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DISCRETE GROUP ACTIONS AND THE MINIMAL PRIMAL IDEAL SPACE

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Abstract.

In this paper the minimal primal ideal space of a C*-algebra A and of a crossed product C*-algebra $A \times_\alpha G$ is investigated. The question is, under what circumstances is it possible to tell whether Min-Primal$(A \times_\alpha G)$ is closed in the space of all proper two-sided closed ideals with the Fell-topology? Positive answers are achieved in a certain class of liminal C*-algebras with group actions that are implemented by essentially inner unitaries.

1. Introduction.

Given a C*-algebra A and a group action $\alpha : G \to \mathrm{Aut}(A)$ it is in general a difficult task to describe the ideal structure of the crossed product C*-algebra $A \times_\alpha G$. A part of the main result of [OR] states if that if A is a continuous trace C*-algebra and G an abelian locally compact group acting pointwise inner on A then the space $\mathrm{Prim}(A \times_\alpha G)$ of the primitive ideals in the crossed product is again a Hausdorff space. Among other things it is shown that if $\pi \times U$ is an irreducible representation of $A \times_\alpha G$, then π is an irreducible representation of A, and the map

$$\mathrm{Res} : \mathrm{Prim}(A \times_\alpha G) \to \mathrm{Prim}(A),\ \ker(\pi \times U) \mapsto \ker(\pi),$$

is a well-defined continuous $\hat{G}$-map where the dual group $\hat{G}$ acts as usual on $A \times_\alpha G$ (hence on $\mathrm{Prim}(A \times_\alpha G)$) and trivially on $\mathrm{Prim}(A)$.

Pointwise inner automorphisms β are π-inner [El], and it can be shown, that such automorphisms on type 1 C*-algebras are essentially inner, i.e. there is an essential ideal $I \subset A$ and a unitary element $u \in M(I)(M(I)$ denotes the multiplier algebra; recall $I \subset A \subset M(A) \subset M(I)$ in a natural way) such that $\beta(x) = uxu^*$ for all $x \in A$.

In this paper we consider essentially inner group actions (i.e. all automorphisms α_s are implemented by a unitary in $M(I)$, where $I \subset A$ is an es-

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sential ideal) of discrete groups on certain classes of liminal C*-algebras which do not necessarily have a Hausdorff primitive ideal space.

Let $\mathrm{Id}(A)$ be the space of all closed and two-sided ideals of the C*-algebra A. If G is a discrete group acting on $A \subset A \times_\alpha G$ identifying $a \in A$ with $\alpha \cdot \delta_0 \in A \times_\alpha G$, 0 being the identity element in G, then

$$\mathrm{Res} : \mathrm{Id}(A \times_\alpha G) \to \mathrm{Id}(A),\ I \mapsto I \cap A,$$

defines a map between ideal spaces. Each ideal is the kernel of some representation $\pi \times U$ (integrated form of some covariant representation) and we have $\pi \times U|_A = \pi$, hence $\mathrm{Res}(\ker(\pi \times U)) = \ker(\pi)$. So this generalizes the above mentioned Res.

The first difficulty arrising is the observation that Res in general neither maps $\mathrm{Prim}(A \times_\alpha G)$ to $\mathrm{Prim}(A)$ nor is $\mathrm{Prim}(A)$ contained in the image of Res. Things are different, if $\mathrm{Prim}(A)$ is replaced by $\text{Min-Primal}(A)$, the space of minimal primal ideals (see [A] or next section). An important property that a C*-algebra can have is the closedness of $\text{Min-Primal}(A)$ with respect to the Fell-topology in $\mathrm{Id}(A)A\}$ (see [AS1]). The main question in this paper is:

What can be said about the space $\text{Min-Primal}(A \times_\alpha G)$ *in terms of A, G and* α? *Under what circumstances has* $\text{Min-Primal}(A \times_\alpha G)$ *the above mentioned closedness property*?

In the case of a continuous trace the restriction map Res between the spaces of primitive ideals (which coincide with the minimal primal ideals) is always open since it is a G-map (see [OR] or [PR] for this). In the context considered here the openness of Res as a map between space of minimal primal ideals turns out to be equivalent to the G-map property, and this is not necessarily the case. But if this is the case and $\text{Min-Primal}(A)$ is closed then so is $\text{Min-Primal}(A \times_\alpha G)$.

The minimal primal ideal space of the crossed product algebra is not always closed (even if $\text{Min-Primal}(A)$ is). But there is a stronger property than the closedness of the primal ideal space (called (EM)), and this property will be inherited from A to $A \times_\alpha G$ under suitable assumptions. As a consequence we have a sufficient condition for the minimal primal ideal space to be closed.

2. Primal Ideals.

Let A be a C*-algebra. $\mathrm{Id}(A)$ carries at least two useful topologies:

1) The Jacobson-topology or weak topology τ_w. A basis of τ_w is given by the sets

$$U(I_1, \ldots, I_n) := \{I \in \mathrm{Id} : I \not\supset I_1, \ldots, I \not\supset I_n\}$$

where $I_1, \ldots, I_n \in \mathrm{Id}(A)$. The restriction of τ_w to $\mathrm{Prim}(A)$ is the Jacobson-topology usually considered on the spectrum $\mathrm{Prim}(A)$ ([Dx2], 3.1). The set of primal ideals, denoted by $\mathrm{Primal}(A)$, is defined to be the τ_w-closure of $\mathrm{Prim}(A)$ in $\mathrm{Id}(A)$. If $I \lhd A$ (i.e. I is a closed and two-sided ideal in A), then the following are equivalent

(i) I is primal

(ii) If $n \in \mathsf{N}$, $I_1, \ldots, I_n \lhd A$, $I_1 \cap \ldots \cap I_n = \{0\}$, then $I_j \subset I$ for some $j \in \{1, \ldots, n\}$.

(iii) There is a net (P_i) in $\mathrm{Prim}(A)$ such that $\cap \lim((P_i)) \subset I$ (here $\lim((P_i))$ denotes the set of limits of the net).

An application of Zorn's lemma shows, that each primal ideal contains a minimal primal ideal; the set of all these ideals will be denoted by Min-Primal(A). A good reference for all this is [A].

2) The Fell-topology or strong topology τ_s is the weakest topology making continuous all maps

$$\mathrm{Id}(A) \to \mathsf{R},\ I \mapsto ||x + I||,\ x \in A.$$

This topology has been introduced by Fell [Fl] describing a topological base. τ_s is always Hausdorff and compact. τ_w and τ_s coincide when restricted to Min-Primal(A) ([A], cor, 4.3). This makes Min-Primal(A) an interesting space. It can be regarded as a substitute for $\mathrm{Prim}(A)$ if $(\mathrm{Prim}(A), \tau_w)$ is not Hausdorff, and in fact $(\mathrm{Prim}(A), \tau_w)$ is Hausdorff iff $\mathrm{Prim}(A) =$ Min-Primal(A). Applications of the minimal primal ideal space in this sense can be found in [Be1], [Be2], [Be3], [Be4].

If we translate ([Sch], Th. 2.2) to ideals, then we get

2.1. Proposition *Let (I_i) be a net in $\mathrm{Id}(A) \setminus \{A\}$ and let $I_i \to P \in \mathrm{Prim}(A)$ with respect to τ_w. Then there are a subnet (I_{i_k}) and primitive ideals $P_{i_k} \supset I_{ik}$ for all k such that $P_{ik} \xrightarrow{k} P$ with respect to τ_w.*

Using this together with ([Dx2], 3.3.) or by the discussion in ([AS2], page 84) we get $I_i \to I$ with respect to τ_w iff $||x + I|| \le \lim\inf_i ||x_i + I_i||$ for all $x \in A$.. This useful characterization again shows $\tau_w \subset \tau_s$.

Now let us characterize minimal primal ideals. The equivalence (i) $\Leftrightarrow$ (iv) is related to ([A], prop. 4.5).

