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ISOMORPHISMS OF HILBERT C*-MODULES AND $*$-ISOMORPHISMS OF RELATED OPERATOR C*-ALGEBRAS

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Abstract.

Let $\mathcal{M}$ be a Banach C*-module over a C*-algebra A carrying two A-valued inner products $\langle .,. \rangle_1$, $\langle .,. \rangle_2$ which induce norms on $\mathcal{M}$ equivalent to the given one. Then the appropriate unital C*-algebras of adjointable bounded A-linear operators on the Hilbert A-modules $\{\mathcal{M}, \langle .,. \rangle_1\}$ and $\{\mathcal{M}, \langle .,. \rangle_2\}$ are shown to be $*$-isomorphic if and only if there exists a bounded A-linear isomorphism S of these two Hilbert A-modules satisfying the identity $\langle .,. \rangle_2 \equiv \langle S(.), S(.) \rangle_1$. This result extends other equivalent descriptions due to L. G. Brown, H. Lin and E. C. Lance. An example of two non-isomorphic Hilbert C*-modules with $*$-isomorphic C*-algebras of "compact"/adjointable bounded module operators is indicated.

Investigations in operator and C*-theory make often use of C*-modules as a tool for proving, especially of Banach and Hilbert C*-modules. Impressing examples of such applications are G. G. Kasparov's approach to K- and KK-theory of C*-algebras [7,18] or the investigations of M. Baillet, Y. Denizeau and J.-F. Havet [1] and of Y. Watatani [17] on (normal) conditional expectations of finite index on W*-algebras and C*-algebras. In addition, the theory of Hilbert C*-modules is interesting in its own.

Our standard sources of reference to Hilbert C*-module theory are the papers [13,9,5,6], chapters in [7,18] and the book of E. C. Lance [11]. We make the convention that all C*-modules of the present paper are left modules by definition. A *pre-Hilbert A-module over a C*-algebra A* is an A-module $\mathcal{M}$ equipped with an A-valued mapping $\langle .,. \rangle : \mathcal{M} \times \mathcal{M} \to A$ which is A-linear in the first argument and has the properties:

$$\langle x,y \rangle = \langle y,x \rangle^* \ , \ \langle x,x \rangle \geq 0 \quad \text{with equality iff} \quad x = 0 \, .$$

The mapping $\langle .,. \rangle$ is called *the A-valued inner product on $\mathcal{M}$*. A pre-Hilbert A-module $\{\mathcal{M}, \langle .,. \rangle\}$ is *Hilbert* if and only if it is complete with respect to the norm $\|.\| = \|\langle .,. \rangle\|_A^{1/2}$. We always assume that the linear structures of A and $\mathcal{M}$ are compatible.

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One of the key problems of Hilbert C*-module theory is the question of isomorphism of Hilbert C*-modules. First of all, they can be isomorphic as Banach A-modules. But there is another natural definition: Two Hilbert A-modules $\{\mathcal{M}_1, \langle .,. \rangle_1\}$, $\{\mathcal{M}_2, \langle .,. \rangle_2\}$ over a fixed C*-algebra A are *isomorphic as Hilbert C*-modules* if and only if there exists a bijective bounded A-linear mapping $S : \mathcal{M}_1 \to \mathcal{M}_2$ such that the identity $\langle .,. \rangle_1 \equiv \langle S(.), S(.) \rangle_2$ is valid on $\mathcal{M}_1 \times \mathcal{M}_1$. In 1985 L. G. Brown presented two examples of Hilbert C*-modules which are isomorphic as Banach C*-modules but which are non-isomorphic as Hilbert C*-modules, cf. [2,12,6]. This result was very surprising since Hilbert space theory, the classical investigations on Hilbert C*-modules like [13,9], G. G. Kasparov's approach to KK-theory of C*-algebras relying on countably generated Hilbert C*-modules and other well-known investigations in this field did not give any indication of such a serious obstacle in the general theory of Hilbert C*-modules. L. G. Brown obtained his examples from the theory of different kinds of multipliers of C*-algebras without identity. Furthermore, making use of the results of the Ph.D. thesis of Nien-Tsu Shen [15] he proved the following: For a Banach C*-module $\mathcal{M}$ over a C*-algebra A carrying two A-valued inner products $\langle .,. \rangle_1$, $\langle .,. \rangle_2$ which induce norms on $\mathcal{M}$ equivalent to the given one the appropriate C*-algebras of "compact" bounded A-linear operators on the Hilbert A-modules $\{\mathcal{M}, \langle .,. \rangle_1\}$ and $\{\mathcal{M}, \langle .,. \rangle_2\}$ are $*$-isomorphic if and only if there exists a bounded A-linear isomorphism S of these two Hilbert A-modules satisfying $\langle .,. \rangle_2 \equiv \langle S(.), S(.) \rangle_1$, cf. [2, Thm. 4.2, Prop. 4.4] together with [6, Prop. 2.3], ([4]). By definition, the set of "compact" operators $\mathrm{K}_A(\mathcal{M})$ on a Hilbert A-module $\{\mathcal{M}, \langle .,. \rangle\}$ is defined as the norm-closure of the set $\mathrm{K}^0_A(\mathcal{M})$ of all finite linear combinations of the operators

$$\{\theta_{x,y} : \theta_{x,y}(z) = \langle z, x \rangle y \text{ for every } x, y, z \in \mathcal{M}\}.$$

