\mathscr{L}(S^4 \bigvee S^4), Z)$$

is

$$H^{Q}_{\mathscr{L}X}(t)) = 1 + (1+t)\sum_{n \geq 1} |\frac{V^{\otimes n}}{S_n}| t^{3n} \tag{3.1.4}$$

where

$$|\frac{V^{\otimes n}}{S_n}| = \frac{1}{n}\sum_{i=1}^{n}(-1)^{i(n-1)} 2^{(i,n)}$$

and over a field of characteristic 2

$$H^*(\mathscr{L}(S^4 \bigvee S^4), Z)$$

has the Hilbert series:

$$H^{Z/2Z}_{\mathscr{L}X}(t) = 1 + (1+t)\sum_{n \geq 1} \left|\frac{V^{\otimes n}}{T_n}\right| t^{3n} \tag{3.1.5}$$

where

$$\left|\frac{V^{\otimes n}}{T_n}\right| = \frac{1}{n}\sum_{i=1}^{n} 2^{(i,n)}.$$

By replacing (3.1.4) and (3.1.5) in (3.1.3) we obtain:

$$\sum_{i\geq 0} v_2(\mathrm{H}_i(\mathscr{L}X, Z))t^i = \sum_{n\geq 1}\left(\left|\frac{V^{\otimes n}}{T_n}\right| - \left|\frac{V^{\otimes n}}{S_n}\right|\right)t^{3n} \tag{3.1.6}$$

and this can be written as

$$\sum_{i\geq 0} v_2(\mathrm{H}_i(\mathscr{L}X, Z))t^i = \sum_{n\geq 2,\text{even}} \frac{2}{n}(2 + 2^{(3,n)} + \cdots\cdots + 2^{(n-1,n)})t^{3n}.$$

By the same argument as above it can easily be seen that

$$\mathrm{H}^*(\mathscr{L}(\mathrm{S}^g \bigvee \mathrm{S}^g \cdots\cdots \bigvee \mathrm{S}^g), k) \quad (g, \text{ even})$$

has only 2-torsion.

3.2. *The ring structure of* $\mathrm{H}^*(\mathscr{L}(\mathbf{S}^4 \vee \mathbf{S}^4), k)$.

3.2.1. *Global observations.*

THEOREM 3.2.1. *Let Q^* denote the indecomposables in* $\mathrm{H}^*(\mathscr{L}\mathbf{S}^4 \vee \mathbf{S}^4), k)$. *The elements $Y_1, Y_2, \ldots\ldots, Y_s$ are representatives of a basis of Q^{3n+1} if and only if the elements $X_1 = \mathrm{B}(Y_1), X_2 = \mathrm{B}(Y_2), \cdots, X_s = \mathrm{B}(Y_s)$ are representatives of a basis of Q^{3n}.*

PROOF. See the proof of Theorem 2.2.2.

Denote the ring $\mathrm{H}^{3*}(\mathscr{L}(\mathbf{S}^{\mathrm{d}} \vee \mathbf{S}^{\mathrm{d}}), k)$ by R and denote $\mathrm{H}^{3*+1}(\mathscr{L}(\mathbf{S}^{\mathrm{d}} \vee \mathbf{S}^{\mathrm{d}}), k)$ by M. Then $M = s^{-1}\bar{R}$ (where $\bar{R} = R/k$ and $(s^{-1}\bar{R})^p = (\bar{R})^{p-1}$) is an R module (using shuffle product) and we have

Theorem 3.2.2. *Let* $\mathrm{X} = \mathbf{S}^4 \vee \mathbf{S}^4$, *then the ring structure of* $\mathrm{H}^*(\mathscr{L}\mathrm{X}, k)$ *is* $R \oplus s^{-1}\overline{R}$, i.e., *the trivial extension of R by $s^{-1}\bar{R}$, where* $R = \mathrm{H}^{3*}(\mathscr{L}(\mathbf{S}^4 \vee \mathbf{S}^4), k)$.

PROOF. See the proof of Theorem 2.2.4.

REMARK 3.2.3. Theorem 3.2.2 is true even for $\mathrm{X} = \mathbf{S}^{d_1} \vee \mathbf{S}^{d_2} \vee \cdots\cdots \vee \mathbf{S}^{d_n} (d_i, \text{ even})$.

Although we are not able to compute the entire ring structure of $R \oplus s^{-1}\bar{R}$ we have the following result, the proof of which is essetially identical to that of Lemma 2.2.6.

LEMMA 3.2.4. *All relations involving elements of degree* $3*+1$ *except commutators of elements of degree* $3*+1$, *are induced by* B^{-1} *from relations involving only elements of degree* $3*$.

THEOREM 3.2.5. *If*

$$H^{3*}(\mathscr{L}(S^4 \bigvee S^4), k) \cong T(W)/I.$$

Then $H^*(\mathscr{L}(S^4 \vee S^4), k)$ *is isomorphic as a ring to*

$$\frac{T(W \oplus W')}{(I + I' + W'^2 + C)}$$

where $W' = B^{-1}(W), I' = 1 \otimes B^{-1}(I)$ and $C = \{XY - YX | X \in W, Y \in W'\}$.

PROOF. See the proof of Theorem 2.2.7.

REMARK 3.2.6. As a result of this section we show that

$$\overline{HC}_*(A) \text{ and } H^{3*}(\mathscr{L}(S^4 \bigvee S^4), k) \tag{3.2.1}$$

have the same ring structure. Here $A = k + V$ is the trivial extension of k by V, where V is a vector space of dimension 2 and $\overline{HC}_*(A)$ is the reduced cyclic homology of A.

In [13], Loday and Quillen have defined a product:

$$HC_n(A) \otimes HC_p(A) \longrightarrow HC_{n+p+1}(A)$$

for a commutative $k-$algebra A as follows:
Let

$$x \in (\beta(A)_{\text{norm}})_{lm} = A \otimes \bar{A}^{m-l} \text{ and } y \in (\beta(A)_{\text{norm}})_{rs} = A \otimes \bar{A}^{s-r}$$

where $\bar{A} = A/k$ and where $\beta(A)_{\text{norm}}$ is a double complex with:

$$HC_n(A) = H_n(\text{Tot}\beta(A)_{\text{norm}})$$

(c.f. [13]. page 571). Define the product as:

$$x \bullet y = \begin{cases} x * B(y) & \text{if } r = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.2}$$

(Note that $x \bullet y \in (\beta(A)_{\text{norm}})_{l+r,m+s+1}$) Then this formula is extended to Tot $\beta(A)_{\text{norm}} \otimes$ Tot $\beta(A)_{\text{norm}}$ by linearity. In (3.2.2) $*$ is the shuffle product and $B : A \otimes \bar{A}^{\otimes n} \longrightarrow A \otimes \bar{A}^{\otimes n+1}$ is defined by:

(3.2.3) $$\begin{aligned} &B(a_0 \otimes a_1 \otimes \cdots \otimes a_n) \\ &= \sum_{i=1}^{n} (-1)^{in} 1 \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}. \end{aligned}$$

LEMMA 3.2.7.

$$B(x * B(y)) = B(x) * B(y).$$

PROOF. ([13]. page 575).

This Lemma shows that the product structure on cyclic homology is compatible with the shuffle product on Hochschild homology on the sense that the following diagram commutes.