2.2. Proposition. *For an ideal $I \in \mathrm{Primal}(A)$ the following are equivalent:*

(i) *I is a minimal primal ideal.*

(ii) id: $(\mathrm{Primal}(A), \tau_w) \to (\mathrm{Primal}(A), \tau_s)$ *is continuous at I.*

(iii) *If the net (P_i) in* $\mathrm{Prim}(A)$ *is τ_w-convergent to I then it also is τ_s-convergent to I.*

(iv) $I \in \mathrm{sep}(\mathrm{Primal}(A), \tau_w))$, i.e. *$I$ is a separated point in the topological* *space* (Primal$(A), \tau_w$) (i.e.: *if* $J \in$ Primal$(A), J \not\supset I$ *then there are disjoint open sets* U_1 *and* U_2 *in* Primal(A) *such that* $I \in U_1, J \in U_2$).

PROOF. "(i) $\Leftrightarrow$ (ii)" is ([A], 4.2) and "(ii) $\Rightarrow$ (iii)" is trivial.

"(iii) $\Rightarrow$ (i)": Let $I_0 \subset I$ be a minimal primal ideal. Since Primal(A) is the τ_w-closure of Prim(A) there is a net (P_i) of primitive ideals converging to I_0, and by (iii) we know that this is τ_s-convergence. So we have for all $x \in A$

$$||x + I|| = \lim_i ||x + P_i|| \geq ||x + I_0||.$$

This implies $I \subset I_0$, hence $I = I_0$ is minimal primal.

In order to prove "(ii) $\Rightarrow$ (iv)" let J be a primal ideal not containing I. Then there is an x in $I \setminus J$ and $\epsilon := \frac{1}{2}||x + J|| > 0$. The set $U_2 = \{P \in \text{Primal}(A); ||x + P|| > \epsilon\}$ is open and contains J. By (ii) there is an open neighbourhood U_1 of I such that $||x + P|| < \epsilon$ for all $P \in U_1$.

"(iv) $\Rightarrow$ (ii)": Let (I_i) be a net in Primal(A) which is τ_w-convergent to I. We must show that each subset (I_{i_j}) has a subnet τ_s-convergent to I. By τ_s-compactness there is τ_s-convergent subnet $I_{i_{j_k}} \to J$ and we have

$$||x + J|| = \lim_k ||x + I_{i_{j_k}}|| \geq \liminf ||x + I_i|| \geq ||x + I||$$

for all elements $x \in A$, hence $J \subset I$. So we must show $I \subset J$. But if this were not the case, then by (iv) we would have two disjoint open subsets U_1 and U_2 containing I and J respectively, and this is impossible since $I_{i_{j_k}} \to J$ and $I_{i_{j_k}} \to I$ with respect to τ_w.

Let us say that A has property (E) if all irreducible representations of A are finite dimensional, and that A has property (Eb) if the dimensions of the irreducible representations are bounded by a fixed finite constant. If A is liminal and $P \lhd A$ is a primitive ideal then we can define dim(P) to be the dimension of any irreducible representation with kernel P. Moreover we may extend the dim-function to Id(A) by the formula

$$\dim(I) := \sum_{P \in \text{supp}(I)} \dim(P).$$

The next lemma will have the consequence, that dim(I) is finite for all primal ideals if A has the property (Eb).

2.3. LEMMA. *Let A be a liminal* C*-*algebra and let* (P_j) *be a net in* Prim(A). *If* $P_j \xrightarrow{\tau_w} I$, *then* $\dim(I) \leq \limsup_j \dim(P_J)$.

PROOF. Since $\dim(A) = 0$ we may assume $I \neq A$. It is enough to consider the case $\limsup_j \dim(P_j) < \infty$ and restricting to a subnet we may assume $\dim(P_j) = n$ for all j. Let $P_j = \ker(\pi_j)$ for some irreducible representation

$\pi_j : A \to M_n = L(\mathbf{C}^n)$. Moreover let e_p be the pth basic vector in $\mathbf{C}^n$ and define

$$f_{p,q,j} := \langle \pi_j(\cdot)e_p, e_q \rangle \quad p, q = 1, \ldots, n$$

which are elements of the unit ball of A. By Alaoglu's theorem we can assume that $(f_{p,q,j})$ is w*-convergent for all p an q in $\{1, \ldots, n\}$, and by linearity we get

$$\forall a \in A : \forall \xi, \eta \in \mathbf{C}^n : \exists \lim_j \langle \pi_j(a)\xi, \eta \rangle.$$

In M_n the weak operator topology and the norm-topology coincide, hence

$$\forall a \in A : \exists \sigma(a) := \lim \pi_j(a) \in M_n.$$

Obviously $\sigma : A \to M_n$ is a representation which is a finite sum of irreducible representations (finite dimensional case), say $\sigma = \sigma_1 \oplus \ldots \oplus \sigma_m$, and we have $\sum_{k=1}^m \dim(\ker(\sigma_k)) \leq n$. Therefore it is sufficient to prove that $\mathrm{supp}(I) \subset \{\ker(\sigma_1), \ldots, \ker(\sigma_m)\}$.

Assume the contrary and consider $Q \in \mathrm{supp}(I)$ which is different from the ideals $\ker(\sigma_k)$. As primitive ideals are maximal and prime, there must be an element a in $\bigcap_{k=1}^m \ker(\sigma_k) \backslash Q$. Of course we have that (P_j) τ_w-converges to Q and therefore we conclude

$$\begin{aligned} 0 < \|a + Q\| &\leq \lim_j \inf \|a + P_j\| = \lim_j \inf \|\pi_j(a)\| \\ &= \|\sigma(a)\| = \max_{k=1,\ldots,m} \|\sigma_k(a)\| = 0. \end{aligned}$$

This contradiction finishes the proof.

We conclude that if A is liminal then

$$\mathrm{Prim}(A)_n := \{P \in \mathrm{Prim}(A); \dim(P) \leq n\}$$

is closed in $\mathrm{Prim}(A)$. This is true for all C*-algebras but requires other techniques ([Dx2], 3.6).

2.4. PROPOSITION. *Let A be a* C*-*algebra with property* (E). *Then the set* $\mathrm{int}(\mathrm{sep}(\mathrm{prim}(A), \tau_w))$ *is τ_w-dense in* $\mathrm{Prim}(A)$ (*hence in* $\mathrm{Primal}(A)$) *and is contained and τ_s-dense in* Min-Primal(A).

PROOF. Let us show , that each non-empty relatively τ_w-open subset U in the primitive ideal space contains a non-empty relatively τ_w-open subset W which entirely consists of separated points. This will prove the τ_w-density.

Since U is a Baire space ([Dx2], 3.4.13) and $U = \bigcup_{n=1}^\infty U \cap \mathrm{Prim}(A)_n$ is countable union of closed sets, there is an integer n such that int $(U \cap \mathrm{Prim}(A)) = U \cap \mathrm{int}(\mathrm{Prim}(A)_n)$ is non-empty. If we choose n minimal with this property, then

$$U \cap \mathrm{int}(\mathrm{Prim}(A)_n) \not\subset \mathrm{Prim}(A)_{n-1}$$
$$\Rightarrow W; = U \cap \mathrm{int}(\mathrm{Prim}(A)_n) \backslash \mathrm{Prim}(A)_{n-1} \neq \emptyset,$$

which is an open set in $\mathrm{Prim}(A)$. We want to show that all $P \in W$ are separated. Let R be a minimal primal ideal and let (P_j) be a net of primitive ideals which converges to R. Then also $P_j \to P(\tau_w)$, hence $P_j \in W$ for large j and this implies $\dim(P_j) = n$ for large j. From lemma 2.3 we know $\sum_{Q \in \mathrm{supp}(R)} \dim(Q) \leq n$. Since P is in $\mathrm{supp}(R)$ and $\dim(P) = n$ we must have $\mathrm{supp}(R) = \{P\}$, i.e. $P = R$. So P is minimal primal, hence separated in $(\mathrm{Primal}(A), \tau_w)$ by prop. 2.2, in particular separated in $\mathrm{Prim}(A)$. This proves the asserted τ_w-density.

First $\mathrm{int}(\mathrm{sep}(\mathrm{Prim}(A))) \subset \text{min-Primal}(A)$ is clear by 2.2. Moreover we have

$$\text{Min-Primal}(A) \subset \mathrm{Primal}(A) = \overline{\mathrm{int}(\mathrm{sep}(\mathrm{Prim}(A)))}^{\tau_w}$$

So for all minimal primal ideals I there is a net in $\mathrm{int}(\mathrm{sep}(\mathrm{Prim}(A)))$ converging to I, and this is automatically τ_s-convergence by prop. 2.2. This completes the proof.

