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METRIC DUALITY IN EUCLIDEAN SPACES

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1. Introduction.

1.1. We consider point sets in the one-point extension $\dot{R}^n = R^n \cup \{\infty\}$ of the euclidean space R^n, $n \geq 2$. Suppose that U is open in $\dot{R}^n$, and let $X = \complement U = \dot{R}^n \setminus U$ be its complement. The Alexander duality gives an isomorphism between the homology of U and the cohomology of X:

$$(1.2) \qquad H_p(U) \approx H^q(X),$$

where $p + q = n - 1$, and H_p and H^q denote the reduced singular homology and CÚech cohomology groups with coefficients in the same group; see [Mu, 74.1]. The isomorphism is functorial with respect to inclusions.

In this paper we study to what extent the *metric* properties of X are determined by the metric properties of U and vice versa. An early example of this was given in 1963 by L.V. Ahlfors [Ah], who characterized the quasicircles in the plane by the three-point property. In 1965 F.W. Gehring [Ge_1] introduced the concept of *linear local connectedness*, abbreviated LLC. We recall the definition. Let M be a metric space with distance written as $|a - b|$. We let $\dot{M} = M \cup \{\infty\}$ denote the one-point extension. Open and closed balls are written as $B(a, r)$ and $\overline{B}(a, r)$. For $c \geq 1$, a set $A \subset \dot{M}$ is said to be c-LLC in M if for each $a \in M$ and $r > 0$ the following two conditions are satisfied:

c-LLC_1: Each pair of points in $A \cap B(a, r)$ can be joined in $A \cap B(a, cr)$.

c-LLC_2: Each pair of points in $A \setminus \overline{B}(a, cr)$ can be joined in $A \setminus \overline{B}(a, r)$.

Depending on what is meant by joining, one can consider pathwise and continuumwise properties $\text{LLC}_1, \text{LLC}_2$, LLC. The property c-LLC_1 is quantitatively equivalent to the property *c-bounded turning*, which (continuumwise) means that each pair $a, b \in A$ can be joined by a continuum $\alpha \subset A$ with diameter at most $c|a - b|$.

For a Jordan curve $X \subset \dot{R}^2$, the Ahlfors condition is equivalent to LLC_1 and also to LLC_2. The components of $U = \dot{R}^2 \setminus X$ are then quasidisks,

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which can be characterized by the LLC property [Ge_3, 3.6]. On the other hand, Gehring and O. Martio [GM, 2.21] proved that a domain $U \subset \dot{R}^2$ is c-LLC if and only if, quantitatively, the components of $X = \dot{R}^2 \setminus U$ are points or closed K-quasidisks.

In this paper we introduce *p-dimensional* analogues of the properties c-LLC_1 and c-LLC_2. The definitions, based on homology for open sets $U \subset \dot{R}^n$ and on cohomology for closed sets $X \subset \dot{R}^n$, are given in 2.2. Sets with these properties are called *outer* and *inner* (p, c)*-joinable*, respectively. The basic result in the theory is the following *duality theorem*, proved in 2.7: Let $p + q = n - 2$, let U be open in $\dot{R}^n$ and let $X = \dot{R}^n \setminus U$. Then U is outer [or inner] (p, c)-joinable in R^n if and only if X is inner [or outer] (q, c)-joinable in R^n. For $n = 2$, $p = q = 0$, this implies the aforementioned result of Gehring--Martio and other well known results.

The proof of the duality theorem is rather easy, and it makes use of the Alexander duality and Mayer--Vietoris sequences. Observe that $p + q$ is $n - 2$ and not $n - 1$ as in the Alexander duality (1.2). Our result should therefore not be regarded as a metric version of the Alexander duality.

Section 3 deals with general properties of joinability. In Section 4 we consider the behavior of the joinability properties under various maps. In fact, the original reason for introducing the LLC property was Gehring's observation [Ge_1] that if $n \geq 3$ and if $f : B^n \to D'$ is K-quasiconformal, then $\complement D'$ is c-LLC with $c = c(K, n)$. Our results imply that $\complement D'$ is indeed (q, c)-joinable for $0 \leq q \leq n - 3$.

The main significance of the joinability properties seems to be based on their relations with *John* and *uniform* domains; also these must be considered in the p-dimensional sense. These are considered in Section 5. As an application we obtain quasimöbius invariance properties of couniform sets, which are partially new also in the case $p = 0$.

1.3. *Terminology and notation.* Let M be a metric space with distance written as $|a - b|$ or $d(a, b)$. Then $\dot{M} = M \cup \{\infty\}$ is a topological space, where the neighborhoods of ∞ are the complements of closed bounded sets of M. We let $d(A, B)$ denote the distance between $A, B \subset M$, and $d(A)$ is the diameter of a set A. In $\dot{M}$ we set $d(a, \infty) = \infty$ and $d(\infty, \infty) = 0$. The numbers $d(A, B)$ and $d(A)$ are then defined for all $A, B \subset \dot{M}$; they may have the value ∞. We agree that $d(A, \varnothing) = \infty$ and $d(\varnothing) = 0$. For $a \in M$ and $r > 0$, balls and spheres are written as

$$B(a, r) = \{x : |x - a| < r\},\ \overline{B}(a, r) = \{x : |x - a| \leq r\},$$
$$S(a, r) = \{x : |x - a| = r\}.$$

For real numbers a, b we write

$$a \wedge b = \min(a,b),\ a \vee b = \max(a,b).$$

The cardinality of a set A is $\#A$.

We fix a nontrivial abelian group $\mathbf{G}$, which will be the same throughout the paper. For a topological space X, we let $H_p(X)$ denote the *reduced* singular homology groups of X with coefficients in $\mathbf{G}$, customarily written as $\tilde{H}_p(X)$ or $\tilde{H}_p(X;\ \mathbf{G})$. By p-chains and p-cycles we mean singular p-chains and *reduced* singular p-cycles with coefficients in $\mathbf{G}$. Each p-chain in X can be uniquely (up to ordering of terms) written in the form

$$g = \gamma_1\sigma_1 + \cdots + \gamma_k\sigma_k,$$

where $\gamma_i \in \mathbf{G}$, $\gamma_i \neq 0$, and $\sigma_1, \cdots, \sigma_k$ are distinct singular p-simplexes in X, that is, continuous maps of the standard p-simplex Δ^p into X. The carrier of σ_i is the image set $|\sigma_i| = \sigma_i\Delta^p$, and the carrier of a p-chain g with the above representation is

$$|g| = |\sigma_1| \cup \cdots \cup |\sigma_k|.$$

We let $H^p(X)$ denote the *reduced* Cÿech (equivalently Alexander--Spanier) cohomology groups of X with coefficients in $\mathbf{G}$. If $A \subset B \subset X$, we use the notation

$$H_p(A) \to H_p(B),\ H^p(B) \to H^p(A)$$

for homomorphisms induced by inclusion, without always mentioning this explicitly.

If a condition A with data v implies a condition A' with data v' so that v' depends only on v, we say that A implies A' *quantitatively*. A symbol (usually c) appearing in both v and v' need not have the same value in both conditions. For example, c-uniformity implies c-LLC quantitatively for domains in R^n, which means that c-uniform domains are c'-LLC with $c' = c'(c)$. If we are dealing with R^n and if v' depends only on v and n, we say that A implies A' *n-quantitatively.*

In most cases, we shall give explicit estimates for the constants.

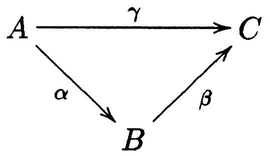
2. Joinability and duality.

2.1. *Summary of Section* 2. We define the joinability properties, which are the basic concepts of this paper. In 2.7 we prove the central duality theorem in $\dot{R}^n$. We also consider duality in some other spaces homeomorphic to S^n.

2.2. *Joinability.* We first introduce some algebraic terminology. Let

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \tag{s}$$

be a (very short) sequence of abelian groups and homomorphisms. We say that the sequence is *fast* if ker $(\beta\alpha) = \ker \alpha$ or, equivalently, ker $(\beta\alpha) \subset \ker \alpha$. Dually, the sequence is *slow* if im $(\beta\alpha) = \operatorname{im} \beta$ or, equivalently, im $\beta \subset \operatorname{im} (\beta\alpha)$. In particular, (s) is fast if $\alpha = 0$ and slow if $\beta = 0$. Setting $\gamma = \beta\alpha$ we also say that the triangle



is fast or slow if the sequence (s) has this property.

Let M be a metric space. Suppose that $A \subset \dot{M} = M \cup \{\infty\}$, $a \in A \setminus \{\infty\}$, $r > 0$ and $c \geq 1$. For each integer p, inclusions induce four sequences

(a) $$H_p(A \cap B(a,r)) \to H_p(A \cap B(a,cr)) \to H_p(A),$$

(b) $$H_p(A \setminus \overline{B}(a,cr)) \to H_p(A \setminus \overline{B}(a,r)) \to H_p(A),$$

(c) $$H^p(A) \to H^p(A \cap \overline{B}(a,cr)) \to H^p(A \cap \overline{B}(a,r)),$$

(d) $$H^p(A) \to H^p(A \setminus B(a,r)) \to H^p(A \setminus B(a,cr)).$$

If the sequence (a) is fast for every $a \in A \setminus \{\infty\}$, $r > 0$, we say that A is *homologically outer* (p,c)*-joinable*. If (b) is fast for all a, r, then A is *homologically inner* (p,c)*-joinable*. If (c) is slow for all a, r, then A is *cohomologically outer* (p,c)*-joinable*. If (d) is slow for all a, r, then A is *cohomologically inner* (p,c)*-joinable*.

We shall abbreviate the words 'homologically' and 'cohomologically' by hlog and cohlog, respectively. We say that A is *hlog* (p,c)*-joinable* if A is both hlog outer (p,c)-joinable and hlog inner (p,c)-joinable. The concept *cohlog* (p,c)*-joinable* is defined analogously.

If A is hlog outer (p,c)-joinable for some $c \geq 1$, we say that A is hlog outer (p)-joinable, and similarly for the other joinability properties.

We shall mainly consider the case where $M = R^n$ and A is either open or closed in $\dot{R}^n$. To simplify terminology, we usually omit the word 'hlog' if A is open in $\dot{R}^n$ and the word 'cohlog' if A is closed in $\dot{R}^n$.

The homological joinability properties can be defined more explicitly in terms of cycles and chains. For example, an open set $U \subset \dot{R}^n$ is outer (p,c)-joinable if and only if, given $a \in U \setminus \{\infty\}$ and $r > 0$, a p-cycle in $U \cap B(a,r)$ bounds in $U \cap B(a,cr)$ whenever it bounds in U.

The choice of open and closed balls in the sequences (a)--(d) was made in order to get the duality theorem in a precise form. However, it is clear that if A has one of the joinability properties, the corresponding sequence is fast or slow also if the open balls are replaced by closed balls or vice versa and if c is replaced by any number $c' > c$.

The concepts defined above depend on the coefficient group $\mathbf{G}$, and in a complete notation one should write $(p, c, \mathbf{G})$-joinable, for example. The results of this paper hold for every coefficient group $\mathbf{G}$, and the reader may think, for example, that $\mathbf{G} = \mathbf{Z}$ everywhere. However, it seems to the author that in some applications it may be useful to let $\mathbf{G}$ be a field.

2.3. *Examples.* To give the reader some idea of joinability, we present some examples without always giving a detailed proof. In the proofs, the duality theorem 2.7 would sometimes be useful.

We consider open sets $U \subset \dot{R}^n$ and thus omit the word 'hlog'. It turns out that the joinability properties are interesting only for $0 \le p \le n-2$; see 2.9. The set U is outer or inner $(0, c)$-joinable if and only if all components of U are c-LLC$_1$ or c-LLC$_2$, respectively; see 3.5. A convex open set $U \subset R^n$ is outer $(p, 1)$-joinable for every p, since the sets $U \cap B(a, r)$ are hlog trivial. A ball $B(x_0, t)$ is $(p, 1)$-joinable for every $p \in [0, n-2]$, since also the sets $B(x_0, t) \setminus \overline{B}(a, r)$ have trivial homology groups for these p.

If $1 \le k \le n-1$, the domain $U = R^k \times B^{n-k}$ is not inner $(k-1)$-joinable, since the $(k-1)$-sphere $S^{k-1}(c)$, $c > 1$, considered as a $(k-1)$-cycle, does not bound in $U \setminus \overline{B}^n$. Let $-1 < t < 1$ and let U be the domain $B^3 \setminus [te, e]$ in R^3, where e is a unit vector. Then U is $(0, 1)$-joinable and inner $(1, c)$-joinable for some $c = c(t)$ but not outer (1)-joinable. The troublesome 1-cycles are small circles around the line segment $[te, e]$. Moreover, $c(t) \to \infty$ as $t \to -1$. However, the limiting case $U = B^3 \setminus [-e, e]$ is (1)-joinable, since the troublesome 1-cycles are no longer boundaries in U. There is a similar situation with $U = B^n \setminus [te, e]$ and $(n-2)$-joinability for each n, including the case $n = 2$.

The complement of the disk $\overline{B}^2$ in $\dot{R}^3$ is (1)-joinable and inner (0)-joinable but not outer (0)-joinable.

Although the point a in the definition of the joinability properties is always a finite point, the point ∞ plays an essential role. For example, a line L in R^n is not hlog inner (0)-joinable but $L \cup \{\infty\}$ has this property.

2.4. *Relative joinability.* Let us consider the situation $A \subset \dot{M}$ as in 2.2. In the definition of the four joinability properties, the center a of the balls was supposed to lie in $A \setminus \{\infty\}$. Hence these properties are *intrinsic* properties of A. The space M plays no role, except that we could include both cases $\infty \in A$ and $\infty \notin A$. Nevertheless, it is often convenient to consider the conditions also at points $a \in M \setminus A$. For example, the duality theorem 2.7 can then be given in a more exact form. We say that A has one of the four properties *in M* (or in $\dot{M}$) if the corresponding condition holds for all $a \in M$. For example, A is hlog outer (p, c)-joinable in M if the sequence (a) of 2.2 is fast for all $a \in M$, $r > 0$. We next show that this relative joinability is in fact quantitatively equivalent to absolute joinability:

2.5. Lemma. *If $p \geq 0$ and if $A \subset \dot{M}$ is hlog outer (p, c)-joinable, then A is hlog outer $(p, 2c+1)$-joinable in M. The corresponding statement is valid for the other three joinability properties as well.*

Proof. Let $a \in M$, $r > 0$. Writing $c' = 2c + 1$ we must show that the sequence

$$H_p(A \cap B(a,r)) \to H_p(A \cap B(a, c'r)) \to H_p(A)$$

is fast. If $A \cap B(a,r) = \varnothing$, the first group is trivial and, consequently, the sequence is fast. If $A \cap B(a,r) \neq \varnothing$, choose a point $x \in A \cap B(a,r)$. Now

$$B(a,r) \subset B(x,2r) \subset B(x,2cr) \subset B(a,c'r),$$

and we obtain the commutative diagram

$$\begin{array}{ccccc} H_p(A \cap B(a,r)) & \longrightarrow & H_p(A \cap B(a,c'r)) & & \\ \big\downarrow & & \big\uparrow & \searrow & \\ & & & & H_p(A). \\ & & & \nearrow & \\ H_p(A \cap B(x,2r)) & \longrightarrow & H_p(A \cap B(x,2cr)) & & \end{array}$$

Since the lower row is fast, so is the upper row.