(3.2.4) $$\begin{array}{ccc} HC_n(A) \otimes HC_m(A) & \stackrel{B\otimes B}{\longrightarrow} & HH_{n+1}(A) \otimes HH_{m+1}(A) \\ \bullet\downarrow & & \downarrow * \\ HC_{n+m+1}(A) & \stackrel{B}{\longrightarrow} & HH_{n+m+2}(A) \end{array}$$

In the above diagram $\bullet$ is the product defined in (3.2.2), $*$ is the shuffle product and the map B is the same map in the long exact sequence:

(3.2.5) $$\cdots\cdots \longrightarrow HH_n(A) \stackrel{I}{\longrightarrow} HC_n(A) \stackrel{S}{\longrightarrow} HC_{n-2}(A) \stackrel{B}{\longrightarrow} HH_{n-1}(A) \longrightarrow \cdots\cdots$$

The cyclic homology $HC_*(A)$ and the reduced cyclic homology $\overline{HC}_*(A)$ of $A = k + V$, the trivial extension of k by V has been computed by many authors e.g. [13] as follows:

$$HC_n(A) = \begin{cases} k + V^{\otimes n+1}/\operatorname{Im} S_{n+1} & \text{if } n = 2t \\ V^{\otimes n+1}/\operatorname{Im} S_{n+1} & \text{otherwise} \end{cases}$$

and

$$\overline{HC}_*(A) = \frac{V^{\otimes n+1}}{\operatorname{Im} S_{n+1}}$$

where $S_n : V^{\otimes n} \longrightarrow V^{\otimes n}$ is defined in (1). Note that by definition of the product $\bullet$ in (3.2.2) we have:

(3.2.6) $$x \in k \subseteq HC_{2t}(A) t \geq 1 \Longrightarrow x \bullet y = 0 \; \forall \, y \in HC_*(A).$$

Consider the following two tables:

(3.2.7)

$HC_*(A)$	V	$\frac{V^{\otimes 2}}{\operatorname{Im} S_2}$	$\frac{V^{\otimes 3}}{\operatorname{Im} S_3}$	$\frac{V^{\otimes 4}}{\operatorname{Im} S_4}$	$\frac{V^{\otimes 5}}{\operatorname{Im} S_5}$	$\frac{V^{\otimes 6}}{\operatorname{Im} S_6}$	$\frac{V^{\otimes 7}}{\operatorname{Im} S_7}$	......
deg	0	1	2	3	4	5	6	.....
dim	2	1	4	4	8	10	20	.....

(3.2.8)

$H^{3n}(\mathscr{L}(S^4 \bigvee S^4), k)$	$1 \otimes V$	$1 \otimes \operatorname{Ker} S_2$	$1 \otimes \operatorname{Ker} S_3$	$1 \otimes \operatorname{Ker} S_4$	$1 \otimes \operatorname{Ker} S_5$	$1 \otimes \operatorname{Ker} S_6$	$1 \otimes \operatorname{Ker} S_7$	..
deg	3	6	9	12	15	18	21	..
dim	2	1	4	4	8	10	20	..

and the product:

$$\frac{V^{\otimes n+1}}{\operatorname{Im} S_{n+1}} \otimes \frac{V^{\otimes m+1}}{\operatorname{Im} S_{m+1}} \bullet \longrightarrow \frac{V^{\otimes n+m+2}}{\operatorname{Im} S_{n+m+2}} \tag{3.2.9}$$

Let z be an element in the image of this product ($\bullet$ was defined in (3.2.2)), i.e., $z = x \bullet y$ for some $x \in \frac{V^{\otimes n+1}}{\operatorname{Im} S_{n+1}}$ and $y \in \frac{V^{\otimes m+1}}{\operatorname{Im} S_{m+1}}$

$$z = x \bullet y = x * B(y).$$

By Lemma 3.2.7:

$$B(z) = B(x \bullet y) = B(x * B(y)) = B(x) * B(y).$$

This shows that $B(z)$ is in the image of the product:

$$(1 \otimes \operatorname{Ker}S_{n+1}) \otimes (1 \otimes \operatorname{Ker}S_{m+1}) * \longrightarrow (1 \otimes \operatorname{Ker}S_{n+m+2}), \tag{3.2.10}$$

where $*$ is the shuffle product. Conversely if $B(z)$ is an element in the image of the product in (3.2.10), then z is in the image of the product in (3.2.9). This shows that

$$\overline{HC}_*(A) \text{and} H^{3*}(\mathscr{L}(S^4 \bigvee S^4), k)$$

have the same ring structure.

3.2.2. *Explicit low-dimensional calculations.*

THEOREM 3.2.8. *The ring* $R = H^{3*}(\mathscr{L}(S^4 \bigvee S^4), k)$ *is isomorphic in degrees less than 14 to the ring*

$$R = \frac{k[a, b, c, d, e, f, g]}{(I)} \tag{3.2.11}$$

where $\deg a = \deg b = 4, \deg c = \deg d = \deg e = \deg f = 9$ *and* $\deg g = 12$, *and where I is generated by the following* 11 *elements:*

$$a^2, b^2, c^2, d^2, e^2, f^2, ac, bf, (ad - bc), (ae - bd), (af - be).$$

PROOF. Recall that $H^*(\mathscr{L}(S^4 \bigvee S^4), k)$ has the following table

$H^*(\mathscr{L}X, k)$	k	$1 \otimes V$	V	$1 \otimes \operatorname{Ker} S_2$	$\frac{V^{\otimes 2}}{\operatorname{Im} S_2}$	$1 \otimes \operatorname{Ker} S_3$	$\frac{V^{\otimes 3}}{\operatorname{Im} S_3}$	$1 \otimes \operatorname{Ker} S_4$	$\frac{V^{\otimes 4}}{\operatorname{Im} S_4}$	......
deg	0	3	4	6	7	9	10	12	13	......
dim	1	2	2	1	1	4	4	4	4	......

In the above table $H(S^4 \bigvee S^4, k) = k + V$, where $V = < x, y >$ is a vector space of dimension 2 and both x and y are in degree 4. We consider $H(S^4 \bigvee S^4, k)$ in dimensions 3, 6, 9, 12.

$$H^3 * H^3 \longrightarrow H^6$$

$$H^3 = \langle a = 1 \otimes x,\, b = 1 \otimes y \rangle, \qquad H^6 = \left\langle \frac{1 \otimes x \otimes y - 1 \otimes y \otimes x}{2} \right\rangle$$

(3.2.12) $$(1 \otimes x) * (1 \otimes x) = 0 \qquad (1 \otimes y) * (1 \otimes y) = 0$$
$$(1 \otimes x) * (1 \otimes y) = 1 \otimes x \otimes y - 1 \otimes y \otimes x$$

In (3.2.12) we have used the shuffle product

$$(1 \otimes a_1) * (1 \otimes a_2) = 1 \otimes a_1 \otimes a_2 - 1 \otimes a_2 \otimes a_1.$$

The product is onto and no generator in degree six is needed.

$$H^9 = \left\langle c = 1 \otimes x \otimes x \otimes x,\ d = \frac{1 \otimes x \otimes x \otimes y + 1 \otimes y \otimes x \otimes x + 1 \otimes x \otimes y \otimes x}{3}, \right.$$

$$\left. e = \frac{1 \otimes x \otimes y \otimes y + 1 \otimes y \otimes x \otimes y + 1 \otimes y \otimes y \otimes x}{3},\ f = 1 \otimes y \otimes y \otimes y \right\rangle.$$