This generalizes ([Dx1], prop. 2) to the non-separable case.

3, Automorphism groups and invariant ideals.

In this paper we consider automorphism groups (i.e. group actions) $\alpha : G \to \mathrm{Aut}(A)$ with the following property:

(A1): There is an essential ideal $I_\alpha \triangleleft A$ and a group homomorphism $u : G \to U(M(I_\alpha))$ such that $\alpha_s(x) = u_s x u_s^*$ for all $x \in A$ and $s \in G$. (Recall that all groups are supposed to be discrete, so there is no continuity condition for u. $U(M(I_\alpha))$ stands for the unitary group of the multiplier algebra $M(I_\alpha)$).

Obviously (A1) implies

(A2): There is an essential ideal $I_\alpha \triangleleft A$ and a map $u : G \to U(M(I_\alpha))$ such that $\alpha_s(x) = u_s x u_s^*$ for all $x \in A$ for all $s \in G$.

3.1. Remark. *In the situation* (A2) *all ideals contained in I_α and all primitive ideals not containing I_α are α-invariant.*

Proof. For all ideals $I \triangleleft I_\alpha$ we have $\alpha_s(I) = \alpha_s(I^3) \subset u_s I_\alpha I I_\alpha u_s^* \subset I_\alpha I I_\alpha \subset I$. To prove the second assertion let P be a primitive ideal not containing I_α. Say $P = \ker(\pi)$ for an irreducible representation π. Then $0 \neq \pi|_{I_\alpha}$

may be extended uniquely to an irreducible representation $\overline{\pi|_{I_\alpha}}$ of $M(I_\alpha)$ and for all $x \in P$ we conclude $\pi(\alpha_s(x)) = \pi(u_s x u_s^*) = \overline{\pi|_{I_\alpha}}(u_s)\pi(x)\overline{\pi|_{I_\alpha}}(u_s^*) = 0$.

3.2. REMARK. If(A, G, α) is a C*-dynamical system such that α satisfies (A1) then there are an essential ideal $I_\alpha \lhd A$ and a map $u : G \to U(M(I_\alpha))$ such that for all irreducible representations $\pi : A \to L(H)$ with $\ker(\pi) \not\supset I_\alpha$ the pair $(\pi, \overline{\pi|_{I_\alpha}} \circ u)$ is a covariant representation of (A, G, α).

This technical version will be helpful later. We will not make use of the simple fact that the converse also holds.

Let (A, G, α) be a C*-dynamical system, G a discrete group. Let

$$K(G, A) := \{x : G \to A; \operatorname{supp}(x) \text{ is finite}\}$$

The following facts are well known (see [G], [T] or the early paper [ZM]):

If $I \lhd A$ is α-invariant then the closure $\overline{K(G,I)}$ of $K(G,I)$ is a closed two-sided ideal in $A \times_\alpha G$ which is generated by I (observe $I \subset A \times_\alpha G$ because G is discrete). We have $\operatorname{Res}(\overline{K(G,I)}) = I$ and the image of Res is precisely the set $\operatorname{Id}_\alpha(A)$ of α-invariant ideals. Moreover if $I \in \operatorname{Id}_\alpha(A)$ and $J \lhd A \times_\alpha G$ then $j \supset \overline{K(G, \operatorname{Res}(J))}$ and $J \supset \overline{K(G,I)} \Leftrightarrow \operatorname{Res}(J) \supset I$.

If G is amenable then the map $\operatorname{Id}_\alpha(A) \to \operatorname{Id}(A \times_\alpha G)$, $I \mapsto \overline{K(G,I)}$ is continuous with respect to the τ_w-topologies. Let us prove that it is also continuous with respect to τ_s.

Let $I_i \to I$ in $(\operatorname{Id}_\alpha(A), \tau_s)$ and let (I_{i_j}) be any subnet such that $(\overline{K(G, I_{i_j})})$ has a τ_s-limit L. We already know $\overline{K(G, I_{i_j})} \to \overline{K(G,I)}$ with respect to the weak topology, and this implies $L \subset \overline{K(G,I)}$. Conversely since Res is τ_s-continuous and τ_s is Hausdorff, we have

$$L \supset \operatorname{Res}(L) = \tau_s - \lim_j \operatorname{Res}(\overline{K(G, I_{i_j})}) = \tau_s - \lim_j I_{i_j} = I.$$

Therefore $L \supset \overline{K(G,I)}$. So $\overline{K(G,I)}$ is the only τ_s-accumulation point of the net $(\overline{K(G, I_i)})$.

3.3. PROPOSITION. *Let G be amenable and let $I \lhd A$ be an essential and α-invariant ideal. Then $\overline{K(G,I)} \lhd A \times_\alpha G$ is essential.*

PROOF. Let $x \in A \times_\alpha G$ with $x \cdot \overline{K(G,I)} = \{0\}$; we must prove $x = 0$. If $y \in I$ then for all $s \in G$ we have $0 = (xy)(s) = x(s)\alpha_s(y)$ and therefore $\alpha_{s^{-1}}(x(s))y = 0$. Here $x(s)$ denotes the generalized Fourier coefficient of x. Since I is essential this yields $x(s) = 0$ for all $s \in G$. By the amenability of G the element x is determined by its generalized Fourier-coefficients and this implies $x = 0$.

4. The restriction map.

Now let us study the restriction map Res: $\mathrm{Id}(A \times_\alpha G) \to \mathrm{Id}(A)$. We already mentioned the continuity of Res (with respect to τ_w and with respect to τ_s) and that Prim(A) does not behave properly under this map. We will establish in this section that things are different if the primitive ideal space is replaced by the minimal primal ideal space. The proof of the following proposition can be taken from ([PR], prop. 2.1)

4.1. PROPOSITION. *Let $\alpha : G \to \mathrm{Aut}(A)$ satisfy* (A2) *with the ideal I_α and let $\pi \times U$ be an irreducible representation of $A \times_\alpha G$ such that $P := \ker(\pi \times U)$ does not contain $\overline{K(G, I_\alpha)}$. Then π is a factor representation, in particular* $\mathrm{Res}(P)$ *is a prime ideal. If A is of type* I *then* $\mathrm{Res}(P) \in \mathrm{Prim}(A)$.

4.2. COROLLARY. *We have* $\mathrm{Res}(\mathrm{Primal}(A \times_\alpha G)) \subset \mathrm{Primal}(A)$ *if G is amenable and α satisfies* (A2).

PROOF. Since G is amenable $\overline{K(G, I_\alpha)}$ is essential in $A \times_\alpha G$ by prop (3.3) implying that every primal ideal $P \lhd A \times_\alpha G$ can be approximated by a net (P_i) of primitive ideals of the crossed product such that $P_i \not\supset \overline{K(G, I_\alpha)}$. By the above proposition all the $\mathrm{Res}(P_i)$ are primal and we have $\mathrm{Res}(P_i) \to \mathrm{Res}(P)$ with respect to τ_w.

If α satisfies (A1) even more is true

4.3. THEOREM. *Let G be amenable and α satisfy* (A1). *Then*

(i) $\mathrm{Min\text{-}Primal}(A) \subset \mathrm{Res}(\mathrm{Min\text{-}Primal}(A \times_\alpha G)) \subset \mathrm{Res}(\mathrm{Primal} A \times_\alpha G)) \subset \mathrm{Primal}(A)$.

(ii) *If A has property* (E) *and* Min-Primal(A) *is closed then* $\mathrm{Min\text{-}Primal}(A) = \mathrm{Res}(\mathrm{Min\text{-}Primal}(A \times_\alpha G))$.