Next assume that A is hlog inner (p, c)-joinable and that $a \in M$, $r > 0$. We must show that the sequence

$$H_p(A \setminus \overline{B}(a, c'r)) \to H_p(A \setminus \overline{B}(a,r)) \to H_p(A)$$

is fast. If $A \cap \overline{B}(a,r) = \varnothing$, the second map is the identity, and the sequence is trivially fast. If $A \cap \overline{B}(a,r) \neq \varnothing$, choose $x \in A \cap \overline{B}(a,r)$. Now

$$\overline{B}(a,r) \subset \overline{B}(x,2r) \subset \overline{B}(x,2cr)) \subset \overline{B}(a,c'r),$$

and we can proceed essentially as in the first case.

The cohlog cases are treated by analogous arguments.

The proof of the duality theorem is based on the following algebraic result:

2.6. LEMMA. *Suppose that the diagram*

$$\begin{array}{ccccccc} & & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\ & \overset{i}{\nearrow} & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \alpha} & \nearrow{\scriptstyle \beta} & \\ A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & & \end{array}$$

of abelian groups and homomorphisms is commutative and has exact horizontal rows. Then the left-hand triangle is slow if and only if the right-hand triangle is fast.

PROOF. The lemma is obtained by elementary diagram chasing. Suppose first that the left-hand triangle is slow and that $c \in C$ with $hc = 0$. Then there is $b \in B$ with $gb = c$. By slowness, there is $a' \in A'$ with $f'a' = jb$. Then $\alpha c = \alpha g b = g' j b = g' f' a' = 0$.

Conversely, assume that the right-hand triangle is fast and that $b \in B$. Since the upper row is exact, we have $hgb = 0$. By fastness, this implies $g' j b = \alpha g b = 0$. By exactness, there is $a' \in A'$ with $f'a' = jb$. Hence the left-hand triangle is slow.

2.7. DUALITY THEOREM. *Suppose that U is an open set in $\dot{R}^n$ and that p is an integer with $0 \le p \le n-2$. Set $X = \complement U = \dot{R}^n \setminus U$ and $q = n - 2 - p$. Then*

(a) *U is outer (p, c)-joinable in R^n if and only if X is inner (q, c)-joinable in R^n,*

(b) *U is inner (p, c)-joinable in R^n if and only if X is outer (q, c)-joinable in R^n,*

(c) *U is (p, c)-joinable in R^n if and only if X is (q, c)-joinable in R^n.*

PROOF. Assume that U is outer (p, c)-joinable in R^n. Let $a \in R^n$, $r > 0$, and set $B_1 = B(a, r)$, $B_2 = B(a, cr)$. Consider the diagram

$$\begin{array}{ccccccc} & & H_{p+1}(U \cup B_1) & \longrightarrow & H_p(U \cap B_1) & \longrightarrow & H_p(U) \oplus H_p(B_1) \\ & \nearrow & \downarrow & & \downarrow & \nearrow & \\ H_{p+1}(U) \oplus H_{p+1}(B_2) & \longrightarrow & H_{p+1}(U \cup B_2) & \longrightarrow & H_p(U \cap B_2). & & \end{array}$$

Here the horizontal rows are parts of exact Mayer--Vietoris sequences. The trivial groups $H_p(B_1)$ and $H_{p+1}(B_2)$ can be omitted. Then all vertical and slanted arrows are induced by inclusions, and hence the diagram is commutative. Now the right-hand triangle is fast. By Lemma 2.6, the left-hand triangle is slow. By Alexander duality (1.2), this sequence is isomorphic to

$$H^q(X) \to H^q(X \setminus B_1) \to H^q(X \setminus B_2).$$

Since this is slow, X is inner (q, c)-joinable in R^n.

All steps of the above proof hold also conversely. Hence (a) is true. Part (b) is proved by an analogous argument with the aid of the diagram

$$\begin{array}{ccccccc} & & H_{p+1}(U \cup \complement\overline{B}_2) & \to & H_p(U \setminus \overline{B}_2) & \to & H_p(U) \oplus H_p(\complement\overline{B}_2) \\ & \nearrow & \downarrow & & \downarrow & \nearrow & \\ H_{p+1}(U) \oplus H_{p+1}(\complement\overline{B}_1) & \to & H_{p+1}(U \cup \complement\overline{B}_1) & \to & H_p(U \setminus \overline{B}_1). & & \end{array}$$

Part (c) is a direct consequence of (a) and (b).

2.8. REMARK. We gave Theorem 2.7 in terms of relative joinability, since the duality then holds with the same constant c. With the aid of 2.5 we obtain an *absolute* version of the duality theorem: If U or X is outer or inner (p, c)-joinable, then X or U is inner or outer $(q, 2c+1)$-joinable (4 cases).

2.9. *The uninteresting cases.* The definitions of the four (p, c)-joinability properties make sense for all integers p. However, for open and closed subsets of $\dot{R}^n$, only the case $0 \le p \le n-2$ gives interesting properties. For completeness, we discuss the situation in the other cases. The easy proofs are omitted.

Assume that A is open or closed in $\dot{R}^n$. Then A is $(p, 1)$-joinable for all $p \le -2$ and $p \ge n$. For $p = n-1$ we have:

(a) A is outer $(n-1)$-joinable if and only if $A = \dot{R}^n$ or $\infty \notin \operatorname{int} A$.

(b) A is inner $(n-1)$-joinable if and only if $A = \dot{R}^n$ or $\operatorname{int} A = \varnothing$.

In the positive cases, A is outer or inner $(n-1, 1)$-joinable.

The case $p = -1$ requires correct understanding of the reduced homology group $H_{-1}(\varnothing)$. In most textbooks this group is either undefined or trivial. I believe, however, that it is both logical and useful to define $H_{-1}(\varnothing) = \mathbf{G}$ as in [Ma, p. 108] or [Br, p. 181]. For example, the reduced Mayer--Vietoris sequence of an excisive couple $\{A, B\}$ is then exact without the restriction $A \cap B \ne \varnothing$, and the Alexander duality (1.2) holds for all integers p, q with $p + q = n-1$ and for all open sets $U \subset S^n$. The duality theorem 2.7 and its proof are valid for all integers p, q with $p + q = n-2$. However, Lemma 2.5 is not valid for $p = -1$. For example, an open set $U \subset \dot{R}^n$ is always outer $(-1, 1)$-joinable, but it is outer (-1)-joinable in R^n if and only if $\overline{U} = \dot{R}^n$ or $U = \varnothing$.

The statements (a) and (b) are easily proved directly, but it is still easier to prove the corresponding statements for (-1)-joinability in $\dot{R}^n$ and make use of the duality theorem.

We remark that the duality theorem is true but rather uninteresting in R^1.

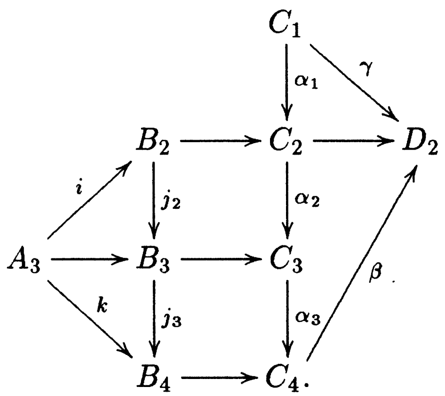
Convention. From now on, we always assume that $0 \leq p \leq n-2$ when we are considering outer or inner (p, c)-joinability of a set in $\dot{R}^n$.

2.10. REMARK. The case $p = n-1$ is more interesting if $\dot{R}^n$ is replaced by S^n. For example, it is easy to show that a compact set $X \subset S^n$ is outer $(n-1, c)$-joinable in S^n if and only if $S^n \setminus X$ is not contained in any ball $B(a, r)$ with $a \in S^n$, $0 < r < 2/c$. On the other hand, X is inner $(n-1)$-joinable if and only if $X = S^n$ or X has no interior points in S^n.

2.11. *Other metrics.* The only property of the euclidean metric needed in the proof of the duality theorem was the fact that the sets $B(a, r)$ and $\complement\overline{B}(a, r)$ are homologically trivial in dimensions $0 \leq p \leq n-1$. Hence the theorem remains valid, for example, in every n-dimensional normed space. Furthermore, we can replace $\dot{R}^n$ by any metric space which is homeomorphic to S^n and satisfies this condition on balls. For example, the duality theorem is valid on S^n. Here the restriction $0 \leq p \leq n-2$ is essential.

More generally, we show that the duality theorem can be extended to a metric space which is homeomorphic to S^n and suitably joinable in its own metric. This result is given in 2.14 and 2.15. We need two generalizations of the algebraic parallelogram lemma 2.6. The proofs are based on routine diagram chasing and therefore omitted.

2.12. LEMMA. *Consider the following diagram of abelian groups and homomorphisms:*



Write $j = j_3 j_2$ and $\alpha = \alpha_3\alpha_2\alpha_1$. Suppose:

(1) *The diagram without the maps i, β is commutative.*

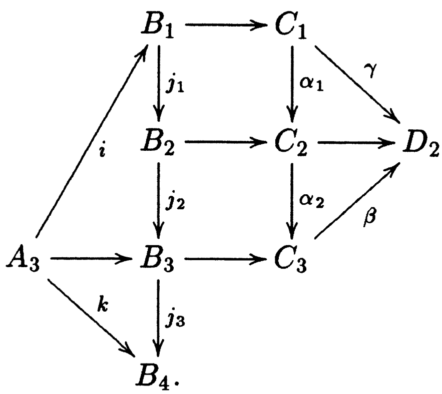
(2) *$k = ji$ and $\gamma = \beta\alpha$.*

(3) *The horizontal rows are exact.*

(4) *The sequence $A_3 \xrightarrow{i} B_2 \xrightarrow{j} B_4$ is slow.*

Then the sequence $C_1 \xrightarrow{\alpha} C_4 \xrightarrow{\beta} D_2$ *is fast.*

2.13. Lemma. *Consider the following diagram of abelian groups and homomorphisms:*



Write $j = j_3 j_2 j_1$ *and* $\alpha = \alpha_2 \alpha_1$. *Suppose:*

(1) *The diagram without the maps* i, β *is commutative.*

(2) $k = ji$ *and* $\gamma = \beta\alpha$.

(3) *The horizontal rows are exact.*

(4) *The sequence* $C_1 \xrightarrow{\alpha} C_3 \xrightarrow{\beta} D_2$ *is fast.*

Then the sequence $A_3 \xrightarrow{i} B_1 \xrightarrow{j} B_4$ *is slow.*

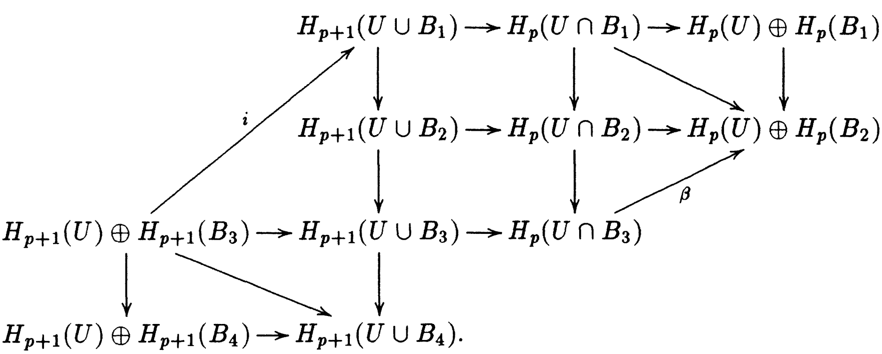
2.14. Theorem. *Let* S *be either a metric space or the one-point extension of a metric space. Suppose that* S *is homeomorphic to* S^n. *Suppose also that* $0 \leq p \leq n-2$ *and that* S *is hlog outer* (p, c_0)*-joinable and hlog outer* $(p+1, c_0)$*-joinable. Let* U *be open in* S, *and write* $X = S \setminus U$, $q = n - 2 - p$. *Then the following conditions are quantitatively equivalent:*

(1) U *is hlog outer* (p, c)*-joinable.*

(2) X *is cohlog inner* (q, c)*-joinable.*

Explicit bounds. Each of the conditions (1), (2) implies the other with $c \mapsto 3cc_0 \vee c_0^2$.

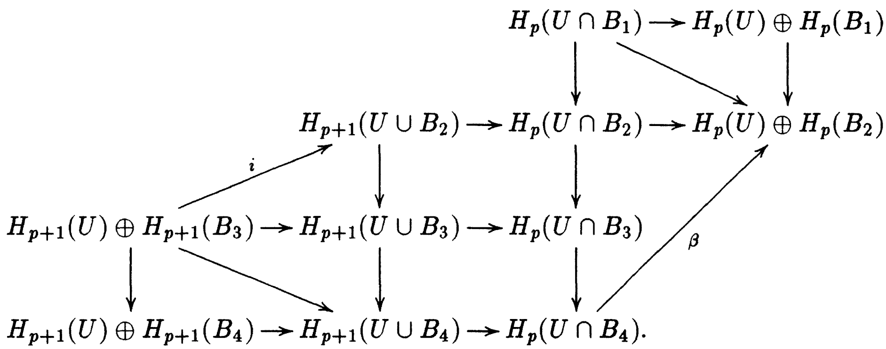
Proof. Suppose that (1) holds. Let $a \in S \setminus \{\infty\}$ and $r > 0$. Set $c' = 1 \vee (3c/c_0)$, $B_1 = B(a, r)$, $B_2 = B(a, c_0 r)$, $B_3 = (a, c'c_0 r)$, $B_4 = B(a, c'c_0^2 r)$. Then $B_1 \subset B_2 \subset B_3 \subset B_4$. We obtain the diagram



Here the horizontal rows are parts of exact Mayer--Vietoris sequences. All vertical arrows are induced by inclusions. The maps i and β are defined by $i(x,y) = u_*x$, $\beta x = (v_*x, 0)$, where u and v are inclusions. The remaining two slanted arrows are defined by commutativity.