The fact that $a^2 = (1 \otimes x)^2 = 0$ and $b^2 = (1 \otimes y)^2 = 0$, implies that

$$H^3 * H^6 = H^3 * H^3 * H^3 = 0$$

and hence no element of degree 9 is obtained by the elements of previous degrees. So we need four generators $a, b, c,$ and d in degree 9. By an easy calculation the shuffle product

$$(1 \otimes a_1 \otimes a_2 \otimes a_3) * (1 \otimes a_4 \otimes a_5 \otimes a_6)$$
$$= \sum_{(3,3)-\text{shuffles}(\sigma)} \text{sign}(\sigma) 1 \otimes a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \cdots\cdots \otimes a_{\sigma^{-1}(12)}$$

implies that:

$$c^2 = d^2 = e^2 = f^2 = 0 \tag{3.2.13}$$

and the shuffle product:

$$(1 \otimes a_1) * (1 \otimes a_2 \otimes a_3 \otimes a_4) = 1 \otimes a_1 \otimes a_2 \otimes a_3 \otimes a_4 \tag{3.2.14}$$
$$- 1 \otimes a_2 \otimes a_1 \otimes a_3 \otimes a_4 + 1 \otimes a_2 \otimes a_3 \otimes a_1 \otimes a_4$$
$$- 1 \otimes a_2 \otimes a_3 \otimes a_4 \otimes a_1$$

implies that

$$ac = bf = 0, \qquad ad = bc, \qquad ae = bd, \qquad \text{and} \qquad af = be. \tag{3.2.15}$$

Put

$$H^{12} = \langle g_1 =$$
$$\frac{1 \otimes x \otimes x \otimes x \otimes y - 1 \otimes y \otimes x \otimes x \otimes x + 1 \otimes x \otimes y \otimes x \otimes x - 1 \otimes x \otimes x \otimes y \otimes x}{4}$$

$$g_2 =$$
$$\frac{1 \otimes x \otimes x \otimes y \otimes y - 1 \otimes y \otimes x \otimes x \otimes y + 1 \otimes y \otimes y \otimes x \otimes x - 1 \otimes x \otimes y \otimes y \otimes x}{4}$$

$$g_3 = \frac{1 \otimes x \otimes y \otimes x \otimes y - 1 \otimes y \otimes x \otimes y \otimes x}{2}$$

$$g_4 =$$
$$\left. \frac{1 \otimes x \otimes y \otimes y \otimes y - 1 \otimes y \otimes x \otimes y \otimes y + 1 \otimes y \otimes y \otimes x \otimes y - 1 \otimes y \otimes y \otimes y \otimes x}{4} \right\rangle.$$

By using the shuffle product (3.2.14), it can easily be proved that the image of the product

$$H^3 * H^9 \longrightarrow H^{12}$$

has dimension 3 and hence one generator in degree 12 is needed. We call this generator g.

REMARK 3.2.9. The above calculations can be also carried out by using the model of Sullivan and Vigué. See [26].

DEFINITION 3.2.10. We say that a local ring (R, m) satisfies the condition M_3 if

$$P_R(z)^{-1} = (1 + 1/z)/A(z) - H_R(-z)/z, \tag{3.2.16}$$

where $k = R/m, A$ is the subalgebra of $\operatorname{Ext}^*_R(k,k)$ generated by $\operatorname{Ext}^1_R(k,k), P_R(z)$, is the Poincaré-Betti series, $A(z)$ is the Hilbert series of A and $H_R(z)$ is the Hilbert series of R.

In the graded case, i.e., when R is a quotient of a polynomial ring

$$k[X_1, X_2, \ldots\ldots X_n]$$

(where the generators X_i have degree 1) by an ideal generated by homogeneous elements, it is clear that the vector space $\operatorname{Ext}^*_R(k,k)$ has an extra grading so that we can introduce a Poincaré-Betti series in two variables

$$P_R(x,y) = \sum_{i\geq 0, j\geq i} \dim_k \operatorname{Ext}^i_R(k,k)^j x^i y^j. \tag{3.2.17}$$

It follows easily that if (R, m) satisfies M_3 then we have the even more precise two-variable version of (3.2.16):

$$P_R(x,y)^{-1} = (1 + 1/x)/A(xy) - H_R(-xy)/x. \tag{3.2.18}$$

We refer the reader to [22] for more details about the condition M_3.

THEOREM 3.2.11. *If the finitely generated ring R satisfies the condition M_3, then the trivial extension ring $R \oplus \bar{R}$ also satisfies the condition M_3.*

PROOF. In the proof of Theorem 2.2.13 we proved that

$$\begin{aligned} &H_{R\oplus\bar{R}}(t) = H_R(t) + (H_R(t) - 1) \quad \text{and} \\ &P_{R\oplus\bar{R}}(x,y) = P_R(x,y)/(1 - (P_R(x,y) - 1)). \end{aligned} \tag{3.2.19}$$

Let A be the subalgebra of $\operatorname{Ext}^*_R(k,k)$ generated by $\operatorname{Ext}^1_R(k,k)$ and A' be the subalgebra of $\operatorname{Ext}^*_{R\oplus\bar{R}}(k,k)$ generated by $\operatorname{Ext}^1_{R\oplus\bar{R}}(k,k)$. Theorem 2.2.13 implies that

$$A'(t) = A(t)/(1 - (A(t) - 1)) \tag{3.2.20}$$

and hence

$$A(t) = 2A'(t)/(1 + A'(t)). \tag{3.2.21}$$

Moreover since R satisfies the condition M_3 we have

$$P_R(x,y)^{-1} = (1 + 1/x)/A(xy) - H_R(-xy)/x. \tag{3.2.22}$$

Now

$$
\begin{aligned}
P_{R\oplus\bar{R}}(x,y)^{-1} &= (P_R(x,y)/(1-(P_R(x,y)-1)))^{-1}\\
&= 2P_R(x,y)^{-1}-1\\
&= 2((1+1/x)/A(xy)-H_R(-xy)/x)-1\\
&= 2(1+1/x)((1+A'(xy))/(2A'(xy)))-2(H_{R\oplus\bar{R}}(-xy)+1)/(2x)-1\\
&= (1+1/x)/(A'(xy))+1+1/x-H_{R\oplus\bar{R}}(-xy)/x-1/x-1\\
&= (1+1/x)/(A'(xy))-H_{R\oplus\bar{R}}(-xy)/x.
\end{aligned}
$$

This shows that $R\oplus\bar{R}$ satisfies the condition M_3.

We devote the last part of this section to prove that the ring $\mathrm{H}^*(\mathscr{L}(S^4\bigvee S^4),k)$ in low dimensions satisfies the condition M_3. In Theorem 3.2.2 we proved that the ring $\mathrm{H}^*(\mathscr{L}(S^4\bigvee S^4),k)$ is the trivial extension of $R=\mathrm{H}^{3*}(\mathscr{L}(\mathrm{S}^4\bigvee \mathrm{S}^4),k)$ and in Theorem 3.2.11 we proved that the trivial extension ring $R\oplus\bar{R}$ satisfies the condition M_3 If the finitely generated ring R satisfies the condition M_3. So in order to show that the whole ring $\mathrm{H}^*(\mathscr{L}(S^4\bigvee S^4),k)$ in low dimensions satisfies the condition M_3 it is enough to show that the ring $R=\mathrm{H}^{3*}(\mathscr{L}(\mathrm{S}^4\bigvee \mathrm{S}^4),k)$ in low dimensions satisfies the condition M_3.

THEOREM 3.2.12. *the ring $\mathrm{H}^{3*}(\mathscr{L}(S^4\bigvee S^4),k)$ in low dimensions satisfies the condition M_3 if all generators are considered to be in degree* 1.

In order to do prove Theorem 3.2.12, we take $R_1=\mathrm{H}^{3*}(\mathscr{L}(S^4\bigvee S^4),k)$ in low dimensions as

$$R_1=\frac{k[a,b,c,d]}{(a^2,b^2,c^2,d^2,ac,(ad-bc))}$$

The Hilbert series of R_1 is

$$H_{R_1}(t)=1+4t+4t^2$$

This shows that $\mathcal{M}^3=0$ ($\mathcal{M}$ is the maximal ideal of R_1) and hence the ring R_1 satisfies automatically the condition M_3.