PROOF. To prove (i) we only have to demonstrate the first inclusion. Let I_α be the essential ideal and $v : G \to U(M(I_\alpha))$ the homomorphism in the definition of (A1). Given $P \in \mathrm{Min\text{-}Primal}(A)$ there is a net (P_i) of primitive ideals such that $I_\alpha \not\subset P_i \to P$. Let $P_i = \ker(\pi)$, π irreducible. Remark 3.2 implies that $\pi \times (\overline{\pi_i|_{I_\alpha}} \circ v)$ is an irreducible representation of $A \times_\alpha G$ and by τ_s-compactness we may assume that

$$\ker(\pi_i \times (\overline{\pi_i|_{I_\alpha}} \circ v)) \xrightarrow{i} Q \in \overline{\mathrm{Prim}(A \times_\alpha G)}^{\tau_s} \subset \mathrm{Primal}(A \times_\alpha G).$$

By τ_s-continuity of Res we conclude

$$\mathrm{Res}(Q) = \lim_i \mathrm{Res}(\ker(\pi_i \times (\overline{\pi_i|_{I_\alpha}} \circ v))) = \lim_i \ker(\pi_i) = P.$$

Therefore if Q_0 is a minimal primal ideal contained in Q we have

$\mathrm{Res}(Q_0) \subset P$. But by the above Corollary $\mathrm{Res}(Q_0)$ is primal and so we get $\mathrm{Res}(Q_0) = P$ since P is minimal primal. This proves (i).

Let U be the interior of the set of separated points in $\mathrm{Prim}(A)$ which is an open and dense subset by prop. 2.4. It is α acts on $\mathrm{Prim}(A)$ by homeomorphisms. Hence the corresponding ideal I_U is essential and α-invariant, and so is $I := I_U \cap I_\alpha$. Therefore $\overline{K(G,I)}$ is an essential ideal by prop. 3.3. So if $Q \in \text{Min-Primal}(A \times_\alpha G)$ is given there is a net (Q_i) of primitive ideals in $A \times_\alpha G$ such that $Q_i \to Q$. By prop. 2.2 this is automatically τ_s-convergence and this implies $\mathrm{Res}(Q_i) \to \mathrm{Res}(Q)$ with respect to τ_s. So by the τ_s-closedness we get the asserted result if we can show that $\mathrm{Res}(Q_i)$ is minimal for all i.

From $Q_i \not\supset \overline{K(G,I)}$ we conclude that $\mathrm{Res}(Q_i)$ is primitive by 4.1 and $\mathrm{Res}(Q_i) \not\supset I$. Now th. 2.4 yields that $\mathrm{Res}(Q_i)$ is minimal primal.

Let us define $\mathcal{M} = \overline{\text{Min-Primal}(A \times_\alpha G)}^{\tau_s} \cap \mathrm{Res}^{-1}(\text{Min-Primal}(A))$.

4.4. Remark. If G is amenable, α satisfies (A1), A has property (E) and Min-Primal(A) is closed then from the above theorem one easily can conclude

$$\mathcal{M} = \overline{\text{Min-Primal}(A \times_\alpha G)}^{\tau_s}.$$

If G is amenable and α satisfies (A1) then $\mathrm{Res} : \mathcal{M} \to \text{Min-Primal}(A)$ is a continuous surjection; even more is true

4.5. Proposition. *Let G be amenable and α satisfies* (A1). *Then* $\mathrm{Res} : \mathcal{M} \to \text{Min-Primal}(A)$ *is a quotient map where $\mathcal{M}$ carries the relative τ_w-topology or the relative τ_s-topology.*

Proof. Since Res is continuous the quotient topologies in both cases are finer than the topology on Min-Primal(A). Conversely let $U \subset \text{Min-Primal}(A)$ such that $\mathrm{Res}^{-1}(U) \cap \mathcal{M}$ is relatively open (τ_w or τ_s; since $\tau_w \subset \tau_s$ we may restrict attention to τ_s) and let us show that U is open in Min-Primal(A). To this end consider a net (I_i) which converges to $I \in U$ and verify that $I_i \in U$ for large indices i.

By the above theorem there are $Q_i \in \text{Min-Primal}(A \times_\alpha G)$ with $\mathrm{Res}(Q_i) = I_i$ and we may assume that this net is τ_s-convergent to the ideal L in the closure of Min-Primal$(A \times_\alpha G)$. Since τ_s is a Hausdorff-topology we get $\mathrm{Res}(L) = \lim_i \mathrm{Res}(Q_i) = \lim_i I_i = I$ and so $L \in \mathrm{Res}^{-1}(U) \cap \mathcal{M}$. By assumption we must have $Q_i \in \mathrm{Res}^{-1}(U) \cap \mathcal{M}$ for large i and this implies $I_i = \mathrm{Res}(Q_i) \in U$ for large i.

It can be shown that in general $\mathrm{Res} : \mathcal{M} \to \text{Min-Primal}(A)$ is not open. This is closely related to the dual action of G as we will see soon. The above theorem and the remark raise the question under what conditions

Min-Primal($A \times_\alpha G$) is closed. Even under the assumptions of the above remark this is not always the case. One sufficient condition for the closedness of Min-Primal($A \times_\alpha G$) will be the openness of Res; the property (EM) will be another such condition.

5. The openness of the restriction map.

In this section we answer the question when the restriction map Res : $\mathcal{M} \to$ Min-Primal(A) is open. The proof of the next proposition can be taken from ([PR], prop. 2.1).

5.1. PROPOSITION. *Let G be abelian, α satisfy* (A1) *with the ideal I_α and let A be of type* I. *If $\pi \times U$ is an irreducible representation of $A \times_\alpha G$ with* $\ker(\pi \times U) \not\supset \overline{K(G, I_\alpha)}$ *then π is irreducible.*

Now let $\hat{\alpha} : \hat{G} \to \mathrm{Aut}(A \times_\alpha G)$ be the dual action. This defines an action

$$\hat{G} \times \mathrm{Id}(A \times_\alpha G) \to \mathrm{Id}(A \times_\alpha G), \quad (\gamma, I) \mapsto \gamma I := \hat{\alpha}_\gamma^{-1}(I),$$

and clearly for a representation $\pi \times U$ of $A \times_\alpha G$ we get

$$\gamma(\ker(\pi \times U)) = \ker(\pi \times \gamma U).$$

The following lemma is well-known or easily established.

5,2, LEMMA. *The action of $\hat{G}$ on* $\mathrm{Id}(A \times_\alpha G)$ *makes* $\mathrm{Id}(A \times_\alpha G)$, $\mathrm{Prim}(A \times_\alpha G)$, $\mathrm{Primal}(A \times_\alpha G)$, *and* Min-Primal$(A \times_\alpha G)$ *a $\hat{G}$-space with respect to the τ_w-topology as well as with respect to the τ_s-topology.*

5.3. PROPOSITION. *Let G be abelian, a satisfy* (A1), *and A have property* (E). *Then there is a subset U of* $\mathrm{Prim}(A)$ *satisfying*

(i) *U is open and dense in* $\mathrm{Prim}(A)$.
(ii) $U \subset \mathrm{int}(\mathrm{sep}(\mathrm{Prim}(A)))$.
(iii) $\mathrm{Res}^{-1}(U) \cap \mathrm{Primal}(A \times_\alpha G) \subset \mathrm{Prim}(A \times_\alpha G)$.
(iv) $U \subset \{P \in \mathrm{Prim}(A); P \not\supset I_\alpha\}$.
(v) *The action of α on* $\mathrm{Id}(A)$ *is trivial on U.*

PROOF. By ([Pd], 6.2.11) there is an essential ideal I_T in A having continuous trace. Let $T \subset \mathrm{Prim}(A)$ be the corresponding open set. Define $S := \mathrm{int}(\mathrm{sep}(\mathrm{Prim}(A)))$ which is an open and dense set in $\mathrm{Prim}(A)$ by 2.4. Then

$$U := T \cap S \cap \{P \in \mathrm{Prim}(A); P \not\supset I_\alpha\}$$

is open and dense. So the properties (i), (ii) and (iv) go without saying. The points in U are primitive ideals not containing I_α and therefore invariant by 3.1.; this implies (v). The difficult point is (iii).