We verify that the diagram with corners omitted satisfies the conditions of 2.13. Clearly the diagram without i and β is commutative. Since S is hlog outer (k, c_0)-joinable for $k = p, p+1$, the maps $H_p(B_1) \to H_p(B_2)$ and $H_{p+1}(B_3) \to H_{p+1}(B_4)$ are zero. This implies 2.13(2). By 2.5, U is hlog outer $(p, 3c)$-joinable in S. Since $c_0c' \geq 3c$, the sequence $H_p(U \cap B_1) \to H_p(U \cap B_3) \xrightarrow{\beta} H_p(U) \oplus H_p(B_2)$ is fast. Hence the sequence $H_{p+1}(U) \to H_{p+1}(U \cup B_1) \to H_{p+1}(U \cup B_4)$ is slow by 2.13. By Alexander duality this is isomorphic to the sequence $H^q(X) \to H^q(X \setminus B_1) \to H^q(X \setminus B_4)$. We have thus proved (2) with the constant $c_0^2c' = 3cc_0 \vee c_0^2$.

Conversely, assume that (2) is true. Let $a \in S \setminus \{\infty\}$ and $r > 0$. Let c' and B_1, B_2, B_3, B_4 be as in the first part of the proof. We obtain the diagram



The maps i and β are defined essentially as in the first part. We can now proceed as before, using 2.12 instead of 2.13, and show that (1) holds with the same constant $3cc_0 \vee c_0^2$.

2.15. THEOREM. *Theorem* 2.14 *remains valid if the words "outer" and "inner" are interchanged.*

PROOF. The proof is obtained from the proof of 2.14 by replacing $B_1, \ldots, B_4$ by $S \setminus \overline{B}_4, \ldots, S \setminus \overline{B}_1$.

3. Basic properties of joinability.

3.1. *Summary of Section* 3. We consider some easy consequences of the definitions of the joinability properties. In particular, we show that (0)-joinability is equivalent to componentwise LLC, defined in 1.1. We also consider the duality theorem in the plane.

3.2. THEOREM. *Suppose that $A \subset \dot{R}^n$ is open or closed. Then A is outer or inner (p, c)-joinable if and only if each component of A is outer or inner (p, c)-joinable, respectively.*

PROOF. Let A_j, $j \in J$, be the components of A. For $p > 0$, the result follows easily from the fact that $H_p(A)$ is the direct sum of the groups $H_p(A_j)$, and $H^p(A)$ is the direct product of the groups $H^p(A_j)$. For $p = 0$, observe that one can replace the reduced groups by unreduced groups in the definition of the $(0, c)$-joinability properties in 2.2; cf. 3.4 and 3.7. Hence the above argument is valid in this case as well.

3.3. *Notation.* Recall that in this paper $H_0(X)$ denotes the reduced 0-dimensional singular homology group of a topological space X. In the next few results we exceptionally need also the unreduced group, which will be written as $\overline{H}_0(X)$. Similarly, $\overline{H}^0(X)$ will be the unreduced 0-dimensional CÚech cohomology group of X. If $X \neq \varnothing$, these groups are isomorphic to $H_0(X) \oplus \mathbf{G}$ and $H^0(X) \oplus \mathbf{G}$, respectively.

We identify the singular 0-simplexes of X with points of the space. Then each 0-chain can be written as a finite sum $g = \sum_i \gamma_i x_i$ where $\gamma_i \in \mathbf{G}$ and $x_i \in X$. Moreover, g is a reduced cycle if and only if $\sum_i \gamma_i = 0$. Given points $x, y \in X$, it is well known that the following conditions are equivalent:

(1) x and y can be joined by a path in X.

(2) $\gamma x - \gamma y$ bounds in X for some nonzero $\gamma \in \mathbf{G}$.

(3) As (2) but for all $\gamma \in \mathbf{G}$.

3.4. LEMMA. *Suppose that X is a topological space and that $A \subset B \subset X$. Then the following conditions are equivalent:*

(1) *The sequence $\overline{H}_0(A) \to \overline{H}_0(B) \to \overline{H}_0(X)$ is fast.*

(2) *The sequence $H_0(A) \to H_0(B) \to H_0(X)$ is fast.*

(3) *Points $x, y \in A$ can be joined by a path in B whenever they can be joined by a path in X.*

PROOF. First observe that if a singular 0-chain bounds in a space, it is a reduced 0-cycle. This obviously implies the equivalence of (1) and (2). Suppose that (2) is true and that x, y are points in A which can be joined by a path in X. Choose an element $\gamma \neq 0$ in $\mathbf{G}$. Then the 0-cycle $\gamma x - \gamma y$ bounds in X. By (2) it bounds in B. Hence (3) holds.

We finally show that (3) implies (1). Let A_i, $i \in I$, be the path components of A. Choose a point $a_i \in A_i$ for each $i \in I$. For a 0-chain g in A we let $[g]$ denote its class in $\overline{H}_0(A)$. Now $\overline{H}_0(A)$ is generated by elements $[\gamma a_i]$, $\gamma \in \mathbf{G}$, $i \in I$. Assume that $u = \sum_i [\gamma_i a_i] \in \overline{H}_0(A)$ is mapped to zero in $\overline{H}_0(X)$. For a path component Y of X set $I(Y) = \{i \in I : a_i \in Y\}$. Then $\sum_{i \in I(Y)} \gamma_i a_i$ can be written as a sum of elements of the form $\gamma x - \gamma y$ where $\gamma \in \mathbf{G}$ and x, y are some points $a_i \in Y$. If (3) is true, these elements bound in B. Hence u is mapped to zero in $\overline{H}_0(B)$, and (1) follows.

3.5. THEOREM. *Let M be a metric space and let $A \subset \dot{M}$. Then the following two conditions are equivalent:*

(1) *A is hlog outer $(0, c)$-joinable in M.*

(2) *Every path component of A is pathwise c-LLC_1.*

The following two conditions are also equivalent:

(3) *A is hlog inner $(0, c)$-joinable in M.*

(4) *Every path component of A is pathwise c-LLC_2.*

PROOF. This follows directly from 3.4.

3.6. *The groups $\overline{H}^0(X)$ and $H^0(X)$.* Let X be a topological space. It is well known that $\overline{H}^0(X)$ can be identified with the group of all locally constant functions $\alpha : X \to \mathbf{G}$; see [Sp, 6.4.5]. Moreover, we can write $H^0(X) = \overline{H}^0(X)/K$, where K is the group of all constant functions $X \to \mathbf{G}$. If $X \neq \varnothing$, we may identify $K = \mathbf{G}$. If $i : A \hookrightarrow X$, then $i^* : \overline{H}^0(X) \to \overline{H}^0(A)$ is the restriction map $i^*\alpha = \alpha | A$. Thus $i^* = 0$ if and only if A is contained in a quasicomponent of X.

We say that two points $x, y \in A$ are *separated* in X if they belong to different quasicomponents, that is, X can be written as a disjoint union of closed sets E, F with $x \in E$, $y \in F$. Equivalently, there is a continuous map $\alpha : X \to \{0, 1\}$ with $\alpha(x) = 0$, $\alpha(y) = 1$.

3.7. LEMMA. *Let X be a topological space and let $A \subset B \subset X$. Then the following two conditions are equivalent:*

(1) *The sequence $\overline{H}^0(X) \to \overline{H}^0(B) \to \overline{H}^0(A)$ is slow.*

(2) *The sequence $H^0(X) \to H^0(B) \to H^0(A)$ is slow.*

Moreover, they imply the condition:

(3) *If points $x, y \in A$ are separated in B, they are separated in X.*

If X is compact metrizable and if A is closed in X, then all three conditions are equivalent.

PROOF. The equivalence of (1) and (2) follows easily from the considerations of 3.6. We show that (1) implies (3). Assume that $x, y \in A$ are separated in B. Then there is a continuous map $\beta : B \to \{0, 1\}$ with $\beta(x) = 0$, $\beta(y) = 1$. By (1) there is $\gamma \in \overline{H}^0(X)$ with $\gamma|A = \beta|A$. Since $\gamma(x) \neq \gamma(y)$, x and y are separated in X. Thus (3) is true.

Suppose that X is compact metrizable, that A is closed in X and that (3) is true. Let $\beta \in \overline{H}^0(B)$ and write $\alpha = \beta|A$. Since A is compact, $\alpha A \subset \mathbf{G}$ is finite. Hence we can express A as a disjoint union of compact sets $A_1, \dots, A_k$ such that $\alpha A_i = \{\gamma_i\}$ for distinct elements $\gamma_1, \dots, \gamma_k$. If $x \in A_i$, $y \in A_j$ with $i \neq j$, then $\beta(x) \neq \beta(y)$, which implies that x and y are separated in B. By (3) they are separated in X. Hence no component of X meets both A_i and A_j. By Lemma 3.8 below, X can be written as a disjoint union of compact sets $X_1, \dots, X_k$ with $A_i \subset X_i$. Define $\gamma : X \to \mathbf{G}$ by setting $\gamma(x) = \gamma_i$ for $x \in X_i$. Then $\gamma \in \overline{H}^0(X)$ and $\gamma|A = \alpha$, and we have proved (1).

3.8. LEMMA. *Suppose that X is a compact metrizable space and that $A_1, \dots, A_k$ are disjoint compact sets in X such that no component of X meets two sets A_i, A_j. Then X can be written as a disjoint union of compact sets $X_1, \dots, X_k$ with $A_i \subset X_i$.*

PROOF. In the case $k = 2$, the lemma is given in [Wh, I(9.3), p. 15]. The general case follows by induction.

3.9. *Terminology.* The LLC concepts were defined in 1.1 using open balls in LLC_1 and closed balls in LLC_2. In the following theorem it is convenient to replace open by closed and closed by open. We let LLC_1' and LLC_2' denote the new concepts. This only means an arbitrarily small change in the parameter c.

3.10. THEOREM. *Let M be a metric space and let $A \subset \dot{M}$ be compact. Then the following two conditions are equivalent:*

(1) *A is cohlog outer $(0, c)$-joinable in M.*

(2) *Every component of A is continuumwise c-LLC_1'.*

The following two conditions are also equivalent:

(3) *A is cohlog inner $(0, c)$-joinable in M.*

(4) *Every component of A is continuumwise c-$0\text{LLC}_2'$.*

PROOF. We prove only the first part, since the proof of the second part is rather similar. Suppose that (1) is true, that C is a component of A and that

$a \in M$, $r > 0$. Let $x, y \in C \cap \overline{B}(a, r)$. By (1), the sequence

$$\text{(a)} \qquad H^0(A) \to H^0(A \cap \overline{B}(a, cr)) \to H^0(A \cap \overline{B}(a, r))$$

is slow. Since x and y are not separated in A, it follows from 3.7 that they are not separated in $A \cap \overline{B}(a, cr)$. Since this set is compact and since the quasi-components of a compact set are components, there is a component of $A \cap \overline{B}(a, cr)$ containing x and y. Hence (2) is true.

Conversely, assume that (2) holds. Let $a \in M$ and $r > 0$. It suffices to show that (a) is slow. Let $x, y \in A \cap \overline{B}(a, r)$ be points which are not separated in A. Since A is compact, these points belong to a component C of A. By (2), there is a continuum α with $\{x, y\} \subset \alpha \subset C \cap \overline{B}(a, cr)$. Hence x and y are not separated in $A \cap \overline{B}(a, cr)$. From 3.7 it follows that (a) is slow.

3.11. THEOREM. (Duality in the plane). *Let U be open in $\dot{R}^2$ and let $X = \dot{R}^2 \setminus U$. Then:*

(a) *The components of U are pathwise c-LLC$_1$ if and only if the components of X are continuumwise c-LLC$'_2$.*

(b) *The components of U are pathwise c-LLC$_2$ if and only if the components of X are continuumwise c-LLC$'_1$.*

(c) *The components of U are pathwise c-LLC if and only if the components of X are continuumwise c-LLC$'$.*

PROOF. This follows directly from 2.7, 3.5 and 3.10.

3.12. REMARKS. 1. In this generality, Theorem 3.11 seems to be new. However, several special cases are well known, at least quantitatively. For example, assume that X is connected. Then the components of U are simply connected domains in $\dot{R}^2$. For such domains, the property c-LLC$_2$ is known to be quantitatively equivalent to the c-John property [NV, 4.5]. Recall that c-LLC$_1$ is quantitatively equivalent to c-bounded turning. Hence we obtain the result [GNV, 5.9]: A continuum $X \subset \dot{R}^2$ is of c-bounded turning if and only if, quantitatively, all components of $\dot{R}^2 \setminus X$ are c-John domains.

Next assume that U is connected. We see that U is c-LLC if and only if each component C of X is c-LLC$'$. One can show that this happens if and only if C is either a point or a closed K-quasidisk, where c and K depend only on each other; cf. [Ge$_2$, Lemma 4]. Consequently, a domain $U \subset \dot{R}^2$ is c-LLC if and only if, quantitatively, it is a K-quasicircle domain. This was proved in [GM, 2.21].

2. If $X \subset \dot{R}^n$ is compact and continuumwise c-LLC$_1$, it is pathwise c'-LLC$_1$ for every $c' > c$; see [NV, 4.3]. I do not know whether the corresponding result is true for LLC$_2$.

4. Joinability and maps.

4.1. *Summary of Section* 4. We study the behavior of the joinability properties under various maps. It is almost obvious that they are quantitatively preserved by L-bilipschitz maps. We show in 4.3 that, more generally, they are quantitatively invariant under η-quasisymmetric maps. Furthermore, hlog (and cohlog) (p,c)-joinability is quantitatively invariant under η-quasimöbius maps. In particular, this is true for K-quasiconformal maps $f:\dot{R}^n \to \dot{R}^n$. In 4.11 we show that, under certain additional conditions, a K-quasiconformal map $f:G\to G'$ quantitatively preserves (p,c)-joinability if $p\ge 1$. For $p=0$ this is not true. Hence the result is relevant only for $n\ge 3$, and it seems to be one reason for the striking difference between the quasiconformal theory in the plane and in higher dimensions.