In this case the Poincaŕe-Betti series of R_1 can be computed as follows. We first use MACAULAY to make a prediction. Introduce R_1 to MACAULAY (more details in [22]), issue the command:
nres R_1 t 8
and then break the computations after a while and issue the new command:
betti t.

MACAULAY produces the following table of graded Betti numbers for R_1 :
(3.2.23)

total :	**1**	**4**	**12**	**33**	**89**	**240**	**649**	**1758**	**4765**
0 :	1	4	12	32	81	200	488	1184	2865
1 :	–	–	–	1	8	40	160	562	1816
2 :	–	–	–	–	–	–	1	12	84

Moreover MACAULAY determines the Hilbert series of R_1 as

$$H_{R_1}(t) = 1 + 4t + 4t^2.$$

Now let

$$f = 1 + 4t + 12t^2 + 32t^3 + 81^4 + 200t^5 + 488t^6 + 1184t^7 + 2865t^8.$$

By using two succesive commands:

convert(″, series); and convert(″, ratpoly);

we convert f into

$$A(t) = -\frac{1}{(t-1)^2(t^2+2t-1)} \tag{3.2.24}$$

where A is the subalgebra of $\mathrm{Ext}^*_{R_1}(k,k)$ generated by $\mathrm{Ext}^1_{R_1}(k,k)$. Substitution of $A(t)$ and $H_{R_1}(t)$ in (3.2.18) (t = xy), we obtain

$$\begin{aligned} P_{R_1}(x,y) &= -\frac{1}{(-1+4xy-4x^2y^2+x^3y^4+x^4y^4)} \\ &= 1 + 4yx + 12y^2x^2 + (32y^3 + y^4)x^3 + (81y^4 + 8y^5)x^4 \\ &+ (200y^5 + 40y^6)x^5 + (488y^6 + 160y^7 + y^8)x^6 \\ &+ (1184y^7 + 562y^8 + 12y^9)x^7 \\ &+ (2865y^8 + 1816y^9 + 84y^{10})x^8 + O(x^9) \end{aligned} \tag{3.2.25}$$

and this determines the table (3.2.23) completely. This was a prediction by Macaulay. The only thing that is needed to be proved is why the subalgebra A generated by $\mathrm{Ext}^1_{R_1}(k,k)$ has the Hilbert series as asserted in (3.2.24). We prove this as follows:

The subalgebra $A = U(\eta)(\eta$ is the Lie subalgebra of the homotopy Lie algebra of R_1) has according to the recipe of Löfwall [15] the presentation:

$$\frac{k\langle T_1, T_2, T_3, T_4\rangle}{([T_1,T_2],[T_2,T_4],[T_3,T_4],[T_1,T_4]+[T_2,T_3])}$$

where $k < T_1, T_2, T_3, T_4 >$ is the free associative k-algebra in variables

T_1, T_2, T_3, T_4 of degree 1 and where $[T_i, T_j] = T_iT_j + T_jT_i$ is the graded commutator.

LEMMA 3.2.13. *The Lie algebra ζ generated by $T_1, T_3, [T_1, T_4]$ is an ideal in η.*

Here η is

$$\frac{L\langle T_1, T_2, T_3, T_4\rangle}{([T_1, T_2], [T_2, T_4], [T_3, T_4], [T_1, T_4] + [T_2, T_3])}$$

where $L < T_1, T_2, T_3, T_4 >$ is the free lie-algebra.

PROOF. We prove the Lemma by induction. Using the Jacobi identity, i.e.,

$$(-1)^{|a||c|}[a, [b, c]] + (-1)^{|b||a|}[b, [c, a]] + (-1)^{|c||b|}[c, [a, b]] = 0$$

we have

$$[T_2, T_1] = 0, \qquad [T_2, T_3] = -[T_1, T_4], \qquad [T_2, [T_1, T_4]] = 0,$$

$$[T_4, T_1] = [T_1, T_4], \qquad [T_4, T_3] = 0, \qquad [T_4, [T_1, T_4]] = -[T_4, [T_2, T_3]] = 0.$$

Now assume for ω in ζ we have $[T_2, \omega]$ and $[T_4, \omega]$ is in ζ, then

$$[T_2, [T_3, \omega]] = [[T_2, T_3], \omega] + [T_3, [T_2, \omega]] = [-[T_1, T_4], \omega] + [T_3, [T_2, \omega]] \in \zeta$$

$$[T_2, [T_1, \omega]] = [T_1, [T_2, \omega]] + [[T_2, T_1], \omega] = [T_1, [T_2, \omega]] \in \zeta$$

$$[T_2, [[T_1, T_4], \omega]] = [[T_1, T_4], [T_2, \omega]] \in \zeta$$

$$[T_4, [T_3, \omega]] = [T_3, [T_4, \omega]] \in \zeta$$

$$[T_4, [T_1, \omega]] = [T_1, [T_4, \omega]] + [[T_4, T_1], \omega] \in \zeta$$

$$[T_4, [[T_1, T_4], \omega]] = [[T_1, T_4], [T_4, \omega]] \in \zeta.$$

Now the exact sequence

$$0 \longrightarrow \zeta \longrightarrow \eta \longrightarrow L(T_2, T_4)/[T_2, T_4] \longrightarrow 0 \tag{3.2.26}$$

shows that

$$A(t) \leq \frac{1}{(1 - 2t - t^2)} \cdot \frac{(1 + t)^2}{(1 - t^2)^2}. \tag{3.2.27}$$

Because

$$U(L(T_2,T_4)/[T_2,T_4])(t) = \frac{(1+t)^2}{(1-t^2)^2} \text{ and } U(\zeta)(t) \le \frac{1}{(1-2t-t^2)}.$$

Equality in (3.2.27) holds if the ideal ζ in (3.2.26) is free. To prove that this ideal is really free let $L(T_2,T_4)/[T_2,T_4]$ acts on $\zeta = L(T_1,T_3,[T_1,T_4])$ by

$$[T_2,T_1]=0, \qquad [T_2,T_3]=-[T_1,T_4], \qquad [T_2,[T_1,T_4]]=0,$$

$$[T_4,T_1]=[T_1,T_4], \qquad [T_4,T_3]=0, \qquad [T_4,[T_1,T_4]]=0.$$

Take the semi-direct product. This semi-direct product is a quotient of

$$\eta = \frac{L\langle T_1,T_2,T_3,T_4\rangle}{([T_1,T_2],[T_2,T_4],[T_3,T_4],[T_1,T_4]+[T_2,T_3])}$$

and has the series

$$\frac{1}{(1-2t-t^2)}\cdot\frac{(1+t)^2}{(1-t^2)^2}.$$

Hence

$$A(t) \ge \frac{1}{(1-2t-t^2)}\cdot\frac{(1+t)^2}{(1-t^2)^2}. \tag{3.2.28}$$

The relations (3.2.27) and (3.2.28) show that $A(t)$ has the desired Hilbert series.