Let $P \in \mathrm{Res}^{-1}(\mathrm{U}) \cap \mathrm{Primal}(\mathrm{A} \times \alpha \mathrm{G})$. We must show that P is primitive, i.e. $\mathrm{supp}(P)$ contains exactly one element. To this end let $Q_1, Q_2 \in \mathrm{supp}(P)$. First we have $\mathrm{Res}(Q_i) \supset \mathrm{Res}(P) \in U$. But the ideals in U are primitive, hence maximal (A is liminal !), and so we conclude $\mathrm{Res}(Q_1) = \mathrm{Res}(P) = \mathrm{Res}(Q_2)$. The Q_i are kernels of some irreducible representations $\pi_i \times U_i$ and we have $\ker(\pi_1) = \ker(\pi_2)$, so without loss of generality we may assume $\pi_1 = \pi_2 =: \pi$ because irreducible representations with the same kernel are equivalent in liminal (type I) C*-algebras (the irreducibility follows from 5.1 and $\mathrm{Res}(Q_i) = \mathrm{Res}(P) \not\supset I_\alpha$). For $s \in G$ and $a \in A$ we get $U_{1,s}\pi(a)U_{1,s}^* = \pi(\alpha_s(a)) = U_{2,s}\pi(a)U_{2,s}^*$, hence $U_{2,s}^* U_{1,s} \in \pi(A)' = \mathrm{C} \cdot 1$, i.e. there is $\gamma_1(s) \in \mathrm{C}$ such that $U_{1,2} = \gamma_1(s)U_{2,s}$. Obviously γ_1 is an element of $\hat{G}$. Define $U := U_2, \gamma_2 := 1$. Then we have

$$Q_1 = \ker(\pi \times \gamma_1 U) \quad \text{and} \quad Q_2 = \ker(\pi \times \gamma_2 U),$$

and the claim reduces to $\gamma_1 = \gamma_2$. The argument from now on are taken from ([OR], prop. 1.5). There the special case $A = \mathscr{C}(X, \mathscr{K})$ is considered where π is a point evaluation; but this is irrelevant. For the convenience of the reader the arguments are repeated here.

Let us assume $\gamma_1 \neq \gamma_2$ and construct a contradiction. Since the dual group is Hausdorff there are open and disjoint subsets $V_1, V_2 \subset \hat{G}$ containing γ_1 and γ_2 respectively. We may assume that these sets are of the form

$$V_i = \{\chi \in \hat{G} : |\chi(s) - \gamma_i)s)| < \epsilon_i \; \forall S \in K_i\}$$

where $\epsilon_i > 0$ and $K_i \subset G$ finite, say $\epsilon_1 = \epsilon_2 =: \epsilon$ and $K_1 = K_2 = \{s_1, \ldots, s_n\}$.

Using these data we will construct open susets $M_1, M_2 \subset \mathrm{Prim}(A \times_\alpha G)$ satisfying $Q_1 \in M_1, Q_2 \in M_2$. Since $Q_1, Q_2 \in \mathrm{supp}(P)$ and P is primal these ideals are in a limit set of convergent net and therefore $M_1 \cap M_2 \neq \emptyset$. This finally will lead to the contradiction $V_1 \cap V_2 \neq \emptyset$.

The ideal I corresponding to U is a continuous trace C*-algebra, hence there are $a_0 \in I_\alpha$ and an open neighbourhood $W \subset U$ of $\mathrm{Res}(P) = \ker \pi$ such that

(1) For all irreducible representations ρ of A with $\ker \rho$ in W the operator $\rho(a_0)$ is a rank 1 projection

This is nothing but the Fell-condition. So $\pi(a_0)$ has the form $\langle \cdot | \xi \rangle \xi$ for some $\xi \in H_\pi$ with $||\xi|| = 1$. Then

$$\phi_i := \langle \pi \times \gamma_i U(\cdot)\xi | \xi \rangle \in P(A \times_\alpha G),$$

where $P(A \times_\alpha G)$ is the set of pure states ($\pi \times \gamma_i U$ is irreducible), If π_{ϕ_i} denotes the GNS-construction with ϕ_i we have $\pi_{\phi_i} \simeq \pi \times \gamma_i u$ which implies $\ker \pi_{\phi_i} = Q_i$. Put

$$x_{s_j} := a_0 w_{s_j}^* \delta_{s_j} \in K(G, A), \ j = 1, \dots, n,$$

where $w : G \to U(M(I\alpha))$ is the map coming from the condition (A1).

Computation of $\phi_i(s_j)$: As $\ker \pi \not\supset I_\alpha$ we get

$$\begin{aligned} U_t \pi(a) U_t^* &= \gamma_i U_t \pi(a)(\gamma_i U_t)^* = \pi(\alpha_t(a)) \\ &= \pi(w_t a w_t^*) = \overline{\pi|_{I_\alpha}}(w_t)\pi(a)\overline{\pi|_{I_\alpha}}(w_t)^* \end{aligned}$$

So $\overline{\pi|_{I_\alpha}}(w_t)^* U_t \in \pi(A)' = \mathbf{C} \cdot 1$, hence there is $\lambda \in \hat{G}$ such that

$$\forall t \in G : U_t = \lambda(t)\overline{\pi|_{I_\alpha}}(w_t), \tag{2}$$

and this implies

$$\begin{aligned} \phi_i(x_{s_j} &= \langle (\pi \times \gamma_i U)(x_{s_j})\xi | \xi \rangle = \left\langle \left(\sum_{t \in G} \pi(x_{s_j}(t) \right) \gamma_i(t) U_i \xi | \xi \right\rangle \\ &= \langle (a_0 w_{s_j}^*) \gamma_i(s_j) U_{s_j} \xi | \xi \rangle \overset{(2)}{=} \gamma_i(s_j)\lambda(s_j)\langle \pi(a_0)\xi | \xi \rangle \\ &= \gamma_i(s_j)\lambda(s_j), \end{aligned} \tag{3}$$

because $\pi(a_0)$ is the rank 1 projection $\langle \cdot | \xi \rangle$.

Definition of M_1 *and* M_2: The canonical map

$$P(A \times_\alpha G) \to (A \times_\alpha G)^\wedge; \phi \mapsto [\pi_\phi]$$

is open by [Dx2], 3.4.11) and so

$$\hat{N}_i := \left\{ [\pi_\psi] : \psi \in P(A \times_\alpha G), \max_{j=1,\dots,n} |\psi(x_{s_j}) - \phi_i(x_{s_j})| < \frac{\epsilon}{2} \right\}$$

is open in $(A \times_\alpha G)^\wedge$ $i = 1, 2$. Since $(A \times_\alpha G)^\wedge$ by definition carries the weak topology of the map $\overline{\ker} : [\pi_\phi] \mapsto \ker \pi_\phi$ we must have $\hat{N}_i = \overline{\ker}^{-1}(N_i)$ for some open subset $N_i \subset \mathrm{Prim}(A \times_\alpha G)$. But then $M_i := N; \cap \mathrm{Res}^{-1}(W) \subset \mathrm{Prim}(A \times_\alpha G)$ are open and we hve $Q_1 \in M_1, Q_2 \in M_2$ by construction. As already explained above we must have $M_1 \cap M_2 \neq \emptyset, \Rightarrow \hat{N}_1 \cap \hat{N}_2 \cap \overline{\ker}^{-1} \mathrm{Res}^{-1}(W) \neq \emptyset$. Let $[\rho \times v]$ be an element in this set, $\rho \times v$ irreducible. Then $\ker(\rho)$ is in $W \subset U$, so $\ker \rho \not\supset I_\alpha$, and therefore ρ is irreducible by 5.1. By definition there exist $\eta_i \in H_{\rho \times v} \| \eta_i \| = 1$, such that the states

$$\psi_i := \langle \rho \times v(\cdot)\eta_i | \eta_i \rangle \in P(A \times_\alpha G)$$

satisfy the inequalities

$$|\psi_i(x_{s_j}) - \phi_i(x_{s_j})| < \frac{\epsilon}{2} \ \forall j = 1, \dots, n, \ \ i = 1, 2. \tag{4}$$

Computation of $\psi(x_{s_j})$: From

$$v_t\rho(a)V_t^* = \rho(\alpha_t(a)) = \rho(w_taw_t^*) = \overline{\rho|_{I_\alpha}}(w_t)\rho(a)\overline{\rho|_{I_\alpha}}(w_t)^*$$

and from the condition (A1) (see 3.2) we again may conclude

$$\exists\gamma\in\hat{G}:\forall t\in G: v_t = \overline{\rho|_{I_\alpha}}(w_t)\gamma(t). \tag{5}$$

Thus we get

$$\begin{aligned}\psi_i(X_{s_j}) &= \langle\rho\times v(x_{s_j})\eta_i|\eta_i\rangle = \left\langle\sum_{t\in G}\rho(x_{s_j}(t))v_t\eta_i|\eta_i\right\rangle \\ &= \langle\rho(a_0w_{s_j}^*)v_{s_j}\eta_i|\eta_i\rangle \quad \text{by definition of } x_{s_j} \\ &\overset{(5)}{=} \langle(\rho(a_0)\eta_i|\eta_i\rangle\gamma(s_j).\end{aligned} \tag{6}$$

Note that $\langle\rho(a_0)\eta_i|\eta_i\rangle\in[0,1]$, since $\rho(a_0)$ is a projection (use $\ker\rho\in W$ and $||\eta_i||=1$).