4.2. *Quasisymmetric maps.* A *growth function* is a homeomorphism $\eta:[0,\infty)\to[0,\infty)$ such that $\eta(t)\ge t$ for all t. Suppose that M and M' are metric spaces, that η is a growth function and that $A\subset M$. An embedding $f:A\to M'$ is said to be *η-quasisymmetric* if

$$\frac{|fa-fx|}{|fb-fx|}\le \eta\Big(\frac{|a-x|}{|b-x|}\Big)$$

for all triples of distinct points x,a,b in A. If $\infty\in A\subset\dot{M}$, an embedding $f:A\to\dot{M}'$ is called η-quasisymmetric if $f(\infty)=\infty$ and if $f\mid A\setminus\{\infty\}$ is η-quasisymmetric. For example, an L-bilipschitz map is η-quasisymmetric with $\eta(t)=L^2t$.

4.3. THEOREM. *Suppose that M and M' are metric spaces, that $A\subset\dot{M}$ and that $f:A\to\dot{M}'$ is η-quasisymmetric. If A is hlog outer (p,c)-joinable, then fA is hlog outer (p,c')-joinable with $c'=c'(c,\eta)$. A similar statement is valid for the other three joinability properties.*

Explicit bound. The theorem is true with any $c'>\eta(c)$.

Proof We prove the case where A is hlog inner (p,c)-joinable; the other three cases are rather similar. Let $c'>\eta(c)$. We show that fA is hlog inner (p,c')-joinable.

Let $x'=fx\in fA\setminus\{\infty\}$ and let $r>0$. Set $B_1'=fA\cap\overline{B}(x',r)$, $B_2'=fA\cap\overline{B}(x',c'r)$ and

$$t=\sup\{|a-x|:a\in f^{-1}B_1'\}.$$

We may assume that $t>0$. The number t is finite, since quasisymmetric maps map bounded sets onto bounded sets. Setting $B_1=A\cap\overline{B}(x,t)$ we have $f^{-1}B_1'\subset B_1$. Hence f defines a map $f_1:A\setminus B_1\to fA\setminus B_1'$.

Setting $B_2 = A \cap \overline{B}(x, ct)$ we show that $fB_2 \subset B_2'$. Let $b \in B_2$ and choose $\lambda > 1$ such that $\eta(\lambda c) = c'$. Next choose $a \in f^{-1}B_1'$ with $|a - x| \geq t/\lambda$. Now

$$|fb - x'| \leq \eta\Big(\frac{|b-x|}{|a-x|}\Big)|fa - x'| \leq \eta\Big(\frac{ct}{t/\lambda}\Big)r = c'r.$$

Hence $fB_2 \subset B_2'$. Thus f^{-1} defines a map $g_2 : fA \setminus B_2' \to A \setminus B_2$, and we obtain the commutative diagram

$$\begin{array}{ccccc} H_p(fA \setminus B_2') & \longrightarrow & H_p(fA \setminus B_1') & \longrightarrow & H_p(fA) \\ \downarrow g_{2*} & & \uparrow f_{1*} & & \uparrow f_* \\ H_p(A \setminus B_2) & \longrightarrow & H_p(A \setminus B_1) & \longrightarrow & H_p(A). \end{array}$$

Here f_* is an isomorphism, and the lower row is fast. It follows that the upper row is fast. Hence fA is hlog inner (p, c)-joinable.

4.4. *Quasimöbius maps.* Let M be a metric space and let a, b, c, d be distinct points in $\dot{M}$. The *cross ratio* of the quadruple (a, b, c, d) is the number

$$|a, b, c, d| = \frac{|a-b||c-d|}{|a-c||b-d|}$$

if all points are finite. If one of them is ∞, the corresponding factors are canceled out.

Let M' be another metric space, let $A \subset \dot{M}$ and let η be a growth function; see 4.2. An embedding $f : A \to \dot{M}'$ is *η-quasimöbius* if

$$|fa, fb, fc, fd| \leq \eta(|a, b, c, d|)$$

for all quadruples (a, b, c, d) of distinct points in A. We recall that a homeomorphism $f : \dot{R}^n \to \dot{R}^n$ is η-quasimöbius if and only if, n-quantitatively, f is K-quasiconformal. Furthermore, an η-quasisymmetric map is θ-quasimöbius with $\theta = \theta_\eta$.

Neither the inner nor the outer joinability properties are invariant under quasimöbius maps. However, we show in 4.6 that (p, c)-joinability is quantitatively preserved by η-quasimöbius maps. We first prove an auxiliary result, for which we introduce some terminology. By an *annulus* in a metric space M we mean a set

$$R = R(a; r_1, r_2) = \{x \in M : r_1 < |x - a| < r_2\},$$

where $a \in M$ and $0 < r_1 < r_2 < \infty$. Let $s(R)$ denote the ratio r_2/r_1. If E and F are subsets of $\dot{M}$ such that one of them is contained in $\overline{B}(a, r_1)$ and the other in $\dot{M} \setminus B(a, r_2)$, we say that R *separates* E and F. For $c \geq 1$, the sets E

and F are *c-separated* in $\dot{M}$ if there is an annulus R with $s(R) \geq c$ separating E and F.

A reader interested only in the case $M = M' = R^n$ of Theorem 4.6 can replace Lemma 4.5 by the simpler version where f is assumed to be a Möbius map of $\dot{R}^n$. In this case the lemma is known to hold with $c_0 = 1$ and $h(c) = c^{1/2} + c^{-1/2} - 1$; see [Ge$_1$, p. 174]. However, even in this case the details are a bit tedious; they are given in [Wa, pp. 71--75]. If $A \subset \dot{R}^n$ and if $f : A \to \dot{R}^n$ is η-quasimöbius, we can factorize $f = ugv$ where g is η-quasisymmetric and u, v are Möbius maps of $\dot{R}^n$. To prove the quasimöbius invariance of (p, c)-joinability in $\dot{R}^n$ it therefore suffices, in view of 4.3, to prove the invariance under Möbius maps. This can be done as in 4.6 replacing 4.5 by the simplified version.

4.5. LEMMA. *Let η be a growth function. Then there are a constant $c_0 \geq 1$ and a homeomorphism $h : [c_0, \infty) \to [1, \infty)$ such that the following is true:*

Let M and M' be metric spaces, let $A \subset \dot{M}$, and let $f : A \to \dot{M}'$ be η-quasimöbius. Let $c > c_0$ and let $E, F \subset A$ be c-separated in A. Then fE and fF are $h(c)$-separated in fA.

Explicit bounds. If $M = R^n$ and f is Möbius, the lemma is true with $c_0 = 1$, $h(c) = c^{1/2} + c^{-1/2} - 1$ and also with $c_0 = 16/9$, $h(c) = 3\sqrt{c}/4$. In the general case one can choose $c_0 = 1 + 6/\eta^{-1}(1/7)$, $h(c) = (7\eta(6/(c-1)))^{-1/2}$.

PROOF. Suppose that $f : A \to \dot{M}'$ is η-quasimöbius and that $E, F \subset A$ are c-separated in A for some $c \geq 3$. We may assume that both sets E, F contain at least two points. Choose an annulus $R = R(a; r, cr)$ separating E and F in A. We may assume that $E \subset \overline{B}(a, r)$ and $F \subset A \setminus B(a, cr)$.

Case 1. $d(fE) \leq d(fF)$. Let $x, y \in E$ with $x \neq y$, $fx \neq \infty$. Let $u, v \in F$ with $u \neq v$, $u \neq \infty$. Consider the cross ratio $\tau = |x, y, u, v|$. Assuming $v \neq \infty$ we obtain

$$\tau \leq 2r \frac{|u-x| + |x-y| + |y-v|}{|x-u||y-v|} \tag{1}$$
$$\leq 2r\left(\frac{1}{|y-v|} + \frac{2r}{|x-u||y-v|} + \frac{1}{|x-u|}\right)$$
$$\leq 2r\left(\frac{1}{(c-1)r} + \frac{2}{(c-1)^2 r} + \frac{1}{(c-1)r}\right)$$
$$\leq \frac{6}{c-1},$$

since $c \geq 3$. This is easily seen to be true also if $v = \infty$.

Suppose first that fF is bounded. Choose $v \in F$ so that

$$(2) \qquad 3|fu - fv| \geq d(fF) \geq d(fE) \geq |fx - fy|.$$

We estimate the cross ratio $f\tau = |fx, fy, fu, fv|$. Since $d(fE) \leq d(fF) < \infty$, none of the points is ∞. Write $t = |fx - fy|/|fx - fu|$ and $\alpha = t/f\tau$. Applying (2) we get

$$\alpha = \frac{|fy - fv|}{|fu - fv|} \leq \frac{|fy - fx| + |fx - fu| + |fu - fv|}{|fu - fv|} \leq 3 + 3\frac{|fx - fu|}{|fx - fy|} + 1 \leq 4 + 3/t.$$

Since $f\tau \leq \eta(\tau)$, these estimates yield

$$(3) \qquad \frac{t^2}{4t + 3} \leq \eta\left(\frac{6}{c - 1}\right).$$

Set $c_0 = 1 + 6/\eta^{-1}(1/7)$. Then $\eta(6/(c_0 - 1)) = 1/7$. Since $\eta(s) \geq s$ for all s, we have $c_0 \geq 43$.

Assume that $c > c_0$. The function $g(s) = s^2/(4s + 3)$ is increasing in $s \geq 0$. Since the right-hand side of (3) is less than $1/7$ and since $g(1) = 1/7$, we have $t < 1$. Hence $t^2 = (4t + 3)g(t) < 7g(t)$. By (3) this implies $t < \varphi(c)$, where $\varphi : [c_0, \infty) \to (0, 1]$ is the decreasing homeomorhism

$$\varphi(c) = \left(7\eta\left(\frac{6}{c - 1}\right)\right)^{1/2}.$$

For a fixed x we consider the number

$$s = \sup\{|fy - fx| : y \in E\}.$$

For all $y \in E$ and $u \in F$ we have

$$|fy - fx| = t|fx - fu| \leq \varphi(c)|fx - fu|.$$

Hence $fE \subset \overline{B}(fx, s)$ and $fF \subset fA \setminus B(fx, s/\varphi(c))$. Setting $h(c) = 1/\varphi(c)$ we get a homeomorphism $\varphi : [c_0, \infty) \to [1, \infty)$ satisfying the condition of the lemma.

Next assume that fF is unbounded. Choose c_0 as above and assume that $c > c_0$. If $\infty \in fF$, we choose $fv = \infty$ and obtain $f\tau = |fx - fy|/|fx - fu| = t$, which yields $t \leq \eta(6/(c - 1))$ and a better estimate than above. Assume that $\infty \notin fF$. If $\infty \in fE$, we choose $fy = \infty$. Then $f\tau = |fu - fv|/|fx - fv| \to 1$ as $fv \to \infty$, and we obtain $1 \leq \eta(6/(c - 1))$ by (1). This implies $c \leq c_0$, a contradiction. Hence $\infty \notin fE$. Now we can choose v such that $f\tau = t|fu - fv|/|fy - fv|$ is arbitrarily close to t, and we again get $t \leq \eta(6/(c - 1))$.

Case 2. $d(fF) < d(fE)$. Now choose $x, y \in F$ and $u, v \in E$, and consider the cross ratio $\tau = |x, y, u, v|$. Since

$$\tau \leq 2r\frac{|x-u|+|u-v|+|v-y|}{|x-u||y-v|} \leq \frac{6}{c-1},$$

the estimate (1) holds again. We can then proceed as in Case 1 and obtain the same c_0 and h. In this case fF lies inside and fE outside of the separating annulus.

4.6. THEOREM. *Suppose that M and M' are metric spaces, that $A \subset \dot{M}$ and that $f : A \to \dot{M}'$ is η-quasimöbius. If A is hlog (p,c)-joinable, then fA is hlog (p,c')-joinable with $c' = c'(c,\eta)$. The corresponding statement holds for cohlog joinability.*

Explicit bounds. If $M = M' = R^n$ and f is Möbius, the theorem is true with $c' = 2c^2$. If $M = M' = R^n$ and f is η-quasimöbius, one can choose $c' = 2\eta(2c^2)^2$. If $\infty \in A$ or $\infty \in fA$, this can be replaced by $2\eta(c)^2$ or $\eta(2c^2)$, respectively. In the general case the theorem holds with any $c' > 1 + 6\eta(7c^2)$.

PROOF. We may assume that $c > 1$. The map $g = f^{-1} : fA \to A$ is η'-quasimöbius with $\eta'(t) = \eta^{-1}(t^{-1})^{-1}$. Let c_0 and $h : [c_0, \infty) \to [1, \infty)$ be the quantities given by Lemma 4.5 for the growth function η'. We show that the theorem holds with any $c' > h^{-1}(c) > c_0$.

We first prove that fA is hlog outer (p,c')-joinable. Let $b \in fA \setminus \{\infty\}$ and $r > 0$. Set $B_1' = fA \cap B(b,r)$, $B_2' = fA \cap B(b,c'r)$. By 4.5, there is an annulus $R(a; s, h(c')s)$ in A separating the sets gB_1' and $g[fA \setminus B_2']$. Thus either $gB_1' \subset \overline{B}(a,s)$ or $gB_1' \subset A \setminus B(a,h(c')s)$.

Case 1. $gB_1' \subset \overline{B}(a,s)$. Write $t = h(c')s/c$. Since $c' > h^{-1}(c)$, we have $t > s$. Set $B_1 = A \cap B(a,t)$, $B_2 = A \cap B(a,ct) = A \cap B(a,h(c')s)$. Now f and g define maps $f_1 : B_2 \to B_2'$ and $g_1 : B_1' \to B_1$, and we obtain the commutative diagram

$$\begin{array}{ccccc}
H_p(B_1') & \longrightarrow & H_p(B_2') & \longrightarrow & H_p(fA) \\
\Big\downarrow g_{1*} & & \Big\uparrow f_{1*} & & \Big\uparrow f_* \\
H_p(B_1) & \longrightarrow & H_p(B_2) & \longrightarrow & H_p(A).
\end{array}$$

Since A is hlog outer (p,c)-joinable, the lower row is fast. Since f_* is an isomorphism, it easily follows that also the upper row is fast.