Now we take another sample namely

$$R_2 = \frac{k[a,b,c,d,e,f]}{(a^2,b^2,c^2,d^2,e^2,f^2,ac,bf,(ad-bc),(ae-bd),(af-be))}.$$

MACAULAY produces the following table of graded Betti numbers for R_2 :
(3.2.29)

total :	**1**	**6**	**26**	**101**	**376**	**1376**	**5001**	**18126**	**65626**
0 :	1	6	26	100	364	1288	4488	15504	53296
1 :	–	–	–	1	12	88	512	2604	12144
2 :	–	–	–	–	–	–	1	18	1186

Moreover MACAULAY determines the Hilbert series of R_2 as

$$H_{R_2}(t) = 1+6t+10t^2+4t^3+t^4.$$

Now let

$f = 1 + 6t + 26t^2 + 100t^3 + 364t^4 + 1288t^5 + 4488t^6 + 15504t^7 + 53296t^8.$

By using two succesive commands:

convert(″, series) ; and convert(″, ratpoly) ;

we convert f into

$$A(t) = -1/(2t-1)(2t^2-4t+1) \tag{3.2.30}$$

where A be the subalgebra of $\mathrm{Ext}^*_{R_2}(k,k)$ generated by $\mathrm{Ext}^1_{R_2}(k,k)$. Substitution of $A(t)$ and $H_{R_2}(-t)$ in (3.2.18) (t = xy), we obtain

$$\begin{aligned}(3.2.31)\quad P(x,y) &= -\frac{1}{(-1+6xy-10x^2y^2+4x^3y^3+x^3y^4)}\\ &=1+6yx+26y^2x^2+(100y^3+y^4)x^3+(364y^4+12y^5)x^4\\ &+(1288y^5+88y^6)x^5+(4488y^6+512y^7+y^8)x^6\\ &+(15504y^7+2604y^8+18y^9)x^7\\ &+(53296y^8+12144y^9+186y^{10})x^8+O(x^9)\end{aligned}$$

and this determines the table (3.2.29) completely. This was a prediction by Macaulay. We prove that the subalgebra A generated by $\mathrm{Ext}^1_{R_2}(k,k)$ has the Hilbert series as asserted in (3.2.30). We prove this as follows:

The subalgebra $A = U(\eta)(\eta$ is the Lie sub algebra of the homotopy Lie algebra of $R_2)$ has according to the recipe of Löfwall [15] the presentation:

$$A = \frac{k\langle T_1, T_2, T_3, T_4, T_5, T_6\rangle}{I}$$

$$I = ([T_1,T_2],[T_3,T_4],[T_3,T_5],[T_3,T_6],[T_4,T_5],[T_4,T_6],[T_5,T_6],$$

$$[T_1,T_4]+[T_2,T_3],[T_1,T_5]+[T_2,T_4],[T_1,T_6]+[T_2,T_5]),$$

where $k\langle T_1, T_2, T_3, T_4, T_5, T_6\rangle$ is the free associative k-algebra in variables $T_1, T_2, T_3, T_4, T_5, T_6$ of degree 1 and where $[T_i, T_j] = T_iT_j + T_jT_i$ is the graded commutator. By changing the orders we rewrite A as

$$A = \frac{k\langle T_1, T_2, T_3, T_4, T_5, T_6\rangle}{I} \tag{3.2.32}$$

where

$$I = ([T_1, T_3], [T_1, T_4], [T_3, T_4], [T_2, T_5], [T_1, T_6], [T_3, T_6], [T_4, T_6], [T_1, T_5] + [T_2, T_3], [T_3, T_5] + [T_2, T_4], [T_2, T_6] + [T_4, T_5]). \tag{3.2.33}$$

In order to compute the Hilbert series of A, we use the program BERGMAN of Jörgen Backelin to compute the associated monomial ring of A, i.e., $gr(A)$ because A and $gr(A)$ have the same Hilbert series. To use BERGMAN first create with a text editor the following file:

```
(setq embdim 6)
(LISPFORMINPUT)
((1 1 3)(1 3 1))
((1 1 4)(1 4 1))
((1 1 6)(1 6 1))
((1 2 5)(1 5 2))
((1 3 4)(1 4 3))
((1 3 6)(1 6 3))
((1 4 6)(1 6 4))
((1 2 3)(1 3 2)(1 1 5)(1 5 1))
((1 2 4)(1 4 2)(1 3 5)(1 5 3))
((1 4 5)(1 5 4)(1 2 6)(1 6 2))
(LISPFORMINPUTEND)
```

Give a name to this file (e.g. TEXT). Then start BERGMAN and write:

```
(ncpbhgroebner ''TEXT'' ''t1'' ''t2'' ''t3'').
```

The file $t1$ gives the Gröbner basis for the ideal I in (3.2.33) as follows:

$$G = ([T_3, T_1], [T_4, T_1], [T_4, T_3], [T_5, T_1] + [T_3, T_2], [T_5, T_2], [T_5, T_3] + [T_4, T_2], [T_6, T_1], [T_6, T_2] + [T_5, T_4], [T_6, T_3], [T_6, T_4], [[T_4, T_2], T_1]] + [T_3^2, T_2]) \tag{3.2.34}$$

and hence the associated monomial ring of A has the form:

$$\mathrm{gr}(A) = \frac{k\langle T_1, T_2, T_3, T_4, T_5, T_6\rangle}{\mathrm{In}(I)}$$

where

$$\mathrm{In}(I) = (T_3T_1, T_4T_1, T_4T_3, T_5T_1, T_5T_2, T_5T_3, T_6T_1, T_6T_2, T_6T_3, T_6T_4, T_4T_2T_1) \tag{3.2.35}$$

The file $t2$ gives the Poincaré-Betti series of $\mathrm{gr}(A)$ (up to degree 4) as

$$P_{gr(A)}(x,y) = 1 + 6xy + 10x^2y^2 + x^2y^3 + 5x^3y^3 + x^3y^4 + x^4y^4. \tag{3.2.36}$$

But by [4] page 843 THÉORÈME 1 the following is an $\mathrm{gr}(A)-$free resolution of k.

$$\begin{aligned} 0 \longrightarrow \mathrm{gr}(A)s_{6431} \longrightarrow & \,\mathrm{gr}(A)s_{431} \oplus \mathrm{gr}(A)s_{531} \oplus \mathrm{gr}(A)s_{631} \oplus \mathrm{gr}(A)s_{641} \\ & \oplus \mathrm{gr}(A)s_{643} \oplus \mathrm{gr}(A)s_{6421} \longrightarrow \mathrm{gr}(A)s_{31} \oplus \mathrm{gr}(A)s_{41} \\ & \oplus \mathrm{gr}(A)s_{43} \oplus \mathrm{gr}(A)s_{51} \oplus \mathrm{gr}(A)s_{52} \oplus \mathrm{gr}(A)s_{53} \\ & \oplus \mathrm{gr}(A)s_{61} \oplus \mathrm{gr}(A)s_{62} \oplus \mathrm{gr}(A)s_{63} \oplus \mathrm{gr}(A)s_{64} \\ & \oplus \mathrm{gr}(A)s_{421} \longrightarrow \mathrm{gr}(A)s_1 \oplus \mathrm{gr}(A)s_2 \oplus \mathrm{gr}(A)s_3 \\ & \oplus \mathrm{gr}(A)s_4 \oplus \mathrm{gr}(A)s_5 \oplus \mathrm{gr}(A)s_6 \longrightarrow R \longrightarrow k, \end{aligned}$$

where $\mathrm{gr}(A)s_{i_1,i_2,\cdots,i_r}$ is the free $\mathrm{gr}(A)-$module generated by $s_{i_1,i_2,\cdots,i_r}$ and where $\mathrm{gr}(A)s_{i_1,i_2,\cdots,i_r}$ is sent to $\mathrm{gr}(A)T_{i_1}s_{i_2,\cdots,i_r}$. This shows that $\mathrm{gr}(A)$ has Global-dimension 4 and hence (3.2.36) is the whole Poincaré-Betti series of $\mathrm{gr}(A)$. This gives

$$A(t) = \mathrm{gr}(A)(t) = \frac{1}{P_{\mathrm{gr}(A)}(-1,t)} = \frac{1}{1 - 6t + 10t^2 - 4t^3},$$

since $\mathrm{gr}(A)$ has monomial relations.