This finally leads us to the desired contradiction: If $\mu\in[0,1]$, then 1 is the nearest point among all points z of the circle and therefore $|1-z|\le|1-\mu|+|\mu-z|\le 2|\mu-z|$ for all these z. This implies

$$\begin{aligned}|\gamma_i(s_j)-\gamma(s_j)\bar{\lambda}(s_j)| &= |1-\gamma_i(s_j)\overline{\gamma(s_j)}\lambda(s_j)| \\ &\le 2|\langle\rho(a_0)\eta_i|\eta_i\rangle-\gamma_i(s_j)\overline{\gamma(s_j)}\lambda(s_j)| \\ &= 2|\langle\rho(a_0)\eta_i|\eta_i\rangle\gamma(s_j)-\gamma_i(s_j)\lambda(s_j)| \\ &= 2|\psi_i(x_{s_j}-\phi_i(x_{s_j}| \qquad \text{by (3) and (6)} \\ &< \epsilon \qquad \text{by (4).}\end{aligned}$$

But this means $\gamma\bar{\lambda}\in V_1\cap V_2$ which is the contradiction we have been looking for.

5.4. Lemma. *Let G be abelian, α satisfy* (A1), *and A have property* (E). *Let $U\subset\mathrm{Prim}(A)$ be as in prop. 5.3. If then $P_1,P_2\in\mathrm{Res}^{-1}(U)\cap\mathrm{Primal}(A\times_\alpha G)$ with $\mathrm{Res}(P_1)=\mathrm{Res}(P_2)$, there is a $\gamma\in\hat{G}$ such that $P_1=\gamma P_2$. Moreover all ideals in $\mathrm{Res}^{-1}(U)\cap\mathrm{Primal}(A\times_\alpha G)$ are minimal primal.* $\mathrm{Res}^{-1}(U)\cap$ Min-Primal$(A\times_\alpha G)$ is dense in Min-Primal$(A\times_\alpha G)$.

Proof. Since the elements in $\mathrm{Res}^{-1}(U)\cap\mathrm{Primal}(A\times_\alpha G)$ are primitive, there are irreducible representations $\pi_j\times U_j$ with $P_j=\ker\pi_j\times U_j$. ker $(\pi_j)\not\supset I_\alpha$ (I_α is the ideal in the definition of (A1)) implies by 5.1 that π_j is irreducible. As $\ker\pi_1=\mathrm{Res}(P_1)=\mathrm{Res}(P_2)=\ker\pi_2$ these must be unitarily equivalent, wlog $\pi_1=\pi_2=:\pi$ For $s\in G$ and $a\in A$ we have $U_{1,s}\pi(a)U_{1,s}^*=\pi(\alpha_2(a))=U_{2,s}\pi(a)U_{2,s}^*$, $U_{1,s}U_{2,s}^*\in\pi(A)'=\mathbf{C}\cdot 1$. So there is a

$\gamma(s) \in \mathbf{C}$ with $U_{1,s} = \gamma(s)U_{2,s}$ and this γ is easily seen to be a character of G. It follows that $P_1 = \ker(\pi \times U_1) = \gamma \ker(\pi \times U_2) = \gamma P_2$ which is the first assertion.

Now let P be in $\mathrm{Res}^{-1}(U) \cap \mathrm{Primal}(A \times_\alpha G)$. If Q is a minimal primal ideal contained in P we must have $\mathrm{Res}(Q) = \mathrm{Res}(P)$ and so $P = \gamma Q$ for some character γ by what has already been proved. This of course implies that P is minimal primal, too.

In order to prove the density assertion let I_u be the ideal corresponding to U which is essential and α-invariant. Then $\overline{K(G, I_U)}$ is essential in $\overline{A \times_\alpha G}$ by prop. 3.3. Now for $P \in \mathrm{Prim}(A \times_\alpha G)$ we have $P \not\supset \overline{K(G, I_U)}$ iff $\mathrm{Res}(P) \not\supset I_U$ iff $\mathrm{Res}(P) \in U$. Hence $\mathrm{Res}^{-1}(U) \cap \mathrm{Prim}(A \times_\alpha G)$ is dense in $\mathrm{Prim}(A \times_\alpha G)$, and this implies that $\mathrm{Res}^{-1}(U) \cap \mathrm{Primal}(A \times_\alpha G)$ is dense in $\mathrm{Primal}(A \times_\alpha G)$. This set is contained in $\text{Min-Primal}(A \times_\alpha G)$ by what we have seen above, and this proves the density result.

5.5. THEOREM. *Let G be abelian, α satisfy* (A1), *and A have property* (E). *Then the following are equivalent:*

(i) $\mathrm{Res} : (\mathcal{M}, \tau_s) \to \mathrm{Min\ Primal}(A)$ *is open.*

(ii) *If $I_1, I_2 \in \mathcal{M}$ with $\mathrm{Res}(I_1) = \mathrm{Res}(I_2)$ there is a $\gamma \in \hat{G}$ such that $I_1 = \gamma I_2$.*

If this is the case and $\text{Min-Primal}(A)$ *is closed then* $\mathcal{M} = \text{Min-Primal}(A \times_\alpha G)$ *is closed.*

PROOF. "(i) $\Rightarrow$ (ii)" Consider $I_1, I_2 \in \mathcal{M}$ with $\mathrm{Res}(I_1) = \mathrm{Res}(I_2) = J$. Let (W_k) and (V_l) be open neighbourhood bases of I_1 and I_2 in $(\mathcal{M}, \tau_s)$ respectively. By assumption $\mathrm{Res}(W_k) \cap \mathrm{Res}(V_l)$ is an open neighbourhood of J in $\text{Min-Primal}(A)$. If $U \subset \text{Min-Primal}(A)$ is as in prop. 5.3 we have $U \cap \mathrm{Res}(W_k) \cap \mathrm{Res}(V_l) \neq \emptyset$, i.e.

$$\exists P_{k,l}^{(1)} \in W_k, P_{k,l}^{(2)} \in V_l : \mathrm{Res}(P_{k,l}^{(1)}) = \mathrm{Res}(P_{k,l}^{(2)}) \in U.$$

By the above lemma we find $\gamma_{k,l} \in \hat{G}$ satisfying $P_{k,l}^{(1)} = \gamma_{k,l} P_{k,l}^{(2)}$. By compactness of $\hat{G}$ we may assume that $\gamma_{k,l}$ converges to some $\gamma \in \hat{G}$. Since $P_{k,l}^{(1)} \xrightarrow{(k,l)} I_1$ and $P_{k,l}^{(2)} \xrightarrow{(k,l)} I_2$ with respect to τ_s we get from the continuity of the $\hat{G}$-action (see 5.2) the desired result $I_1 = \gamma I_2$.

"(ii) $\Rightarrow$ (i)": Suppose that $\mathrm{Res} : (\mathcal{M}, \tau_s) \to \text{Min-Primal}(A)$ is not open. Then there are an open subset $W \subset \mathcal{M}$, an ideal $I \in W$, and a net (J_i) in $\text{Min-Primal}(A) \backslash \mathrm{Res}(W)$ with $J_i \to \mathrm{Res}(I)$. Let $I_i \in \mathcal{M}$ with $\mathrm{Res}(I_i) = J_i$. Restricting to a subnet we may assume $I_i \to \tilde{I}$ in $(\mathrm{Id}(A \times_\alpha G), \tau_s)$. The continuity of Res yield $\mathrm{Res}(\tilde{I}) = \mathrm{Res}(I)$. This implies $\tilde{I}$ in $\mathcal{M}$ and by assumption we can find a character γ such that $I = \gamma \tilde{I} = \lim_i \gamma I_i$. So we must have $\gamma I_i \in W$ for large indices i and therefore $J_i = \mathrm{Res}(I_i) = \mathrm{Res}(\gamma I_i) \in \mathrm{Res}(W)$

for these indices. This contradiction finishes the proof of the reverse direction.

