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ON A PROPERTY OF THE FOURIER TRANSFORM

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Let, as usual, $C_0(\mathsf{R})$ denote the supremum normed Banach space of contiuous functions tending to zero at infinity.

Very few of the elements of $C_0(\mathsf{R})$ are Fourier transforms of integrable functions. One way of stating this more precisely is to observe that the set of Fourier transforms is a first category subset of $C_0(\mathsf{R})$. Another indication of the smallness of the set of Fourier transforms is given by the following result.

THEOREM. *If X is a closed reflexive subspace of $C_0(\mathsf{R})$ consisting entirely of Fourier transforms of $L^1(\mathsf{R})$ functions then X is finite dimensional*

PROOF. Suppose that X is a closed reflexive infinite dimensional subspace of $C_0(\mathsf{R})$ and that every element of X is the Fourier transform of some $f \in L^1(\mathsf{R})$.

Let $Y = \{f \in L^1(\mathsf{R}) : \hat{f} \in X\}$. Then clearly Y is a closed subspace of $L^1(\mathsf{R})$. By the open mapping theorem we have for some $K > 0$ and every $f \in Y$

$$\|\hat{f}\|_\infty \geq K\|f\|_1 \tag{1}$$

Furthermore, we have for some $a, L > 0$ and every $f \in Y$

$$\sup_{|x|\leq a} |\hat{f}(x)| \geq L\|\hat{f}\|_\infty \tag{2}$$

In order to prove this we assume that (2) is false. Letting $\{\epsilon_n\}_{n=1}^\infty$ be a sequence of positive numbers such that $\sum_{n=1}^\infty \epsilon_n < 1$ we choose inductively a sequence $\{\hat{f}\}_{n=1}^\infty$ in X as follows.

We let $\hat{f}_1$ be any element of X such that $\|\hat{f}_1\|_\infty = 1$, and having chosen $\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n$ we let α_n be so large that

$$\sup_{|x|\geq\alpha_n} |\hat{f}_k(x)| \leq \epsilon_n \tag{3}$$

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for every $k \leq n$. Then, by our assumption that (2) does not hold for any $a, L > 0$ we can find $\hat{f}_{n+1} \in X$ with $\|\hat{f}_{n+1}\|_\infty = 1$ such that

$$\sup_{|x| \leq \alpha_n} |\hat{f}_{n+1}(x)| \leq \epsilon_n \tag{4}$$

Having thus, by induction, constructed a sequence $\{\hat{f}_n\}_{n=1}^\infty$ we let b_n be such that $|\hat{f}_n(b_n)| = \|\hat{f}_n\|_\infty = 1$. Given any positive integer N and numbers $a_1, a_2, \ldots, a_N$ we let k be such that $|a_k| = \max_{1 \leq n \leq N} |a_n|$. We then have, by (3) and (4)

$$\left\| \sum_{n=1}^N a_n \hat{f}_n \right\|_\infty \geq |a_k| \left(\left|\hat{f}_k(b_k)\right| - \sum_{n \neq k} \left|\hat{f}_n(b_k)\right| \right) \geq \max_{1 \leq n \leq N} |a_n| \left(1 - \sum_{n=1}^\infty \epsilon_n \right) \tag{5}$$

Also, clearly we have

$$\left\| \sum_{n=1}^N a_n \hat{f}_n \right\|_\infty \leq \max_{1 \leq n \leq N} |a_n| \left(1 + \sum_{n=1}^\infty \epsilon_n \right) \tag{6}$$

(5) and (6) imply, together with (1) that

$$\left\| \sum_{n=1}^N a_n f_n \right\|_1 \geq \max_{1 \leq n \leq N} |a_n| \left(1 - \sum_{n=1}^\infty \epsilon_n \right) \tag{7}$$

$$\left\| \sum_{n=1}^N a_n f_n \right\|_1 \leq K^{-1} \max_{1 \leq n \leq N} |a_n| \left(1 + \sum_{n=1}^\infty \epsilon_n \right) \tag{8}$$

This means that the mapping $\epsilon_n \to f_n$, where $\{\epsilon_n\}_{n=1}^\infty$ is the canonical basis of c_0, is an isomorphism between c_0 and the closed subspace of $L^1(\mathsf{R})$ spanned by $\{f_n\}_{n=1}^\infty$. However, as is well known, no subspace of $L^1(\mathsf{R})$ can be isomorphic to c_0. This follows e.g. from the fact that $L^1(\mathsf{R})$ is weakly sequentially complete (see [1], p. 290), which c_0 is not. Thus we have obtained a contradiction and the proof of (2) is complete.

Next we observe that for every $J, M > 0$ we must have for some $f \in Y$

$$\sum_{n=-\infty}^{\infty} \left| \int_{nJ}^{(n+1)J} f\, dx \right| < M \int |f| dx \tag{9}$$

For otherwise the mapping $f \to (x_n)$, where $x_n = \int_{nJ}^{(n+1)J} f\, dx$ would be an isomorphism between Y and an infinite dimensional subspace of l_1. As is well known (see [2], p.53) every infinite dimensional subspace of l_1 contains an isomorphic copy of l_1, and is thus in particular not reflexive, contradicting the reflexivity of Y. Thus for every positive J, M (9) holds for some $f \in Y$.

Now let $f \in Y$ be such that (9) holds with $M = KL/2$ and $J = M/a$, where K and L have the same meaning as in (1) and (2). We then have by (9)

$$\sup_{|x|\leq a}\left|\hat{f}(x)\right| \leq \sup_{|x|\leq a}\sum_{n=-\infty}^{\infty}\left|\int_{nM/a}^{(n+1)M/a} e^{-itx}f(t)dt\right| \leq$$

$$\sup_{|x|\leq a}\left(\sum_{n=-\infty}^{\infty}\int_{nM/a}^{(n+1)M/a}|e^{-itx}-e^{ixnM/a}||f(t)|dt+\sum_{n=-\infty}^{\infty}\left|\int_{nM/a}^{(n+1)M/a}f(t)dt\right|\right)\leq$$

$$\sum_{n=-\infty}^{\infty} aM/a\int_{nM/a}^{(n+1)M/a}|f(t)|dt+\sum_{n=-\infty}^{\infty}\left|\int_{nM/a}^{(n+1)M/a}f(t)dt\right| < KL\int_{-\infty}^{\infty}|f(t)|dt$$

On the other hand, by (1) and (2) we have

$$\sup_{|x|\leq a}\left|\hat{f}(x)\right| \geq KL\int|f|dt,$$

a contradiction which completes the proof of the theorem.

The result we have just proved is no longer true if we drop the reflexivity condition. This is easily seen by considering the infinite dimensional subspace of $C_0(\mathsf{R})$ consisting of the functions $(\sum_{n=0}^{\infty} a_n e^{-i2^n x})e^{-x^2/2}$ with $\sum_{n=0}^{\infty}|a_n| \leq \infty$. By the theory of lacunary Fourier series (see [3], p. 247) this is a closed subspace of $C_0(\mathsf{R})$. Also, each $(\sum_{n=0}^{\infty} a_n e^{i2^n x})e^{-x^2/2}$ is the Fourier transform of $1/\sqrt{2\pi}\sum_{n=0}^{\infty} a_n e^{-(x+2^n)^2/2} \in L^1(\mathsf{R})$.

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