To prove that R_2 has really the Poincaré-Betti series as asserted in (3.2.31), it is enough to prove that this ring satisfies the condition M_3. In order to do this we rewrite R_2 and A here and replace R_2 by R and the generators $T_1, T_2, T_3, T_4, T_5, T_6$ of A by X, Y, Z, U, V, W for simplicity. So:

$$R = \frac{k[a,b,c,d,e,f]}{(a^2, b^2, c^2, d^2, e^2, f^2, ac, bf, (ad - bc), (ae - bd), (af - be))}$$

and

$$A = \frac{k\langle X, Y, Z, U, V, W\rangle}{I}$$

where

$$\begin{aligned} I = (& [X,Z], [X,U], [Z,U], [Y,V], [X,W], [Z,W], [U,W], [X,V] \\ & + [Y,Z], [Z,V] + [Y,U], [Y,W] + [U,V]). \end{aligned}$$

The complex $R^* \otimes A$ (for more details look at [22] page 306) is the following (X, Y, Z, U, V, W is the basis for A^1 dual to a, b, c, d, e, f):

$$
0 \longrightarrow R_4^* \otimes A \xrightarrow{d_4=\begin{pmatrix} W \\ U \\ Z \\ X \end{pmatrix}} R_3^* \otimes A \xrightarrow{d_3=\begin{pmatrix} U & W & 0 & 0 \\ Z & 0 & W & 0 \\ 0 & Z & U & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ X & 0 & 0 & W \\ 0 & 0 & 0 & 0 \\ 0 & X & 0 & U \\ 0 & 0 & 0 & 0 \\ 0 & 0 & X & Z \end{pmatrix}} R_2^* \otimes A \longrightarrow
$$

$$
\xrightarrow{d_2=\begin{pmatrix} Z & U & W & V & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Z & V & 0 & U & 0 & W & 0 \\ X & 0 & 0 & Y & 0 & U & V & W & 0 & 0 \\ 0 & X & 0 & 0 & 0 & Z & Y & 0 & V & W \\ 0 & 0 & 0 & X & Y & 0 & Z & 0 & U & 0 \\ 0 & 0 & X & 0 & 0 & 0 & 0 & Z & Y & U \end{pmatrix}} R_1^* \otimes A \xrightarrow{d_1=\begin{pmatrix} X & Y & Z & U & V & W \end{pmatrix}} A
$$

Let α be in the Ker of d_4, i.e., $d_4(\alpha) = 0$. We have:

$W\alpha = 0$
$U\alpha = 0$
$Z\alpha = 0$
$X\alpha = 0.$

Then Lemma B.7 ([22] page 310) gives $\alpha = 0$ and this shows that the homology is zero in degree 4. Now let $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be in the Ker of d_3, i.e., $d_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 0$. We have:

$$
\begin{aligned}
U\alpha_1 + W\alpha_2 \qquad\qquad &= 0 \\
Z\alpha_1 \qquad + W\alpha_3 \qquad &= 0 \\
Z\alpha_2 + U\alpha_3 \qquad &= 0 \\
X\alpha_1 \qquad\qquad + W\alpha_4 &= 0 \\
X\alpha_2 \qquad + U\alpha_4 &= 0 \\
X\alpha_3 + Z\alpha_4 &= 0
\end{aligned}
$$

Lemma B.8 ([22] page 310) gives

$$(\alpha_1, \alpha_2) = (W, U)t_1$$

$$(\alpha_1, \alpha_3) = (W, Z)t_2$$

$$(\alpha_2, \alpha_3) = (U, Z)t_3$$

$$(\alpha_1, \alpha_4) = (W, X)t_4$$

$$(\alpha_2, \alpha_4) = (U, X)t_5$$

$$(\alpha_3, \alpha_4) = (Z, X)t_6.$$

Using once more Lemma B.7 ([22] page 310) we get $t_1 = t_2 = t_3 = t_4 = t_5 = t_6$ and this implies that

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (Wt_1, Ut_1, Zt_1, Xt_1)$$

and hence the homology is zero in degree 3 too. Next we prove that the homology is not zero in degree 2. In order to do this, we compute the Poincaré-Betti series of A. In (3.2.36) we computed the Poincaré-Betti series of $\mathrm{gr}(A)$ as

$$P_{\mathrm{gr}(A)}(x,y) = 1 + 6xy + 10x^2y^2 + x^2y^3 + 5x^3y^3 + x^3y^4 + x^4y^4.$$

The Hilbert series of R determines the $|\mathrm{Tor}^A_{i,i}(k,k)|$ as

$$|\mathrm{Tor}^A_{1,1}(k,k)| = 6, \quad |\mathrm{Tor}^A_{2,2}(k,k)| = 10, \quad |\mathrm{Tor}^A_{3,3}(k,k)| = 4, \quad |\mathrm{Tor}^A_{4,4}(k,k)| = 1.$$

To determine the Poincaré-Betti series of A completely, we use the spectral sequence starting with $\mathrm{Tor}^{\mathrm{gr}}_{*,*}(A)(k,k)$ and converging to $\mathrm{Tor}^A_{*,*}(k,k)$ (cf. [2]). Now by (3.2.36) we have only two nonzero element off the diagonal namely

$$\mathrm{Tor}^{\mathrm{gr}(A)}_{2,3}(k,k), \qquad \text{and} \qquad \mathrm{Tor}^{\mathrm{gr}(A)}_{3,4}(k,k).$$

The complex

$$0 = \mathrm{Tor}^{\mathrm{gr}(A)}_{1,3}(k,k) \longrightarrow \mathrm{Tor}^{\mathrm{gr}(A)}_{2,3}(k,k) \,{}^{d}\!\longrightarrow \mathrm{Tor}^{\mathrm{gr}(A)}_{3,3}(k,k) \longrightarrow \mathrm{Tor}^{\mathrm{gr}(A)}_{4,3}(k,k) = 0,$$

where d is the differential in the spectral sequence, shows that the map d is the inclusion map, because we know that $\mathrm{Tor}^{\mathrm{gr}(A)}_{3,3}(k,k)$ has dimension 5 and $\mathrm{Tor}^A_{3,3}(k,k)$ has dimension 4, and hence $\mathrm{Tor}^A_{2,3}(k,k) = 0$.

To compute $\mathrm{Tor}^A_{3,4}(k,k)$ we use the complex

$$0 = \mathrm{Tor}^{\mathrm{gr}(A)}_{2,4}(k,k) \longrightarrow \mathrm{Tor}^{\mathrm{gr}(A)}_{3,4}(k,k) \,{}^{f}\!\longrightarrow \mathrm{Tor}^{\mathrm{gr}(A)}_{4,4}(k,k) \longrightarrow \mathrm{Tor}^{\mathrm{gr}(A)}_{5,4}(k,k) = 0.$$

Here the map f is zero because both $\mathrm{Tor}^{\mathrm{gr}(A)}_{4,4}(k,k)$ and $\mathrm{Tor}^{(A)}_{3,3}(k,k)$ has dimension 1. This shows that $|\mathrm{Tor}^A_{3,4}(k,k)| = 1$ and hence the Poincaré-Betti series of A is

$$P_A(x,y) = 1 + 6xy + 10x^2y^2 + 4x^3y^3 + x^3y^4 + x^4y^4.$$

Now we go back to the complex $R^* \otimes A$. The existence of a nonzero element, i.e., $\mathrm{Tor}^A_{3,4}(k,k)$ off the diagonal shows that $\mathrm{H}_2(R^* \otimes A)$ is different from zero. Theorem B.3 ([22] page 306) now implies that R satisfies M_3. As a consequence, we obtain the following result.