By 4.3 we have Min-Primal$(A) \subset \mathcal{M}$. Conversely each $P \in \mathcal{M}$ contains a minimal primal ideal Q which implies $\mathrm{Res}(P) \supset \mathrm{Res}(Q)$. But both $\mathrm{Res}(P)$ and $\mathrm{Res}(Q)$ are minimal primal hence they are equal. By the above result we have $P = \gamma Q$ for some character γ and so P is minimal primal, too, This also proves the closedness assertion.

Res is not open in general (even under stronger assumptions) as will be seen in remark 6.6. I do not know whether the equivalence of (i) and (ii) holds under weaker hypotheses.

6. The property (EM).

6.1. The property (E) in general does not carry over from A to $A \times_\alpha G$ and it is easy to see that (Eb) does. There are examples showing that the slightly stronger property "(Eb) and the minimal primal ideal space is closed" is not inherited from A to $A \times_\alpha G$. In this section it will be shown that the following stronger property (EM) is in fact inherited. A is said to have the property (EM) iff A has (Eb), Min-Primal(A) is closed, and the map $I \mapsto \dim(I)$ is continuous on MinPrimal(A).

So this will be another theorem yielding the closedness of the minimal primal ideal space of the crossed product algebra. We start with a series of lemmata.

6.2. Lemma. *Let A have property* (E), *α satify* (A2),$\{Q_1, \dots, Q_r\} \subset$ Prim(A) *be an α-invariant set with $Q_1, \dots, Q_r$ pairwise different, and G act transitively on this set. Then*

(i) $\bigcap_{j=1}^r Q_j$ *is primal and α-invariant.*

(ii) $\mathcal{J} := \{J \in \mathrm{Id}(A \times_\alpha G) : J \cap A = \bigcap_{j=1}^r Q_j\}$ *has maximal elements and these are primitive.*

(iii) *If P is any primitive ideal in $A \times_\alpha G$ with* $\mathrm{Res}(P) = \bigcap_{j=1}^r Q_j$, *then*

$$\dim P \geq r \cdot \dim Q_1 = \sum_{j=1}^r \dim Q_j.$$

Proof. (i): Let $I_\alpha \lhd A$ be the essential ideal in the definition of (A2). Then there is a net (P_i) of primitive ideals such that $I_\alpha \not\subset P_i \to Q_1$. For all $j \in \{1, \dots, r\}$ we can find $s_j \in G$ with $Q_j = \alpha_{s_j}^{-1}(Q_1)$ by the assumed transitivity. By 3.1 the P_i are α-invariant implying $P_i = \alpha_{s_j}^{-1} P_i \xrightarrow{i} \alpha_{s_j}^{-1}(Q_1) = Q_j$. So $\{Q_1, \dots, Q_r\}$ is contained in the limit of some convergent net. This proves the first part, the α-invariant being clear.

(ii): Since the image of Res consists of all invariant ideals, (i) gives us that

$\{\mathcal{J}\}$ is non-empty. $\{\mathcal{J}\}$ obviously is inductively ordered and so we have maximal elements by Zorn's lemma. Let P be such a maximal element and let us show that it is primitive.

Since $P \neq A \times_\alpha G$ it must be contained in some primitive ideal R. $\Rightarrow$

$$\operatorname{Res}(R) \supset \operatorname{Res}(P) = \bigcap_{j=1}^{R} Q_j \quad \Rightarrow$$

$$\operatorname{supp}(\operatorname{Res}(R)) \subset \operatorname{supp}\left(\bigcap_{j=1}^{T} Q_j\right) = \{Q_1, \ldots, Q_r\}.$$

This last equality stems from the liminality of A (all primitive ideals are maximal) and the fact that primitive ideals are prime.

But then supp(Res(R)) is an α-invariant subset in $\{Q_1, \ldots, Q_r\}$ and by the assumed transitivity we have equality implying $\operatorname{Res}(R) = \bigcap_{j=1}^{R} Q_j$. Hence R is in $\{\mathcal{J}\}$ and the maximality proves $P = R$.

(iii): Now let $P \triangleleft A \times_\alpha G$ be primitive with $P \cap A = \bigcap_{j=1}^{r} Q_j$. There are $\pi, \pi_1, \ldots, \pi_r$ irreducible representations such that $P = \ker \pi$ and $Q_j = \ker \pi_j$, and the π_j's are pairwise inequivalent. By ([Dx2], 4.2.5) we get

$$\pi(A) \cong A/(a \cap \ker \pi) = A \Big/ \left(\bigcap_{j=1}^{R} Q_j\right) \cong \bigoplus_{j=1}^{r} A/Q_j \cong \bigoplus_{j=1}^{r} \pi_j(A).$$

So H_π is a Hilbert space such that $L(H_\pi)$ contains a subalgebra isomorphic to $\bigoplus_{j=1}^{r} \pi_j(A) \cong \bigoplus_{j=1}^{r} L(H_{\pi_j})$ (π_j is finite dimensional !). So

$$\dim P = \dim H_\pi \geq \sum_{j-1}^{r} \dim H_{\pi_j} = \sum_{j=1}^{r} \dim Q_j.$$

Clearly $\dim(Q_1) = \dim(Q_j)$ for all j.

6.3. Lemma. *Let G be abelian, α satisfy* (A1), *A have property* (E), *and $U \subset \operatorname{Prim}(A)$ as in prop.* 5.3. *Furthermore let $I \triangleleft A$ be an α-invariant primal ideal and (P_i) a net in* $\operatorname{Prim}(A \times_\alpha G)$ *such that* $\operatorname{Res}(P_i) \in U$ *and* $\operatorname{Res}(P_i) \to I$.

If then $Q \in \operatorname{Prim}(A \times_\alpha G)$ with $\operatorname{Res}(Q) \supset I$, there is subnet (P_{i_k}) and a character $\gamma \in \hat{G}$ such that $P_{i_k} \xrightarrow{k} \gamma Q$ with respect to τ_w,

Proof. We have $\overline{K(G, \operatorname{Res}(P_i))} \to \overline{K(G, I)} \subset \overline{K(G, \operatorname{Res}(Q))} \subset Q$. So by Schochetman's theorem 2.1 there are a subset (P_{i_k}) and primitive ideals $R_{i_k} \in \operatorname{supp}\overline{K(G, \operatorname{Res}(P_{i_k}))}$ satisfying $R_{i_k} \to Q$. Since $\operatorname{Res}(R_{i_k}) \supset \operatorname{Res}(P_{i_k})$ we even have equality because $\operatorname{Res}(P_{i_k})$ is primitive by 5.3 and therefore maximal. 5.4 provides us with characters $\gamma_k \in \hat{G}$ satisfying $P_{i_k} = \gamma_k R_{i_k}$. Restricting to another subnet we may assume $\gamma_k \to \gamma \in \hat{G}$ by compactness of $\hat{G}$ and $R_{i_k} \to R(\tau_s)$ by τ_s-compactness of $\mathrm{Id}(A \times_\alpha G)$.

Now 5.2 implies $P_{i_k} = \gamma_k R_{i_k} \to \gamma R$. As $R_{i_k} \to Q(\tau_w)$, we have $R \subset Q$ implying $\gamma R \subset \gamma Q$, hence $P_{i_k} \to \gamma Q(\tau_w)$.