Case 2. $gB_1' \subset A \setminus B(a,h(c')s)$. Set $B_1 = A \cap \overline{B}(a,s)$, $B_2 = A \cap \overline{B}(a,cs)$. Now f and g define maps $f_1 : A \setminus B_1 \to B_2'$ and $g_1 : B_1' \to A \setminus B_2$, and we have the diagram

$$\begin{array}{ccccc} H_p(B_1') & \longrightarrow & H_p(B_2') & \longrightarrow & H_p(fA) \\ \downarrow{\scriptstyle g_{1*}} & & \uparrow{\scriptstyle f_{1*}} & & \uparrow{\scriptstyle f_*} \\ H_p(A \setminus B_2) & \longrightarrow & H_p(A \setminus B_1) & \longrightarrow & H_p(A). \end{array}$$

Since A is hlog inner (p, c)-joinable, the lower row is fast. Hence the upper row is again fast, and we have proved that fA is hlog outer (p, c')-joinable. The proofs for the inner joinability and for the cohlog case are rather similar.

4.7. THEOREM. *For a set $A \subset \dot{R}^n$, the following conditions are quantitatively equivalent:*

(1) *A is hlog (p, c)-joinable.*

(2) *fA is hlog outer (p, c)-joinable for every Möbius map of $\dot{R}^n$.*

(3) *fA is hlog inner (p, c)-joinable for every Möbius map of $\dot{R}^n$.*

The statement is also true with hlog replaced by cohlog.

Explicit bounds. (1) implies (2) and (3) with $c \mapsto 2c^2$. Each of (2) and (3) implies (1) with $c \mapsto 2c + 1$. See also 4.16.

PROOF. By 4.6, (1) implies (2) and (3) quantitatively. Suppose that (2) is true. With $f = \text{id}$ it implies that A is hlog outer (p, c)-joinable. Let $a \in R^n$ and let $r > 0$. Choose an inversion f of $\dot{R}^n$ which interchanges the spheres $S(a, r)$ and $S(a, 3cr)$. By (2) and 2.5, the set fA is hlog outer $(p, 3c)$-joinable in $\dot{R}^n$. Hence the sequence

$$H_p(fA \cap B(a, r)) \to H_p(fA \cap B(a, 3cr)) \to H_p(fA)$$

is fast. This is isomorphic to the sequence

$$H_p(A \setminus \overline{B}(a, 3cr)) \to H_p(A \setminus \overline{B}(a, r)) \to H_p(A).$$

Hence A is hlog inner $(p, 3c)$-joinable. The proofs for (3) $\Rightarrow$ (1) and for the cohlog case are rather similar.

4.8. *Quasiballs.* If $f : \dot{R}^n \to \dot{R}^n$ is K-quasiconformal, the set fB^n is an *open K-quasiball*, and $f\overline{B}^n$ is a *closed K-quasiball*. By [AVV, 5.23], f is η-quasimöbius with $\eta = \eta_K$. Hence it follows from 4.6 that open and closed quasiballs are (p, c)-joinable with $c = c(K)$ for all $0 \le p \le n - 2$.

4.9. *Quasiconformal maps.* Suppose that D and D' are domains in $\dot{R}^n$, and let $f : D \to D'$ be a K-quasiconformal map (homeomorphism). Suppose also that D is (p, c)-joinable. We shall study under which conditions D' is (p, c')-joinable with $c' = c'(c, K, n)$. If f extends to a K-quasiconformal map $g : \dot{R}^n \to \dot{R}^n$, this follows from 4.6 as in 4.8. In the general case, D' need not be $(0, c')$-joinable even if D is a ball. For example, D' can be the infinite tube $B^{n-1} \times R^1$, which is not inner (0)-joinable. However, Gehring [Ge_1, p. 174] observed that if $n \geq 3$ and if $D = B^n$, then $\complement D'$ is c'-LLC with $c' = c'(K, n)$. By the duality theorem, this means that D' is $(n-2, c')$-joinable. We shall show in 4.13 that D' is indeed (p, c')-joinable for all $1 \leq p \leq n-2$. (A still stronger result will be given in 5.27.) This will follow from the more general result 4.11, in which D is only assumed to be QED and to satisfy a homological triviality condition.

We recall that a domain $D \subset \dot{R}^n$ is called c-QED (quasiextremal distance domain) if

$$M(E, F) \leq cM(E, F; D)$$

for each pair of continua $E, F \subset D$; here $M(E, F)$ is the modulus of the family of all paths joining E and F in $\dot{R}^n$, and $M(E, F; D)$ is the modulus of the subfamily whose members are in D. The concept was introduced by Gehring and Martio [GM], who proved the n-quantitative implications c-uniform $\Rightarrow$ c-QED $\Rightarrow$ c-LLC [GM, 2.11, 2.18].

We need a topological auxiliary result:

4.10. Lemma. *Suppose that $0 \leq p \leq n-1$, that U is open in $\dot{R}^n$ with $H_{p+1}(U) = 0$, that F is closed in U and that z is a p-cycle in $U \setminus F$, not bounding in $U \setminus F$. Then there is a component F_0 of F such that z does not bound in $U \setminus F_0$.*

Proof. Let Φ be the family of all open sets $V \subset U$ such that $U \setminus F \subset V$ and such that z does not bound in V. Thus $U \setminus F \in \Phi$. With the aid of Brouwer's reduction theorem [HW, p. 161] we show that Φ contains a maximal element. It suffices to show that if $V_1 \subset V_2 \subset \ldots$ is an ascending sequence in Φ, then the union V of the sequence is in Φ. Assume that $V \notin \Phi$. Then $z = \partial g$ for some $g \in S_{p+1}(V)$. Since $|g|$ is compact, $|g| \subset V_j$ for some j, and we get the contradiction $V_j \notin \Phi$.

Let V be a maximal member of Φ. It suffices to show that the set $A = U \setminus V$ is connected, since then A is contained in a component of F. If A is not connected, we can express A as a disjoint union $A = A_1 \cup A_2$, where the sets A_j are nonempty and closed in A and hence in U. The sets $V_j = U \setminus A_j$ are open. Since V is maximal in Φ, z bounds in V_1 and in V_2. Consider the exact Mayer--Vietoris sequence

$$H_{p+1}(V_1 \cup V_2) \to H_p(V_1 \cap V_2) \to H_p(V_1) \oplus H_p(V_2).$$

Here $V_1 \cup V_2 = U$ and $V_1 \cap V_2 = V$. The class u of z in $H_p(V)$ is mapped to zero by the second map. Since $H_{p+1}(U) = 0$, this implies $u = 0$. Hence z bounds in V, which is a contradiction.

4.11. THEOREM. *Suppose that* $1 \leq p \leq n-2$, *that* $D \subset \dot{R}^n$ *is a* c-QED (p, c)-*joinable domain and that* $f : D \to D'$ *is* K-*quasiconformal. Suppose also that* $H_{p+1}(D) = 0$. *Then* D' *is* (p, c')-*joinable with* $c' = c'(c, K, n)$. *Moreover,* $\complement D'$ *is* $(n-2-p, c'')$-*joinable with* $c'' = 2c' + 1$.

PROOF. Observe that the hypotheses imply that $n \geq 3$. The last statement follows from the first part and from the absolute duality theorem 2.8. By 4.7 it suffices to show that D' is outer (p, c')-joinable, since the hypothesis is Möbius invariant. To this end, let $y_0 \in D'$, $r > 0$, and let z' be a p-cycle in $D' \cap B(y_0, r)$ bounding in D'. Suppose that $c' > 1$ is such that z' does not bound in $D' \cap B(y_0, c'r)$. It suffices to find an upper bound $c' \leq c_1(c, K, n)$.

Set $F = f^{-1}[D' \setminus B(y_0, c'r)]$. The map f^{-1} carries z' to a p-cycle z, which bounds in D but not in $D \setminus F$. By 4.10 there is a component F_0 of F such that z does not bound in $D \setminus F_0$. We can write $z = z_1 + \ldots + z_k$ such that the sets $|z_j|$ are the components of $|z|$. Since $p \geq 1$, each z_j is a p-cycle, and some z_j does not bound in $D \setminus F_0$. We may assume that z_1 does not bound in $D \setminus F_0$. We shall prove the inequalities

(1) $d(|z_1|, F_0) \leq cd(|z_1|)$,

(2) $d(|z_1|, F_0) \leq cd(F_0)$.

Assume that (1) is false. Fix a point $a \in |z_1|$. Write $s = d(|z_1|)$ and $\lambda = d(|z_1|, F_0)/cs$. Then $\lambda > 1$ and $|z_1| \subset B(a, \lambda s)$. Since D is (p, c)-joinable, z_1 bounds in $D \cap B(a, c\lambda s) \subset D \setminus F_0$, which is a contradiction.

Next assume that (2) is false. Now fix $a \in F_0$ and set $s = d(F_0)$. Then $|z_1| \subset D \setminus \overline{B}(a, cs)$. Since D is (p, c)-joinable, z_1 bounds in $D \setminus \overline{B}(a, s) \subset D \setminus F_0$, which is again a contradiction.

Let $1 < t < c'$ and write $U = f^{-1}[D' \setminus \overline{B}(y_0, tr)]$. Then $F \subset U$. Choose points $a, b \in F_0$ such that $d(a, |z_1|) \leq 2d(F_0, |z_1|)$ and $|a - b| \geq d(F_0)/3$. Since F_0 is a connected subset of the open set U, we can join the points a, b by an arc $\alpha \subset U$. Then $d(\alpha) \geq |a - b| \geq d(F_0)/3$ and $d(\alpha, |z_1|) \leq d(a, |z_1|) \leq 2d(F_0, |z_1|)$. Hence (1) and (2) yield

$$d(\alpha, |z_1|) \leq 6c(d(\alpha) \wedge d(|z_1|)).$$

Since D is c-QED, a standard modulus estimate [GM, 2.6] gives $M(\alpha, |z_1|; D) \geq q(c, n) > 0$. On the other hand, the sets $f\alpha$ and $f|z_1|$ are separated by the annulus $R(y_0; r, tr)$, and hence

$$M(f\alpha, f|z_1|; D') \leq \omega_{n-1}(\log t)^{1-n},$$

where ω_{n-1} is the area of S^{n-1}. Since f is K-quasiconformal, these estimates imply

$$q(c,n) \le K\omega_{n-1}(\log t)^{1-n}.$$

Letting $t \to c'$ gives the desired bound $c' \le c_1(c,K,n)$.

4.12. Corollary. *Let $D \subset \dot{R}^n$ be a c-QED domain such that $\complement D$ is c-LLC, and let $f : D \to D'$ be K-quasiconformal. Then $\complement D'$ is c'-LLC with $c' = c'(c,K,n)$.*

Proof. The condition LLC implies that $\complement D$ is connected and hence $H_{n-1}(D) = 0$. The result follows from 4.11 together with 2.7 and 3.5.

4.13. Corollary. *Let $n \ge 3$ and let $f : B^n \to D'$ be a K-quasiconformal map. Then D' is (p,c')-joinable for all $1 \le p \le n-2$ with $c' = c'(K,n)$, and hence $\complement D'$ is $(q, 2c'+1)$-joinable for all $0 \le q \le n-3$.*

4.14. Example. We show that 4.11 is not true without the condition $H_{p+1}(D) = 0$. Let $D_0 \subset R^3$ be the cylinder $B^2 \times (-1,1)$, let $1/2 < t < 1$, and let C be the suspension (double cone) of $\overline{B}^2(1/2)$ with $\pm te_3$ as suspension points, $e_3 = (0,0,1)$. Then $D = D_0 \setminus C$ is a domain in R^3.

The components C and $\complement D_0$ of $\complement D$ are closed K_0-quasiballs with a universal K_0. By 4.8, they are $(0,c)$-joinable with a universal c. By 2.8, D is $(1,3c)$-joinable for all t. An elementary but lengthy cases-and-subcases argument shows that D is c_1-uniform in the ordinary sense with a universal c_1. Hence D is c_2-QED with a universal c_2 by [GM, 2.18]. Thus D satisfies the conditions of 4.11 for $p = 1$ except that $H_2(D) \ne 0$. Let D'_0 be the infinite tube $B^2 \times R^1$, and let $f_0 : D_0 \to D'_0$ be a K-quasiconformal map such that $f_0 B^2 = B^2$ and $f_0(x)_3 \to \pm\infty$ as $x \to \pm e_3$. Then f_0 defines a K-quasiconformal map of D onto a domain D', and D' is not outer $(1,c')$-joinable with any c' independent of t.

4.15. Question. Suppose that $D \subset R^3$ is a hlog trivial $(1,c)$-joinable domain and that $f : D \to D'$ is K-quasiconformal. Is D' $(1,c')$-joinable with $c' = c'(K)$? In the possible counterexample D cannot be $(0,c)$-joinable, since then 5.22 would imply that D is c'-uniform and hence c''-QED, contrary to 4.11.

4.16. *Relative versions.* Since each joinability property is, by 2.5, quantitatively equivalent to the corresponding relative property in the space M, all results of this section have relative versions, which follow directly from the corresponding absolute results. However, one can often get better bounds for the constants by rewriting the proof in the relative setting. In 4.7, for example, (2) implies (1) with the same constant c if joinability is understood to be relative to R^n.

5. Joinability, John and uniformity.

5.1. *Summary of Section* 5. We study the relations between the joinability properties of an open set $U \subset \dot{R}^n$ and the John and uniform properties of U, all taken in the p-dimensional sense. In order to get better results we slightly generalize the definition of hlog (p, c)-uniform domains by setting the uniformity condition only on *null-homologous* p-cycles. Moreover, the set U is not required to be connected. These sets U are called *weakly* (p, c)-uniform, and the weakly (p, c)-John sets are defined analogously. We show in 5.5 that weakly (p)-John sets are inner (p)-joinable and in 5.6 that weakly (p)-uniform sets are (p)-joinable. The converse results 5.21 and 5.22 need some additional conditions. Some applications are given in 5.24--5.27.

5.2. *Weakly John and uniform sets.* We first recall the known results in the case $p = 0$. A c-uniform domain $D \subset \dot{R}^n$ is c'-LLC with $c' = c'(c)$. The implications "uniform $\Rightarrow$ QED $\Rightarrow$ LLC" of [GM] give this with $c' = c'(c, n)$, but a direct proof with $c' = c'(c)$ is easy; cf. [Vä$_1$, 4.7].

The converse is false as in seen from the annuli $R(a; r_1, r_2)$ with r_2/r_1 close to 1 or from the domain between two parallel planes in R^3. However, the converse is true for simply connected planar domains D, for which c-LLC implies that either $\#\partial D \le 1$ or D is a K-quasidisk with $K = K(c)$ [Ge$_2$, Lemma 4].