THEOREM 3.2.14. *The whole ring* $\mathrm{H}^*(\mathscr{L}(S^4 \bigvee S^4), k)$ *in low dimensions satisfies the condition* M_3 *if all generators are considered to be in degree* 1.

PROOF. Consider the generators X_i's of $\mathrm{H}^{3*}(\mathscr{L}(\mathrm{S}^4 \bigvee \mathrm{S}^4), k)$ and the generators Y_i's of $\mathrm{H}^{3*+1}(\mathscr{L}(\mathrm{S}^4 \bigvee \mathrm{S}^4), k)$ as bigraded elements with bidegree $(1, \deg X_i\text{'s})$ and $(1, \deg Y_i\text{'s})$ respectively. Then using the notation of definition 2.2.13, we have (by Theorem 3.2.2)

$$\mathrm{H}^*(\mathscr{L}(\mathrm{S}^4 \bigvee \mathrm{S}^4), k) = \mathrm{H}^{3*}(\mathscr{L}(\mathrm{S}^4 \bigvee \mathrm{S}^4), k) \oplus s^{0,-1}\overline{\mathrm{H}^{3*}(\mathscr{L}(\mathrm{S}^4 \bigvee \mathrm{S}^4), k)}$$

In other words the ring structure of $\mathrm{H}^*(\mathscr{L}(\mathrm{S}^4 \bigvee \mathrm{S}^4), k)$ is the trivial extension of the ring $\mathrm{H}^{3*}(\mathscr{L}(\mathrm{S}^4 \bigvee \mathrm{S}^4), k)$ by the module $\overline{s^{0,-1}\mathrm{H}^{3*}(\mathscr{L}(\mathrm{S}^4 \bigvee \mathrm{S}^4), k)}$. Now if we only consider this new degree 1 (the first degree), then $\mathrm{H}^*(\mathscr{L}(\mathrm{S}^4 \bigvee \mathrm{S}^4), k)$ is nothing but $R \oplus \bar{R}$ where $R = \mathrm{H}^{3*}(\mathscr{L}(\mathrm{S}^4 \bigvee \mathrm{S}^4), k)$. Now the proof follows of Theorem 3.2.11.

4. $\mathrm{S}^4, \mathrm{S}^5$, and the EMSS of the path fibration.

Recall (cf. introduction) that

$$\mathrm{E}^2_{-p,q} = \mathrm{Ext}^p_{\mathrm{H}^*(\mathscr{L}\mathrm{X},k)}(k,k)_q \Longrightarrow (\mathrm{H}_*(\Omega\mathscr{L}\mathrm{X}, k)) \tag{4.1}$$

and if this spectral sequence degenerates, then

$$\dim_k(\mathrm{H}_n(\Omega\mathscr{L}\mathrm{X}, k)) = \sum_{p\geq 0} \dim_k \mathrm{Ext}^p_{\mathrm{H}^*(\mathscr{L}\mathrm{X},k)}(k,k)_{p+n}, \tag{4.2}$$

where the sum is finite.

THEOREM 4.1. *The Eilenberg-Moore spectral sequence* (4.1) *does not degenerate when* $\mathrm{X} = \mathrm{S}^4$.

PROOF. (cf. [28]) The series U(L) (universal enveloping algebra of L), where L is a graded lie algebra

$$\mathrm{L} = \mathrm{L}_1 + \mathrm{L}_2 + \mathrm{L}_3 + \cdots\cdots \qquad |\,\mathrm{L}_i\,| = l_i \qquad i = 1, 2, 3, \cdots\cdots$$

can be written as

$$\text{(4.3)} \qquad \mathrm{U(L)}(z) = \frac{(1+z)^{l_1}}{(1-z^2)^{l_2}} \frac{(1+z^3)^{l_3}}{(1-z^4)^{l_4}} \cdots\cdots.$$

Now set $\mathrm{H}_*(\Omega \mathrm{X}, k) = \mathrm{U(L)}$, then

$$\mathrm{H}_*(\Omega \mathscr{L}\mathrm{X}, k) = \mathrm{U}(\mathrm{L} \oplus \mathrm{SL}).$$

›where $(\mathrm{SL})_n = \mathrm{L}_{n+1}$. In other words if $g = \mathrm{L} \oplus \mathrm{SL} = g_1 + g_2 + g_3 + \cdots\cdots$, then

$$| g_1 | = l_1 + l_2, \qquad | g_2 | = l_2 + l_3, \qquad | g_3 | = l_3 + l_4, \cdots\cdots.$$

This last equality can be easily seen in cohomology case (cf. [28] page 183 and [26]). We have replaced cohomology with homology because here k is a field of characteristic zero. This implies

$$\text{(4.4)} \quad \mathrm{H}_*(\Omega \mathscr{L}\mathrm{X}, k)(z) = \mathrm{U}(\mathrm{L} \oplus \mathrm{SL})(z) = \frac{(1+z)^{l_1+l_2}}{(1-z^2)^{l_2+l_3}} \frac{(1+z^3)^{l_3+l_4}}{(1-z^4)^{l_4+l_5}} \cdots\cdots.$$

Now let $\mathrm{X} = \mathbf{S}^4$, then

$$\text{(4.5)} \qquad \mathrm{H}_*(\Omega \mathrm{X}, k)(z) = \frac{1}{1-z^3} = \frac{1+z^3}{1-z^6}$$

and hence

$$\text{(4.6)} \quad \mathrm{H}_*(\Omega \mathscr{L}\mathrm{X}, k)(z) = \frac{(1+z^3)(1+z^5)}{(1-z^2)(1-z^6)}$$
$$= 1 + z^2 + z^3 + z^4 + 2z^5 + 2z^6 + 2z^7 + 3z^8 + \cdots\cdots.$$

We have the following table for $\mathrm{H}^*(\mathscr{L}\mathbf{S}^4, k)$ as a particular case of the table (3.1.1).

$\mathrm{H}^*(\mathscr{L}\mathrm{S}^4, k)$	k	$1 \otimes x$	x	$1 \otimes x^{\otimes 3} l$	$x^{\otimes 3}$	$1 \otimes x^{\otimes 5}$	$x^{\otimes 5}$	...
deg	0	3	4	9	10	15	16	...
dim	1	1	1	1	1	1	1	...

To complete the proof of Theorem 4.1, we need

LEMMA 4.2. *The product in* $\mathrm{H}^*(\mathscr{L}\mathrm{S}^4, k)$ *is* 0.

PROOF. Easy. Recall that $x^{\otimes n} * x^{\otimes m} = 0$ (where $*$ is the shuffle product) because the product in $\mathrm{H}^*(\mathrm{X}, k)$ is 0.

THEOREM 4.3. $\mathrm{H}^*(\mathscr{L}\mathbf{S}^4, k)$ *is isomorphic as a ring to*

$$R = \frac{k[X_1, X_2, X_3, \ldots\ldots]}{(X_i X_j, i \leq j)}.$$

PROOF. Follows easily by lemma 4.2.

THEOREM 4.4. $\mathrm{Ext}_{\mathrm{H}^*(\mathscr{L}\mathrm{S}^4,k)}(k,k)$ *is generated by elements of degree one, i.e.,* $\mathrm{Ext}^1(k,k)$. *It has the following explicit form*

$$\mathrm{Ext}_{\mathrm{H}^*(\mathscr{L}\mathrm{S}^4,k)}(k,k) = [\mathrm{Ext}^1_{\mathrm{H}^*(\mathscr{L}\mathrm{S}^4,k)}(k,k)] = k\langle T_1, T_2, T_3, \ldots\ldots\rangle.$$

where the bidegree of T_i, *is* $(1, \deg X_i)$.