6.4. Lemma. *Let G be abelian, α satisfy* (A1), *A satisfy* (EM)‹ *and $U \subset \mathrm{Prim}(A)$ as in prop.* 5.3. *Then the following holds*

(i) *If (P_i) is a net in $\mathrm{Res}^{-1}(U) \cap \mathcal{M}$ and $P_i \to J(\tau_s)$ then*

$$\lim_i \dim P_i = \sum_{Q \in \mathrm{supp}(J)} \dim Q = \sum_{R \in \mathrm{supp}(\mathrm{Res}(J))} \dim R.$$

(ii) *We have equality in lemma* 6.2. (iii), i.e.
If $\{Q_1, \ldots, Q_r\} \subset \mathrm{Prim}(A)$ is an α-invariant set (pairwise different Q_i's) on which α operates transitively and P is a primitive ideal in $A \times_\alpha G$ such that $\mathrm{Res}(P) = \bigcap_{j=1}^r Q_j$, we have

$$\dim P = r \cdot \dim Q_1 = \sum_{j=1}^r \dim Q_j.$$

Proof. (i) First note $\mathrm{Res}(P_i) \in U \subset \mathrm{Prim}(A) \cap \text{min-Primal}(A)$, hence

$$\mathrm{Res}(J) = \tau_s - \mathrm{Res}(P_i) \in \text{Min-Primal}(A),$$

becaue Min-Primal(A) is closed by (EM); and we have

$$\lim_i \dim \mathrm{Res}(P_i) = \sum_{R \in \mathrm{supp}(\mathrm{Res}(J))} \dim R. \tag{1}$$

By proposition 5.1 we know $\dim P_i = \dim \mathrm{Res}(P_i)$ ($P_i = \ker(\pi \times U)$ holds for some irreducible $\pi \times U$; then $\mathrm{Res}(P_i) = \ker \pi$ and the irreducibility of π yields the equality of the dimensions). This together with (1) implies

$$\lim_i \dim P_i = \sum_{R \in \mathrm{supp}(\mathrm{Res}(J))} \dim R \tag{2}$$

and from lemma 2.3 we get

$$\lim_i \dim P_i \geq \sum_{Q \in \mathrm{supp}(J)} \dim Q. \tag{3}$$

We only have to prove equality here.

supp $\mathrm{Res}(J)$ is finite (because A has property (Eb) and therefore this set decomposes into a disjoint union supp $\mathrm{Res}(J) = C_i \cup \ldots \cup C_r$, where the sets C_j are α-invariant and α operates transitively on them. By lema 6.2 (ii) there are $Q_j \in \mathrm{Prim}(A \times_\alpha G)$ with $\mathrm{Res}(Q_j) = \cap C_j$ and lemma 6.2. (iii) says

$$\dim Q_j \geq \sum_{R \in C_j} \dim R. \tag{4}$$

Since $\mathrm{Res}(Q_j) = \cap C_j \supset \mathrm{Res}(J) \in \text{Min-Primal}(A)$ and because of lemma 6.3 we may assume $P_i \to \gamma_j Q_j$ for some $\gamma_j \in \hat{G}$ (take r times a subnet if necessary). But we also have $P_i \to J$ with respect to the τ_s-topology implying for all $x \in A \times_\alpha G$

$$\|x + J\| = \lim_i \|x + P_i\| \geq \|\gamma_j Q_j\|$$

From this we see $J \subset \gamma_j Q_j$ and then

$$\mathrm{supp}(J) \supset \{\gamma_1 Q_1, \ldots, \gamma_r Q_r\}. \tag{5}$$

Because $\mathrm{Res}(\gamma_j Q_j) = \mathrm{Res}(Q_j) = \cap C_j$ the ideals $\gamma_1 Q_1, \ldots, \gamma_r Q_r$ must be pairwise different and this leads to

$$\begin{aligned}\sum_{Q \in \mathrm{supp}(J)} \dim Q &\overset{(5)}{\geq} \sum_{j=1}^{R} \dim(\gamma_j Q_j) = \sum_{j=1}^{R} \dim Q_j \\ &\overset{(4)}{\geq} \sum_{j=1}^{r} \sum_{R \in C_j} \dim R = \sum_{R \in \mathrm{supp}(\mathrm{Res}(J))} \dim R \\ &\overset{(2)}{=} \lim_i \dim P_i.\end{aligned}$$

Together with (3) the desired equality follows.

(ii) Since the last chain of inequalities ends up in an equality we conclude

$$\sum_{j=1}^{r} \dim Q_j = \sum_{j=1}^{r} \sum_{R \in C_j} \dim R.$$

By (4) or lemma 6.2 (iii) we then have

$$\dim Q_j = \sum_{R \in C_j} \dim R, \quad j \in \{1, \ldots, r\}$$

which proves (ii).

6.5. Theorem. *Let G be abelian, α satisfy* (A1), *and A have property* (EM). *Then also $A \times_\alpha G$ has the property* (EM), *in particular* $\text{Min-Primal}(A \times_\alpha G)$ *is closed and we have* $\mathcal{M} = \text{Min-Primal}(A \times_\alpha G)$

Proof. It was already mentioned that $A \times_\alpha G$ has (Eb). Next we show that the minimal primal ideal space is closed. To this end consider an ideal J in $\overline{\text{Min-Primal}(A \times_\alpha G)}^{\tau_s} \setminus \{A \times_\alpha G\}$. Let U be as in 5.3, hence there is a net (P_i) in $\mathrm{Res}^{-1}(U)$ such that $P_i \to J$ with respect to τ_s. Let I be a minimal primal ideal contained in J. $\Rightarrow \mathrm{Res}(I) \subset \mathrm{Res}(J)$. As $\mathrm{Res}(P_i) \to \mathrm{Res}(J)$ and Min-Primal(A) is closed $\mathrm{Res}(J)$ must be minimal primal implying $\mathrm{Res}(I) = \mathrm{Res}(J)$. Now lemma 6.4 leads to

$$\sum_{Q \in \mathrm{supp}(J)} \dim Q = \sum_{R \in \mathrm{supp}(\mathrm{Res}(J))} \dim R = \sum_{R \in \mathrm{supp}(\mathrm{Res}(I))} \dim R = \sum_{Q \in \mathrm{supp}(I)} \dim Q.$$

Since $\mathrm{supp}(I) \supset \mathrm{supp}(J)$ we must have equality, hence $I = J$ proving that J is minimal primal.

Now let us verify the continuity of the map

$$\dim : \text{Min-Primal}(A \times_\alpha G) \to \mathsf{N}.$$

Let P_i converge to J in the minimal primal ideal space. Since N with the discrete topology is a regular space we may assume that the ideals P_i all lie in some dense subset, say in $\mathrm{Res}^{-1}(U) \cap \text{Min-Primal}(A \times_\alpha G)$ (see lemma 5.4 for this). We have to show

$$\sum_{Q \in \mathrm{supp}(P_i)} \dim Q \to \sum_{Q \in \mathrm{supp}(J)} \dim Q.$$

But the left side is nothing but $\dim P_i$ and the continuity assertion follows from the proceeding lemma. This proves that the crossed product algebra has property (EM); the rest goes without saying.

6.6. Remark. The property (EM) does not imply openess of Res in general. Consider

$$A := \left\{ x \in C([0,2], M_3) : x(1) = \begin{pmatrix} \lambda_1(x) & 0 & 0 \\ 0 & \lambda_2(x) & 0 \\ 0 & 0 & \lambda_3(x) \end{pmatrix} \right\},$$

where $C([0,2], M_3)$ stands for the algebra of continuous functions on the interval $[0,2]$ with values in the 3×3-matrices. The irreducible representations of are given by

$$\pi_t(x) = x(t),\ t \in [0,2] \setminus \{1\}, \quad \lambda_1, \lambda_2, \lambda_3.$$

$I := \ker(\lambda_1) \cap \ker(\lambda_2) \cap \ker(\lambda_3)$ is an essential ideal in A. Let

$$v(t) := \begin{cases} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } t < 1 \\ \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } t > 1 \end{cases}$$

Then $v \in M(I)$ and the conjugation with v defines an $\mathsf{Z}/(4\mathsf{Z})$-action on A

satisfying (A1). Tedious computations show that Res is not open, in fact it can be shown that there are eight minimal primal ideals in $A \times Z/(4Z)$ which are mapped to I by Res. Since $Z/(4Z)$ cannot act transitively on eight elements, Res is not open by 5.2. The details are left to the reader.

Another useful property is "quasistandard" (see [AS1}) which means that Min-Primal(A) is closed and each primitive ideal containes exactly one minimal primal ideal. But even if A additionally satisfies (EM) then $A \times_\alpha G$ in general fails to be quasistandard as can be shown by examples.

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