A c-John domain is c-LLC$_2$ with $c' = c'(c)$. The proof is easy but seems to belong to the folklore. The converse is again true for simply connected planar domains [NV, 4.5] and, more generally, for K-quasiconformal images of B^n [He, 3.1].

Uniform domains of order p have recently been studied by J. Heinonen and S. Yang [HY] and by P. Alestalo [Al]. In [HY] the definition is based on homotopy; in [Al] both homotopical and homological versions are considered. Since we do not consider homotopical uniformity in this paper, we shall simplify the terminology by omitting the word "homologically".

We first give a somewhat generalized definition for (p, c)-uniformity. Indeed, we allow that the open set U is not connected, and the uniformity condition is only required to hold for null-homologous cycles.

Suppose that U is open in $\dot{R}^n$, that $0 \le p \le n-2$ and that $c \ge 1$. We say that U is *weakly* (p, c)*-John* if for every p-cycle z bounding in U there is a $(p+1)$-chain g such that $\partial g = z$ and

$$\text{(L)} \qquad d(x, |z|) \le c d(x, \partial U) \quad \text{for all} \quad x \in |g|.$$

If, in addition, g satisfies the condition

$$\text{(T)} \qquad d(|g|) \le c d(|z|),$$

the set U is *weakly (p,c)-uniform*. We say that (L) is the *lens condition* and (T) is the *turning condition*. Together they are called *uniformity conditions*.

As with joinability, U is said to be weakly (p)-John or weakly (p)-uniform if it is weakly (p,c)-John or weakly (p,c)-uniform for some c, respectively.

We see at once that U is weakly (p,c)-John or weakly (p,c)-uniform if and only if each component of U has this property.

It is possible to characterize these properties purely in terms of homology, without mentioning cycles and chains; see 5.10 and 5.14.

These properties depend also on the coefficient group $\mathbf{G}$, and in a complete notation, one should write $(p,c,\mathbf{G})$-uniform etc. Compared with the terminology of [Al], we see that U is a homologically (p,c)-uniform domain in the sense of [Al] if and only if

(1) U is weakly (p,c)-uniform,
(2) $H_p(U)=0$,
(3) U is connected,
(4) $\infty \notin U$,
(5) $\mathbf{G}=\mathbf{Z}$.

Furthermore, U is weakly $(0,c)$-uniform if and only if, quantitatively, each component of U is a c-uniform domain in the ordinary distance cigar sense. The corresponding statement holds in the John case.

We say that U is *(p,c)-uniform* if it satisfies the conditions (1),(2),(3). Similarly, U is *(p,c)-John* if it satisfies these conditions with "uniform" replaced by "John" in (1).

We have not excluded the case $\infty \in U$. The conditions (L) and (T) make sense in the natural way also in this case; recall that $d(a,\infty)=\infty$ for $a \in R^n$ and $d(\infty,\infty)=0$. In particular, (L) is true for all $c \geq 1$ whenever $x=\infty \in |g|$. However, we show in 5.4 that removing the point ∞ from U does not essentially change the John and uniform properties of U. The proof is based on the following topological observation, which will be useful also later:

5.3. LEMMA. *Suppose that $0 \leq p \leq n-2$ and that $A, V \subset \dot{R}^n$ such that V is open, $H_p(V)=0$, and $\complement V \subset \operatorname{int} A$. Then the map $H_p(A \cap V) \to H_p(A)$ is an isomorphism.*

PROOF. Since $A \cup V = \dot{R}^n$, the lemma follows from the exactness of the Mayer--Vietoris sequence

$$H_{p+1}(A \cup V) \to H_p(A \cap V) \to H_p(A) \oplus H_p(V) \to H_p(A \cup V).$$

5.4. THEOREM. *Let U be an open set in $\dot{R}^n$ with $\infty \in U$ and let $c > 1$. Then:*
(1) *U is weakly (p,c)-John if and only if $U \setminus \{\infty\}$ is weakly (p,c)-John.*

(2) *U is weakly (p,c)-uniform if and only if $U \setminus \{\infty\}$ is weakly (p,c)-uniform.*

PROOF. Write $U_0 = U \setminus \{\infty\}$ and suppose first that U is weakly (p,c)-John. Let z be a p-cycle bounding in U_0. Then $z = \partial g$ for some chain g in U satisfying the lens condition 5.2(L). If $\infty \notin |g|$, there is nothing to prove. Assume that $\infty \in |g|$. Choose $R > 0$ such that the ball $B(R)$ contains $|z| \cup \complement U$. Set $\lambda = (c+1)/(c-1)$ and $A = |g| \cup \complement B(\lambda R)$. By 5.3, $z = \partial g_0$ for some g_0 in $A \setminus \{\infty\}$. We show that g_0 satisfies the lens condition in U_0. Only points $x \in |g_0| \setminus B(\lambda R)$ need to be considered. For these we have

$$\frac{d(x,|z|)}{d(x,\partial U_0)} \le \frac{|x|+R}{|x|-R} \le \frac{\lambda+1}{\lambda-1} = c.$$

Hence U_0 is weakly (p,c)-John.

Conversely, assume that U_0 is weakly p-John. Let z be a p-cycle bounding in U. If $\infty \notin |z|$, then z bounds in U_0 by 5.3. Hence $z = \partial g_0$ with some $(p+1)$-chain g_0 in $U_0 \subset U$ satisfying the lens condition. If $\infty \in |z|$, we may assume that $|z| \ne \{\infty\}$. Fix a point $a \in |z| \setminus \{\infty\}$. Choose a ball $B(a,R)$ containing $\complement U$, and set $\lambda = c/(c-1)$, $A = |z| \cup \complement B(a, 2\lambda R)$. By 5.3, there is a chain g_1 in A such that $\partial g_1 = z - z_0$ with $\infty \notin |z_0|$. Moreover, the p-cycle z_0 bounds in U_0 by 5.3. Hence $z_0 = \partial g_0$ with some g_0 satisfying the lens condition in U_0. Writing $g = g_0 + g_1$ we have $\partial g = z$. We show that g satisfies the lens condition in U.

Suppose that $x \in |g| \subset |g_0| \cup |g_1|$ and that $x \notin |z|$. Set $t = |x-a|/R$.

Case 1. $t \ge \lambda$. Now $d(x,\partial U) \ge (t-1)R$, and hence

$$d(x,|z|) \le |x-a| = Rt \le t d(x,\partial U)/(t-1) \le c d(x,\partial U).$$

Case 2. $t \le \lambda$. Since $|g_1| \cap B(a,\lambda R) \subset |z|$, we have $x \in |g_0|$. Since $|z_0| \subset A$, we obtain

$$d(x,|z|) = d(x,A) \le d(x,|z_0|) \le c d(x,\partial U).$$

Hence U is weakly p-John.

The proof for the uniform case is similar but simpler, because the turning condition excludes the case $\infty \in |g|$ in the first part of the proof.

5.5. THEOREM. *If an open set $U \subset \dot{R}^n$ is weakly (p,c)-John, U is inner $(p, 2c+1)$-joinable in R^n.*

PROOF Write $c' = 2c+1$. Let $a \in R^n$ and $r > 0$. Let z be a p-cycle in $U \setminus \overline{B}(a, c'r)$ bounding in U. We must show that z bounds in $U \setminus \overline{B}(a,r)$. Since U is weakly (p,c)-John, $z = \partial g$ for some $(p+1)$-chain g satisfying the lens condition in U. If $|g| \cap \overline{B}(a,r) = \varnothing$, there is nothing to prove. In the opposite case we fix a point $x \in |g| \cap \overline{B}(a,r)$. Now

$$2cr = c'r - r < d(x, |z|) \leq cd(x, \partial U),$$

and hence $\overline{B}(a,r) \subset \overline{B}(x,2r) \subset U$. Applying 5.3 with $V = \complement\overline{B}(a,r)$ we see that z bounds in $U \setminus \overline{B}(a,r)$.

5.6. THEOREM. *If an open set $U \subset \dot{R}^n$ is weakly (p,c)-uniform, U is $(p, 2c+1)$-joinable in $\dot{R}^n$.*

PROOF. The inner joinability follows from 5.5 and the outer joinability follows easily from the turning condition 5.2(T).

5.7. *Discussion.* The converse results for 5.5 and 5.6 are not true. For example, the (p)-John property is easily destroyed by a countable set, which has no effect on the joinability properties. However, results in the converse direction seem to be extremely useful in the applications. It is clear that some topological conditions are needed. Indeed, if $H_{n-1}(U) = 0$ or, equivalently, $X = \complement U$ is connected, then the converses of 5.5 and 5.6 are quantitatively true in the case $p = n-2$. This can be proved with the Mayer--Vietoris technique. However, for $p \leq n-3$ the situation becomes more complicated. Let us consider the case $n = 3$, $p = 0$. Suppose that $U \subset \dot{R}^3$ is a domain with $X = \complement U \subset S^2$. The following examples show that there are at least four essentially different ways in which the $(0,c)$-John property of U can break down:

(1) X consists of a finite but large number of points scattered in S^2. Alternatively, X consists of two closed caps so that $S^2 \setminus X$ is a narrow neighborhood of the equator.

(2) X is a fairly dense net of lines of latitude and longitude. Alternatively, $X = S^2 \setminus (D_1 \cup D_2)$ where D_1 and D_2 are small open caps centered at the poles.

(3) $X = S^2 \setminus D$ where D is a small open cap.

(4) X is a long spiral-like arc from the north pole to the south pole.

In (1) and (2), U is inner (p,c)-joinable for $p = 0, 1$ with a reasonable c, but the topological properties of U and X are complicated, because $H_0(X) \neq 0$ in (1) and $H_1(X) \neq 0$ in (2). In the last two cases, X and U are topologically very simple, but the inner $(0,c)$-joinability of U breaks down in (3) and the inner $(1,c)$-joinability in (4).

We shall prove a result (Theorem 5.21), which implies that an open set U in $\dot{R}^3$ is weakly c'-John if $H^0(X) = H^1(X) = 0$ and if U is inner $(0,c)$- and $(1,c)$-joinable; $c' = c'(c)$. Replacing inner joinability by joinability we obtain an analogous sufficient condition for weak $(0,c')$-uniformity.

Unfortunately, I have not been able to prove the cases $p \leq n-3$ without deeper tools than those used so far in this paper. Indeed, we shall make use of results on the canonical homomorphisms in the Čech theory, obtained by E.E. Floyd, E. Dyer and O. Jussila in the fifties and early sixties. This theory will be summarized in 5.15. Before that, we characterize the John and uniform properties of an open set U with the aid of inflations of ∂U.

5.8. *Relative inflation.* For $A \subset R^n$ and $r > 0$, the ordinary r-inflation of A is

$$B(A,r) = \{x \in R^n : d(x,A) < r\} = \bigcup_{a \in A} B(a,r).$$

Let V be open in R^n with $V \neq R^n$. For $x \in V$ we set $\delta(x) = d(x, \partial V) = d(x, \complement V)$. For $A \subset V$ and $0 < t < 1$, the *t-inflation* of A *relative to* V is the open set

$$B(A,t,V) = \bigcup_{a \in A} B(a, t\delta(a)).$$

The definition makes sense also if V is open in $\dot{R}^n$ and if $R^n \cap \partial V \neq \emptyset$, $\infty \notin A$. If $\infty \in A$, we set

$$B(A,t,V) = B(A \setminus \{\infty\}, t, V) \cup \{\infty\},$$

but this set is not open in general.

5.9. Lemma. *Suppose that $A \subset B \subset U \subset \dot{R}^n$ and that U is open, A is compact, and $\#A \geq 2$. Then the following conditions are quantitatively equivalent:*

(1) $d(x,A) \leq cd(x, \partial U)$ for all $x \in B$,

(2) $B \cap B(\partial U, 1/c, \complement A) = \emptyset$.

Explicit bounds. Each of the conditions implies the other with the constant $c \mapsto c+1$.

Proof. We may clearly assume that $R^n \cap \partial U \neq \emptyset$. Assume that (1) holds. For $x \in R^n$ write $\delta(x) = d(x,A)$. Assume that $x \in B \cap B(\partial U, 1/(c+1), \complement A)$. If $x = \infty$, this is clearly impossible. If $x \neq \infty$, there is $z \in \partial U \setminus \{\infty\}$ such that $(c+1)|x-z| < \delta(z)$. Since

$$\delta(z) \leq \delta(x) + |x-z| \leq cd(x, \partial U) + |x-z| \leq (c+1)|x-z|,$$

this is a contradiction,

Conversely, assume that (2) is true. Let $x \in B$. If $x = \infty$, then $d(x, \partial U) = \infty$, and (1) holds. If $x \neq \infty$, then for all $z \in \partial U \setminus \{\infty\}$ we have $|x-z| \geq \delta(z)/c$, and hence

$$\delta(x) \leq \delta(z) + |x-z| \leq (c+1)|x-z|.$$

Thus (1) follows with $c \mapsto c+1$.

5.10. THEOREM. *Let U be an open set in $\dot{R}^n$. Then the following conditions are quantitatively equivalent:*

(1) *U is weakly (p,c)-John.*

(2) *If z is a p-cycle bounding in U and if $\#|z| \geq 2$, then z bounds in $U \setminus B(\partial U, 1/c, \complement|z|)$.*

(3) *If $A \subset U$ is compact with $\#A \geq 2$, the sequence*

$$H_p(A) \to H_p(U \setminus B(\partial U, 1/c, \complement A)) \to H_p(U)$$

is fast.

Explicit bounds. The conditions (2) and (3) are equivalent with the same constant c, and each of (2) and (1) implies the other with $c \mapsto c+1$.

PROOF. The conditions (2) and (3) are obviously equivalent with the same constant c. The quantitative equivalence of (1) and (2) follows almost directly from 5.9.

5.11. *Möbius inflation.* In order to obtain a variation of 5.10 for weakly uniform open sets we introduce a new kind of inflation, based on cross ratios. Let V be open in $\dot{R}^n$ with $\#\partial V \geq 2$. For $a \in V$, $y \in \complement V$ and $0 < t < 1$ we set

$$Q(a,t,V,y) = \{x \in V : |x,a,y,z| < t \quad \text{for all} \quad z \in \complement V\}.$$

This set is Möbius invariant: If φ is a Möbius map of $\dot{R}^n$, then $Q(\varphi a, t, \varphi V, \varphi y) = \varphi Q(a,t,V,y)$. It is often convenient to normalize the situation so that $a = \infty$ or $y = \infty$. Since $|x,a,\infty,z| = |x-a|/|a-z|$ and $|x,\infty,y,z| = |y-z|/|x-y|$, we get

$$(1) \qquad Q(a,t,V,\infty) = B(a, td(a, \complement V)), \; Q(\infty,t,V,y) = \complement\overline{B}(y, M(y)/t),$$

where

$$M(y) = \max\{|y-z| : z \in \complement V\}.$$

From these considerations we see:

(2) $Q(a,t,V,y) \subset V$.