PROOF. Theorem 4.3 shows that the ring $\mathrm{H}^*(\mathscr{L}\mathbf{S}^4, k)$ is a Koszul algebra and hence $\mathrm{Ext}_{\mathrm{H}^*(\mathscr{L}\mathrm{S}^4,k)}(k,k)$ is generated by elements of degree one. The second part follows by easy calculations (using Lemma 2.2.18).

Now by Theorem 4.4 we have

$$(4.7)\qquad \sum_{i\geq 0}\left(\sum_{p\geq 0}\dim_k \mathrm{Ext}^p_{\mathrm{H}^*(\mathscr{L}\mathrm{X}}, k)(k,k)_{p+i}\right) z^i$$
$$= \frac{1}{1-\sum_{i\geq 1}(\mid \mathrm{H}_i^+(\mathscr{L}\mathrm{X},k)\mid -1)z^i}$$
$$= 1 + z^2 + z^3 + z^4 + 2z^5 + 2z^6 + 3z^7 + 5z^8 + 6z^9 + \cdots\cdots$$

Comparing $\dim_k(\mathrm{H}_n(\Omega\mathscr{L}\mathrm{X},k))$ and $\sum_{p\geq 0}\dim_k \mathrm{Ext}^p_{\mathrm{H}^*(\mathscr{L}\mathrm{X},k)}(k,k)_{p+n}$ for $n = 7$, we obtain

$$\dim_k(\mathrm{H}_7(\Omega\mathscr{L}\mathrm{X},k)) = 2 \qquad (\text{by}(4.6))$$

and

$$\sum_{p\geq 0}\dim_k \mathrm{Ext}^p_{\mathrm{H}^*(\mathscr{L}\mathrm{X},k)}(k,k)_{p+7} = \dim_k \mathrm{Ext}^3_{\mathrm{H}^*(\mathscr{L}\mathrm{X},k)}(k,k)_{10} = 3 \qquad (\text{by}(4.7)).$$

Now (4.1) implies

$$\mathrm{E}^2_{-3,10} = \mathrm{Ext}^3_{\mathrm{H}^*(\mathscr{L}\mathrm{X},k)}(k,k)_{10} \Longrightarrow \mathrm{H}^*_7(\Omega\mathscr{L}\mathrm{X},k)$$

and hence

$$\dim \mathrm{E}^\infty_{-3,10} = \dim_k(\mathrm{H}_7(\Omega\mathscr{L}\mathrm{X},k)) = 2 \qquad \text{and} \qquad \dim \mathrm{E}^2_{-3,10} = 3.$$

This implies that we have a non zero differential in the spectral sequence (E^r, d^r). Notice that d^r has bidegree $(-r, r-1)$.

Comparing two complexes

$$\mathrm{E}^2_{-1,9} \xrightarrow{d^2_{-1,9}} \mathrm{E}^2_{-3,10} \xrightarrow{d^2_{-3,10}} 0 = \mathrm{E}^2_{-5,11}$$

and

$$0 = \mathrm{E}^r_{r-3,-r+11} \xrightarrow{d^r_{-1,9}} \mathrm{E}^r_{-3,10} \xrightarrow{d^r_{-3,10}} = \mathrm{E}^r_{-3-r,r+9} = 0 \qquad \forall \qquad r \geq 3,$$

we see that $d^2_{-1,9} \neq 0$ and hence the Eilenberg–Moore spectral sequence does not degenerate at E^2 though it seems to converges very quickly.

THEOREM 4.5. *The Eilenberg – Moore spectral sequence* (4.1) *degenerates when* $\mathrm{X} = \mathbf{S}^5$.

PROOF. We have $\mathrm{H}_*(\Omega \mathrm{S}^5, k)(z) = \dfrac{1}{1 - z^4}$ and hence by (4.4)

$$(4.8) \quad \mathrm{H}_*(\Omega \mathscr{L} \mathrm{S}^5, k)(z) = \frac{(1 + z^3)}{(1 - z^4)}$$
$$= 1 + z^3 + z^4 + z^7 + z^8 + z^{11} + z^{12} + z^{15} + z^{16} + \cdots\cdots$$

Now as a particular case of Theorem 2.1.1, we have the following table

$\mathrm{H}^*(\mathscr{L}\mathrm{S}^5, k)$	k	$1 \otimes x$	x	$1 \otimes x^{\otimes 2}$	$x^{\otimes 2}$	$1 \otimes x^{\otimes 3}$	$x^{\otimes 3}$	$\cdots\cdots$
deg	0	4	5	8	9	12	13	$\cdots\cdots$
dim	1	1	1	1	1	1	1	$\cdots\cdots$

To complete the proof of Theorem 4.5, we need:

LEMMA 4.6. *The algebra* $\mathrm{H}^*(\mathscr{L}\mathbf{S}^5, k)$ *is generated by* $1 \otimes x$ *and* x.

PROOF. Lemma 2.2.1 implies

$$1 \otimes x^{\otimes n} * 1 \otimes x^{\otimes m} = \frac{(n+m)!}{n!m!} 1 \otimes x^{\otimes n+m}$$

and

$$1 \otimes x^{\otimes n} * x^{\otimes m} = \frac{(n+m-1)!}{(n-1)!m!} x^{\otimes n+m}$$

where $*$ means shuffle product.

By Lemma 4.6 above we get:

$$\mathrm{H}^*(\mathscr{L}\mathbf{S}^5, k) = R = \frac{k[X_1, X_2]}{X_2^2} \qquad (\deg X_1 = 4, \qquad \deg X_2 = 5).$$

This ring is a complete intersection and

$$\mathrm{Ext}_{\mathrm{H}^*(\mathscr{L}\mathbf{S}^5,k)}(k,k) = [\mathrm{Ext}^1_{\mathrm{H}^*(\mathscr{L}\mathbf{S}^5,k)}(k,k)] = \frac{k\langle \mathrm{T}_1, \mathrm{T}_2\rangle}{(\mathrm{T}_1^2, \mathrm{T}_1\mathrm{T}_2 + \mathrm{T}_2\mathrm{T}_1)},$$

where the monomial T_1 generates

$$\mathrm{Ext}^1_{\mathrm{H}^*(\mathscr{L}\mathrm{S}^5,k)}(k,k)_4$$

and the monomial T_2 generates

$$\mathrm{Ext}^1_{\mathrm{H}^*(\mathscr{L}\mathrm{S}^5,k)}(k,k)_5.$$

Counting monomials in $\mathrm{Ext}_{\mathrm{H}^*(\mathscr{L}\mathrm{S}^5,k)}(k,k)$ we obtain

deg	1	2	3	4	5	6	......
Monomials	T_1, T_2	T_1T_2, T_2^2	$T_1T_2^2, T_2^3$	$T_1T_2^3, T_2^4l$	$T_1T_2^4, T_2^5$	$T_1T_2^5, T_2^6$	......

The fact that the monomials $\mathrm{T}_1\mathrm{T}_2{}^n$ generate $\mathrm{Ext}^{n+1}_{\mathrm{H}^*(\mathscr{L}\mathrm{S}^5,k)}(k,k)_{5n+4}$ together with (4.8) imply that

$$\dim_k(\mathrm{H}_n(\Omega\mathscr{L}\mathrm{X},k)) = \sum_{p\geq 0} \dim_k \mathrm{Ext}^p_{\mathrm{H}^*(\mathscr{L}\mathrm{X},k)}(k,k)_{p+n}$$

and hence the Eilenberg–Moore spectral sequence (4.1) degenerates in this case.

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