(3) In all formulas above we can replace $\complement V$ by ∂V.

(4) The set $Q(a,t,V,y)$ is a *generalized open ball*, that is, a Möbius image of B^n.

We next consider the dependence of $Q(a,t,V,y)$ on y. We normalize $a = \infty$. Clearly $M(y') \leq 2M(y)$ for all $y, y' \in \complement V$. Suppose that $t < 1/3$. If $x \in Q(\infty,t,V,y)$, then

$$|x - y'| \ge |x - y| - |y - y'| > M(y)/t - M(y) \ge M(y')/3t.$$

Hence $x \in Q(\infty, 3t, V, y')$, and we obtain:

If $t < 1/3$, $a \in V$ and $y, y' \in \complement V$, then

$$Q(a, t, V, y) \subset Q(a, 3t, V, y'). \tag{5}$$

For $A \subset V$, $t > 0$ and $y \in \complement V$ we define the *Möbius t-inflation* of *A relative to* (V, y) by

$$Q(A, t, V, y) = \bigcup_{a \in A} Q(a, t, V, y).$$

Also this set is a Möbius invariant: $Q(\varphi A, t, \varphi V, \varphi y) = \varphi Q(A, t, V, y)$ for each Möbius map φ on $\dot{R}^n$. The point ∞ plays no special role. By (5) we have

$$Q(A, t, V, y) \subset Q(A, 3t, V, y') \tag{6}$$

whenever $t < 1/3$, $A \subset V$ and $y, y' \in \complement V$. From (1) we see that

$$Q(A, t, V, \infty) = B(A, t, V) \tag{7}$$

whenever $A \subset V \subset R^n$.

5.12. Lemma. *Suppose that $A \subset B \subset U \subset R^n$ and that U is open, A is compact, and $\#A \ge 2$. Then the following conditions are quantitatively equivalent:*

(1) *$d(B) \le cd(A)$ and $d(x, A) \le cd(x, \partial U)$ for all $x \in B$.*
(2) *$Q(\partial U, 1/c, \complement A, y) \cap B = \varnothing$ for some $y \in A$.*
(3) *$Q(\partial U, 1/c, \complement A, y) \cap B = \varnothing$ for all $y \in A$.*

Explicit bounds. (2) $\Rightarrow$ (3) with $c \mapsto 3c$, (3) $\Rightarrow$ (1) with $c \mapsto 4(c+1)$, and (1) $\Rightarrow$ (3) with $c \mapsto 24c^2 + 1$.

Proof. Trivially (3) implies (2), and the quantitative implication (2) $\Rightarrow$ (3) follows from 5.11(6). The condition (3) can be rewritten as follows: For all $a \in \partial U$, $x \in B \setminus A$, $y \in A$, there is $z \in A$ such that $|x, a, y, z| \ge 1/c$. This is the same as (iii) of [Al, 2.1] and hence quantitatively equivalent to (1). The explicit bounds follow from the proof of [Al, 2.1].

5.13. Remark. Lemma 5.12 is not true if $U \subset R^n$ is replaced by $U \subset \dot{R}^n$, since (3) does not imply the first inequality of (1). However, the proof of [Al, 2.1] shows that it still implies the second condition of (1) with $c \mapsto 4(c+1)$.

5.14. Theorem. *For an open set $U \subset \dot{R}^n$, the following conditions are quantitatively equivalent:*

(1) *U is weakly (p, c)-uniform.*

(2) *If z is a p-cycle bounding in U and if $\#|z| \geq 2$, then z bounds in $U \setminus Q(\partial U, 1/c, \complement|z|, y)$ for some $y \in |z|$.*

(3) *As* (2) *but for all $y \in |z|$.*

(4) *If $A \subset U$ is compact with $\#A \geq 2$, the sequence*

$$H_p(A) \to H_p(U \setminus Q(\partial U, 1/c, \complement A, y)) \to H_p(U)$$

is fast for some $y \in A$.

(5) *As* (4) *but for all $y \in A$.*

Explicit bounds. The implications (2) $\Leftarrow$ (3) $\Leftrightarrow$ (5) $\Rightarrow$ (4) $\Leftrightarrow$ (2) are true with the same constant c. Moreover, (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) with $c \mapsto 3c$, (1) $\Rightarrow$ (2) with $c \mapsto 24c^2 + 1$, and (3) $\Rightarrow$ (1) with $c \mapsto 10c$.

PROOF. The quantitative equivalences (2) $\Rightarrow$ (3) and (4) $\Leftrightarrow$ (5) follow from 5.11(6), and (2) $\Leftrightarrow$ (4) is obvious. If $\infty \notin U$, the equivalence (1) $\Leftrightarrow$ (2) follows from 5.12. Suppose that $\infty \in U$ and that (1) holds. Let z be as in (2). By (1), $z = \partial g$ for some $(p+1)$-chain g satisfying the uniformity conditions in U. If $|z| \subset R^n$, then also $|g| \subset R^n$ by the turning condition. Applying 5.12 to the open set $U \setminus \{\infty\}$ we obtain (2). Assume that $\infty \in |z|$. We prove that $|g| \cap Q(\partial U, 1/(c+1), \complement|z|, \infty) = \varnothing$. It suffices to show that for all $x \in |g| \setminus \{\infty\}$, $a \in \partial U$ there is $w \in |z|$ such that

$$\text{(A)} \qquad \frac{1}{c+1} \leq |x, a, \infty, w| = \frac{|x-a|}{|a-w|}.$$

Pick $w \in |z|$ with $|x - w| = d(x, |z|)$. Then

$$|x - a| \geq d(x, \partial U) \geq d(x, |z|)/c = |x - w|/c$$

by the lens condition. Since

$$|a - w| \leq |a - x| + |x - w| \leq (c+1)|x - a|,$$

(A) follows.

We finally show that (3) implies (1) also in the case $\infty \in U$. Let z be a p-cycle bounding in U. We may assume that $\#|z| \geq 2$. By (3), $z = \partial g$ with $|g| \subset U \setminus Q(\partial U, 1/c, \complement|z|, y)$ for all $y \in |z|$. By 5.12 and 5.13, g satisfies the lens condition in U with a constant $c' = c'(c)$. If $\infty \in |z|$, the turning condition is trivially satisfied. If $\infty \notin |z|$, write $d = d(|z|)$ and fix $y \in |z|$. If now $|g| \subset B(y, 4cd)$, g satisfies the turning condition with the constant $8c$. If $|g| \not\subset B(y, 4cd)$, we may assume that there is a point $x_0 \in |g| \cap S(y, 4cd)$.

We show that $\complement U \subset B(y, 3cd)$. Let $a \in \partial U$. Since $x_0 \notin Q(a, 1/c, \complement|z|, y)$, there is $w \in |z|$ with $|x_0, a, y, w| \geq 1/c$. Since $|y - w| \leq d$ and $|x_0 - a| \leq 5cd + |a - w|$, this implies $|a - w| \leq 5cd/3 < 2cd$. It follows that $\partial U \subset B(y, 3cd)$, and hence $\complement U \subset B(y, 3cd)$.

Write $A = |g| \cup \complement B(y, 4cd)$, $V = B(y, 5cd)$. By 5.3, $z = \partial g_1$ for some g_1 with $|g_1| \subset A \cap V$. The chain g_1 satisfies the turning condition $d(|g_1|) \leq 10cd$. The lens condition of g implies the same condition at points $x \in |g_1| \cap B(y, 4cd) \subset |g|$. If $x \in |g_1| \setminus B(y, 4cd)$, then $d(x, |z|) \leq |x - y| \leq 5cd \leq 5d(x, \partial U)$, and the theorem is proved.

5.15. *Nerves.* We give a summary of the theory of nerves and canonical homomorphisms needed in the sequel. Let X be a metrizable topological space and let $\mathcal{U} = (U_i)_{i \in I}$ be an indexed covering of X. For a multi-index $\mathbf{i} = (i_0, \dots, i_p) \in I^{p+1}$ we set $U_{\mathrm{i}} = U_{i_0} \cap \dots \cap U_{i_p}$. The *nerve* of $\mathcal{U}$ is the simplicial complex $N(\mathcal{U})$ whose ordered p-simplexes are the multi-indexes i for which $U_{\mathrm{i}} \neq \emptyset$. The reduced cohomology groups of $N(\mathcal{U})$ with coefficients in $\mathbf{G}$ are written as $H^p(\mathcal{U})$.

If $Y \subset X$, a covering $\mathcal{V}V = (V_j)_{j \in J}$ of Y is a *refinement* of $\mathcal{U}$ if there is a map $u : J \to I$, called a $(\mathcal{V}, \mathcal{U})$*-projection* or simply a *projection*, such that $V_j \subset U_{u(j)}$ for all $j \in J$. A projection u induces homomorphisms $u^* : H^p(\mathcal{U})$ $toH^p(\mathcal{V})$, which are independent of the choice of u. If $\mathcal{U}$ is an open covering of X, there are canonical homomorphisms $\pi = \pi_{\mathcal{U}} : H^p(\mathcal{U}) \to H^p(X)$. These maps can also be defined if $\mathcal{U}$ is a locally finite closed covering, but we shall work solely with open coverings. Recall that $H^p(X)$ is the reduced CÚech cohomology group of X.

Suppose that $u : J \to I$ is a $(\mathcal{V}, \mathcal{U})$-projection. For $\mathbf{j} = (j_0, \dots, j_p)$ we write $u\mathbf{j} = (uj_0, \dots, uj_p)$. For an integer $q \geq 0$ we say that the projection u is q-*strong* if the inclusions $V_{\mathrm{j}} \subset U_{u\mathrm{j}}$ induce zero maps $H^k(U_{u\mathrm{j}}) \to H^k(V_{\mathrm{j}})$ for all $k \leq q$ and for all simplexes $\mathbf{j}$ in $N(\mathcal{V})$.

We need the following basic result on canonical homomorphisms:

5.16. THEOREM. *Suppose that X is a metrizable space, that $X \supset X_{-1} \supset X_0 \supset \dots \supset X_q$ and that $\mathcal{U}_k = (U_i)_{i \in I_k}$ is a covering of X_k with sets U_i open in X_k, $-1 \leq k \leq q$. Suppose also that $u_k : I_k \to I_{k-1}$ is a q-strong $(\mathcal{U}_k, \mathcal{U}_{k-1})$-projection for $0 \leq k \leq q$. Consider the commutative diagram*

$$\begin{array}{ccc} H^j(\mathcal{U}_q) & \xrightarrow{\pi_q} & H^j(X_q) \\ \uparrow{\scriptstyle u^*} & & \uparrow{\scriptstyle \alpha^*} \\ H^j(\mathcal{U}_{-1}) & \xrightarrow[\pi_{-1}]{} & H^j(X_{-1}), \end{array}$$

where $u = u_0 \dots u_q$, $\alpha : X_q \hookrightarrow X_{-1}$, and the maps π_{-1}, π_q are canonical. Then $\operatorname{im} \alpha^* \subset \operatorname{im} \pi_q$ *for $0 \leq j \leq q$ and* $\ker \pi_{-1} \subset \ker u^*$ *for $0 \leq j \leq q + 1$.*

5.17. COMMENTS. For finite closed coverings, Theorem 5.16 is given in

[Dy, p. 129]; see also [Fl, p. 320] and [Ju_1, p. 39]. The case of arbitrary open coverings seems to belong to the folklore. However, Corollary 5.18 below is given in [Go, p. 213] for arbitrary open coverings and for locally finite closed coverings. We prefer to work with open coverings, since various inflations of a set have natural open coverings consisting of open balls. Shrinking these balls slightly and choosing sufficiently large finite subfamilies one could obtain the results of this paper by using only finite closed coverings, but the proofs would be more complicated and less elegant. An expository preprint [Ju_2] has been recently written by O. Jussila.

5.18. COROLLARY. *Suppose that $\mathcal{U} = (U_i)_{i\in I}$ is an open covering of a metrizable space X such that $H^j(U_{\mathbf{i}}) = 0$ for all $\mathbf{i} \in I^{p+1}$, $p \geq 0$, and for all $j \leq q$. Then the canonical homomorphisms $\pi : H^j(\mathcal{U}) \to H^j(X)$ are bijective for $j \leq q$ and injective for $j = q+1$.*

5.19. *Notation.* Suppose that $X \subset V \subset \dot{R}^n$, where V is open and has at least one finite boundary point. Recall from 5.8 that if $\infty \notin X$, the t-inflation $B(X, t, V)$ of X relative to V was defined as the union of all balls $B(x, td(x, \partial V))$, $x \in X$. If $\infty \in X$, we simply added ∞ to the set $B(X \setminus \{\infty\}, t, V)$. In the following crucial lemma, however, we want to inflate also the point ∞ by a set $\complement\overline{B}(R)$, $R > 0$ and write

$$B(X, t, R, V) = B(X \setminus \{\infty\}, t, V) \cup \complement\overline{B}(R).$$

5.20. LEMMA. *Suppose that $X \subset V \subset \dot{R}^n$ where X is compact and V is open with $0 \notin V$. Suppose also that $0 \leq q \leq n-2$ and that*

(1) *X is outer (j, c)-joinable for $0 \leq j \leq q$,*

(2) *$H^j(X) = 0$ for $0 \leq j \leq q$.*

Let $0 < t \leq 1/2$ and $c' > 2(c+2)^{q+1}$. Then, if $\infty \notin X$, the sequence

$$H^{q+1}(B(X, t, V)) \to H^{q+1}(B(X, t/c', V)) \to H^{q+1}(X)$$

is fast. If $\infty \in X$ and $R > 0$, then ker $i \subset$ ker j *in the diagram*

$$\begin{array}{ccc} H^{q+1}(B(X, t, R/2, V)) & \xrightarrow{\ i\ } & H^{q+1}(X \cup \complement\overline{B}(R)) \\ \big\downarrow j & & \\ H^{q+1}(B(X, t/c', 3^{q+1}R, V)). & & \end{array}$$

PROOF. We prove the case $\infty \in X$; the case $\infty \notin X$ is similar but simpler; the balls $B(R)$ do not occur at all. For $a \in R^n$ set $\delta(a) = d(a, \partial V)$. We start with

Fact 1. If $a \in R^n$ and $B(a, t\delta(a))$ meets $B(R)$, then $B(a, t\delta(a)) \subset B(3R)$.

Since $0 \notin V$, and $t \leq 1/2$, we have $\delta(a) \leq |a| \leq R + t\delta(a) \leq R + \delta(a)/2$. Hence the diameter of the ball $B(a, t\delta(a))$ is $2t\delta(a) \leq \delta(a) \leq 2R$, and Fact 1 follows.

Set $X(R) = X \cup \complement\overline{B}(R)$. For $x \in X$ define $U(x, t, R) = X(R) \cap B(x, t\delta(x))$ if $x \neq \infty$ and $U(\infty, t, R) = \complement\overline{B}(R)$. Then the family $\mathcal{U}(t, R)$ of all $U(x, t, R)$, $x \in X$, is an open covering of $X(R)$. If $t' < t$ and $R' > R$, then $\mathcal{U}(t', R')$ is a refinement of $\mathcal{U}(t, R)$, and a natural $(\mathcal{U}(t', R'), \mathcal{U}(t, R))$-projection is given by the identity map of the index set X. We next prove:

Fact 2. If $(c+2)t' < t$ and $R' \geq 3R$, this projection is q-strong in the sense of 5.15.

Define $b > c$ by $(b+2)t' = t$. Let $x_0, \ldots, x_k$ be distinct points in X with

$$E(t', R') = \bigcap_{i=0}^{k} U(x_i, t', R') \neq \varnothing.$$

We must show that the map $H^j(E(t, R)) \to H^j(E(t', R'))$ is zero for $0 \leq j \leq q$.

Suppose first that $x_i = \infty$ for some i. Then $E(t', R')$ is the intersection of $\complement\overline{B}(R')$ and the balls $B(x_s, t'\delta(x_s))$, $s \neq i$, and hence cohlog trivial. Next assume that $x_i \in R^n$ for all i. We may assume that $\delta(x_0) \leq \delta(x_i)$ for all $0 \leq i \leq k$. If $B(x_0, t\delta(x_0))$ does not meet $B(R)$, then $E(t, R)$ is the intersection of all balls $B(x_i, t\delta(x_i))$ and hence cohlog trivial. Assume that $B(x_0, t\delta(x_0))$ meets $B(R)$. Then $B(x_0, t\delta(x_0)) \subset B(3R) \subset B(R')$ by Fact 1. Since $|x_i - x_0| \leq 2t'\delta(x_i)$ and since $t = (b+2)t'$, we have $B(x_0, bt'\delta(x_0)) \subset B(x_i, t\delta(x_i))$ for all i. Hence

$$E(t', R') \subset U(x_0, t', R') = X \cap B(x_0, t'\delta(x_0)) \subset X \cap B(x_0, bt'\delta(x_0)) \subset E(t, R).$$

Since $b > c$, it follows from (1) and (2) that the maps

$$H^j(X \cap B(x_0, bt'\delta(x_0))) \to H^j(X \cap B(x_0, t'\delta(x_0)))$$

are zero; observe that we are using open balls and therefore needed a constant $b > c$. Hence the maps $H^j(E(t, R)) \to H^j(E(t', R'))$ are zero as well, and Fact 2 is proved.

Define $b' > c$ by $c' = 2(b'+2)^{q+1}$. For $k = -1, \ldots, q$ set $t_k = (b'+2)^{-k-1}t$ and $R_k = 3^{k+1}R$. Thus $t_{-1} = t$, $t_k = (b'+2)t_{k+1}$, $t_q = 2t/c'$, $R_{-1} = R$ and $R_{k+1} = 3R_k$. By Fact 2, the coverings $\mathcal{U}(t_k, R_k)$ satisfy the condition of 5.16 with the substitution $X \mapsto X(R)$, $X_k \mapsto X(R_k)$; now $u_k : X \to X$ is the identity projection. Hence $\ker \pi_{-1} \subset \ker u^*$ in the diagram

$$\begin{array}{ccc} H^{q+1}(\mathcal{U}(t_q, R_q)) & \xrightarrow{\pi_q} & H^{q+1}(X(R_q)) \\ \uparrow{\scriptstyle u^*} & & \uparrow \\ H^{q+1}(\mathcal{U}(t, R)) & \xrightarrow{\pi_{-1}} & H^{q+1}(X(R)); \end{array}$$

here u is again the identity projection.

Set $P(t, R) = B(X, t, R, V)$. For $x \in X$ define $V(x, t, R) = B(x, t\delta(x))$ if $x \neq \infty$ and $V(\infty, t, R) = \complement \overline{B}(R)$. Then the family $\mathscr{V}(t, R)$ of all $V(x, t, R)$, $x \in X$, is an open covering of $P(t, R)$. Since finite intersections of the members of $\mathcal{V}(t, R)$ are cohlog trivial, it follows from 5.18 that the canonical homomorphisms $\pi : H^j(\mathscr{V}(t, R)) \to H^j(P(t, R))$ are bijective for all j. The identity map of X defines simplicial maps

$$N(\mathscr{V}(t/2, R)) \xrightarrow{\varphi} N(\mathscr{U}(t, R)) \xrightarrow{\psi} N(\mathscr{V}(t, R/2))$$

for all $0 < t < 1$ and $R > 0$. We obtain the commutative diagram

$$\begin{array}{ccccccc} H^{q+1}(\mathcal{V}(t, R/2)) & \xrightarrow{\psi^*} & H^{q+1}(\mathcal{U}(t, R)) & \xrightarrow{u^*} & H^{q+1}(\mathcal{U}(t_q, R_q)) & \xrightarrow{\varphi^*} & H^{q+1}(\mathcal{V}(t_q/2, R_q)) \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_{-1}} & & \downarrow{\scriptstyle \pi_q} & & \downarrow{\scriptstyle \pi_1} \\ & & H^{q+1}(X(R)) & \longrightarrow & H^{q+1}(X(R_q)) & & \\ & \nearrow{\scriptstyle i} & & & & \nwarrow & \\ H^{q+1}(P(t, R/2)) & & & \xrightarrow{\quad j \quad} & & & H^{q+1}(P(t_q/2, R_q)) \end{array}$$

where each π_j is canonical. We proved above that $\ker \pi_{-1} \subset \ker u^*$ and that π_0 and π_1 are bijective. Since $t_q/2 = t/c'$ and $R_q = 3^{q+1}R$, the lemma follows by simple diagram chasing.

5.21. THEOREM. *Suppose that* $0 \leq p \leq n-2$ *and that* U *is an open set in* $\dot{R}^n$ *such that*

(1) U *is inner* (k, c)*-joinable for* $p \leq k \leq n-2$,

(2) $H_k(U) = 0$ *for* $p+1 \leq k \leq n-1$.

Then U *is weakly* (p, c')*-John with* $c' = c'(c, n-p)$.

Explicit bounds. The theorem is true with any $c' > 4(2c+3)^{n-p-1}$. If (1′) below is true, one can choose any $c' > 4(c+2)^{n-p-1}$.

PROOF. Set $X = \complement U$ and $q = n-2-p$. Replacing c by $2c+1$ we see by Alexander duality (1.2) and by 2.8 that we may assume:

(1′) X is outer (j, c)-joinable for $0 \leq j \leq q$,

(2′) $H^j(X) = 0$ for $0 \leq j \leq q$.

We assume that $\infty \in X$. The proof for the case $\infty \in U$ is similar but easier; it makes use of the simpler case of 5.20.

Let z be a p-cycle bounding in U with $\#|z| \geq 2$. By 5.10 it suffices to find $c' = c'(c, q)$ such that z bounds in $U \setminus B(\partial U, 1/c', V) = B(X, 1/c', V)$ where $V = \complement|z|$. We show that this is true for each $c' > 4(c+2)^{q+1}$. Choose c'' with $4(c+2)^{q+1} < 2c'' < c'$. For $0 < t < 1$ and $R > 0$ we set $P(t, R) = B(X, t, R, V)$ and $\overline{P}(t, R) = \operatorname{cl} P(t, R)$. Choose $R > 0$ such that z bounds in $U \cap B(R/2)$. Consider the diagram

$$\begin{array}{ccccc}
H^{q+1}(\overline{P}(1/2, R/2)) & & & & \\
\downarrow{\scriptstyle i_1} & & & & \\
H^{q+1}(P(1/2, R/2)) & \xrightarrow{i_2} & H^{q+1}(X \cup \complement B(R)) & \xrightarrow{i_3} & H^{q+1}(X \cup \complement\overline{B}(R)) \\
\downarrow{\scriptstyle j} & & & & \\
H^{q+1}(P(1/2c'', 3^{q+1}R)) & \xrightarrow{k} & H^{q+1}(\overline{P}(1/c', 3^{q+2}R)) & &
\end{array}$$

where all maps are induced by inclusions. Here $\ker(i_3 i_2) \subset \ker j$ by 5.20, and hence $\ker(i_2 i_1) \subset \ker(k j i_1)$. By Alexander duality this means that $\ker \alpha \subset \ker \beta$ in the diagram

$$\begin{array}{ccc}
H_p(\complement\overline{P}(1/2, R/2) & \xrightarrow{\alpha} & H_p(U \cap B(R)) \\
\downarrow{\scriptstyle \beta} & & \\
H_p(\complement\overline{P}(1/c', 3^{q+2}R)). & &
\end{array}$$

Hence z bounds in $\complement\overline{P}(1/c', 3^{q+2}R) \subset U \setminus B(\partial U, 1/c', V)$.

5.22. THEOREM. *Suppose that $0 \leq p \leq n-2$ and that U is an open set in $\dot{R}^n$ such that*

(1) *U is (k, c)-joinable for $p \leq k \leq n-2$,*

(2) *$H_k(U) = 0$ for $p+1 \leq k \leq n-1$.*

Then U is weakly (p, c')-uniform with $c' = c'(c, n-p)$.

Explicit bounds. The theorem holds with any $c' > 40(18c^2+2)^{n-p-1}$. If (1) is replaced by "(1′) $X = \complement U$ is (j, c)-joinable for $0 \leq j \leq n-2-p$" one can choose any $c' > 40(2c^2+2)^{n-p-1}$.

PROOF. Let z be a p-cycle bounding in U with $\#|z| \geq 2$ and let $y \in |z|$. By 5.14 it suffices to find $c' = c'(c, n-p)$ such that z bounds in

$U \setminus Q(\partial U, 1/c', V, y)$ where $V = \complement|z|$ and Q is the Möbius inflation defined in 5.11. Since Q is Möbius invariant and since (k,c)-joinability is quantitatively Möbius invariant by 4.6, we may normalize $y = \infty$. Since $Q(\partial U, t, V, \infty) = B(\partial U, t, V)$ by 5.11(7), the theorem follows from 5.21 and 5.10. Observe that we needed only the easy case $\infty \in U$ of 5.21.

5.23. *Uniform holes.* Suppose that X is a compact set in $\dot{R}^n$. By a *hole* of X we mean a component of $U = \complement X$. By the *Swiss cheese conjecture* we mean the following statement: If all holes of X are c-uniform domains and if $f : X \to \dot{R}^n$ is η-quasimöbius, then the holes of fX are c'-uniform with $c' = c'(c, \eta, n)$ or maybe even $c'(c, \eta)$. The case where X has only one hole was proved in [Vä$_2$, 5.6], and the case where X has precisely two holes with X as a common boundary was announced in [Vä$_2$, 5.10]. The proofs made use of compact families of compact sets in $\dot{R}^n$, and they did not give explicit bounds for $c'(c, \eta, n)$.

We show that the results of this paper can be applied to prove the Swiss cheese conjecture, generalized to weakly (p,c)-uniform sets, in several cases including the case $X = S^{n-1}$, and with explicit bounds.

5.24. THEOREM. *Suppose that X is a compact set in $\dot{R}^n$ and that $0 \le p \le n-2$. Suppose also that*

(1) *$U = \complement X$ is weakly (k,c)-uniform for $p \le k \le n-2$,*

(2) *$H^j(X) = 0$ for $0 \le j \le n-2-p$.*

Let $f : X \to \dot{R}^n$ be η-quasimöbius. Then $\complement fX$ is weakly (k,c')-uniform for $p \le k \le n-2$ with $c' = c'(c, \eta, n-p)$.

Explicit bounds. The theorem is true with any $c' > 40[8\eta(2(2c+1)^2)^4 + 2]^{q+1}$, $q = n-2-p$. If X is (j,c)-joinable for $0 \le j \le q$, one can choose any $c' > 40[8\eta(2c^2)^4 + 2]^{q+1}$ or, if f is η-quasisymmetric, any $c' > 40[2\eta(c)^2 + 2]^{q+1}$.

PROOF. By (1) and 5.6, U is (k, c_1)-joinable in $\dot{R}^n$ for $p \le k \le n-2$ with $c_1 = 2c+1$. Set $q = n-2-p$. By the duality theorem 2.7, X is (j, c_1)-joinable for $0 \le j \le q$. By the quasimöbius invariance 4.6, fX is (j, c_2)-joinable for $0 \le j \le q$ with $c_2 = c_2(c, \eta)$. By the absolute duality theorem 2.8 we see that $U' = \complement fX$ is (k, c_3)-joinable for $c_3 = 2c_2 + 1$. The condition (2) of 5.22 holds for U' by Alexander duality, and the theorem follows from 5.22.

5.25. COROLLARY. *If $f : S^{n-1} \to \dot{R}^n$ is η-quasimöbius, then $\complement fS^{n-1}$ is weakly (p,c)-uniform for $0 \le p \le n-2$ with $c = c(\eta, n)$. In particular, the components of $\complement fS^{n-1}$ are c-uniform domains in the ordinary sense.*

Explicit bounds. The corollary holds with any $c' > 40[8\eta(2)^4 + 2]^{n-1}$. If f is η-

quasisymmetric, one can choose any $c' > 40[2\eta(1)^2 + 2]^{n-1}$. If f is L-bilipschitz, one can choose $c' = 10 \cdot 4^n L^{4n-4}$.

5.26. Remark. P. MacManus [MM] has recently proved the case $p = 0$ of 5.25 with c independent of n.

5.27. Theorem. *Let $n \geq 3$ and let $f : B^n \to D'$ be a K-quasiconformal map. Then D' is (p, c')-uniform for $1 \leq p \leq n-2$ with $c' = c'(K, n)$.*

Proof. This follows directly from 4.13 and 5.22.

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