It is a C*-subalgebra and a two-sided ideal of $\mathrm{End}^*_A(\mathcal{M})$, the set of all adjointable bounded A-linear operators on $\{\mathcal{M}, \langle .,. \rangle\}$, what is the multiplier C*-algebra of $\mathrm{K}_A(\mathcal{M})$ by [9, Thm. 1]. Note, that in difference to the well-known situation for Hilbert spaces, the properties of an operator to be "compact" or to possess an adjoint depend heavily on the choice of the A-valued inner product on $\mathcal{M}$. These properties are not invariant even up to isomorphic Hilbert structures on $\mathcal{M}$, in general, cf. [6]. We make the convention that operators T which are "compact"/adjointable with respect to some A-valued inner product $\langle .,. \rangle_i$ will be marked $T^{(i)}$ to note where this property arises from. The same will be done for sets of such operators.

In 1994 E. C. Lance showed that two Hilbert C*-modules are isomorphic as Hilbert C*-modules if and only if they are isometrically isomorphic as Banach C*-modules ([10]) opening the geometrical background of this functional-analytical problem and extending a central result for C*-algebras: C*-algebras are isometrically multiplicatively isomorphic if and only if they are $*$-isomorphic, [8, Thm. 7, Lemma 8].

At the contrary, non-isomorphic Hilbert structures on a given Hilbert A-module $\mathcal{M}$ over a C*-algebra A can not appear at all if $\mathcal{M}$ is *self-dual*, i. e. every bounded module map $r : \mathcal{M} \to A$ is of the form $\langle ., a_r \rangle$ for some element $a_r \in \mathcal{M}$ (cf. [5, Prop. 2.2,Cor. 2.3]), or if A is unital and $\mathcal{M}$ is *countably generated*, i. e. there exists a countably set of generators inside $\mathcal{M}$ such that the set of all finite A-linear combinations of generators is norm-dense in $\mathcal{M}$ (cf. [2, Cor. 4.8, Thm. 4.9] together with [7, Cor. 1.1.25] and [6, Prop. 2.3]).

Now, we come to the goal of the present paper: Whether for a Banach C*-module $\mathcal{M}$ over a C*-algebra A carrying two A-valued inner products $\langle .,. \rangle_1$, $\langle .,. \rangle_2$ which induce norms on $\mathcal{M}$ equivalent to the given one the appropriate C*-algebras $\mathrm{End}_A^{(1,*)}(\mathcal{M})$ and $\mathrm{End}_A^{(2,*)}(\mathcal{M})$ of all adjointable bounded A-linear operators on $\mathcal{M}$ are $*$-isomorphic, or not? This question is non-trivial since even non-$*$-isomorphic non-unital C*-algebras can possess a common multiplier C*-algebra:

Consider the commutative AW*-algebra $A = \mathrm{L}^\infty([0,1]) \oplus \mathrm{D}([0,1])$, where $\mathrm{L}^\infty([0,1])$ is the W*-algebra of all Lebesgue-measurable, essentially bounded functions on [0,1] factorized by the subset of those functions which equal to zero outside a set of measure zero, and where $\mathrm{D}([0,1])$ is the Dixmier algebra of all Borel functions on [0,1] factorized by the subset of all those functions which equal to zero outside a meager set. By the Gel'fand theorem the commutative AW*-algebra A can be described as the set of continuous functions on the disjoint union of two compact Hausdorff spaces: K_1 -- a hyperstonean one, K_2 -- a stonean one without any non-trivial normal measure and with a dense subset of first category, (cf. [16, III,Thm. 1.17]). Both these compact spaces possess accumulation points $x_1 \in K_1$, $x_2 \in K_2$. Set

$$B_1 = \{f \in A : f(x_1) = 0\} \quad , \qquad B_2 = \{f \in A : f(x_2) = 0\} .$$

The sets B_1 and B_2 are two-sided norm-closed ideals of A with trivial orthogonal complements. Since A is an AW*-algebra their multiplier C*-algebras are injectively embedded in A as C*-subalgebras and coincide with their sets of two-sided multipliers estimated with respect to A, [14]. Consequently, $\mathrm{M}(B_1) \equiv \mathrm{M}(B_2) \equiv A$, whereas B_1 is not $*$-isomorphic to B_2 by the Stone-Weierstraß theorem and by the very different properties of the compact spaces K_1, K_2. (Note, that there are topological reasons for this phenomenon which are of separate interest.)

At the contrary, if one of two non-unital C*-algebras with $*$-isomorphic multiplier C*-algebras is separable then the initial C*-algebras are $*$-isomorphic by [3], what fits with the result on countably generated Hilbert C*-modules by [7, Cor. 1.1.25]. That is, additional arguments are needed to describe the relation between the multiplier C*-algebras of non-$*$-isomorphic C*-algebras of "compact" operators on some Banach C*-modules carrying non-isomorphic C*-valued inner products. One quickly realizes that the techniques of multiplier theory are not suitable to shed some more light on this general situation. One has to turn back to C*-theory and to the properties of $*$-isomorphisms, as well as to the theory of Hilbert C*-modules.

THEOREM. *Let A be a C*-algebra and $\mathcal{M}$ be a Banach A-module carrying two A-valued inner products $\langle .,. \rangle_1$, $\langle .,. \rangle_2$ which induce norms equivalent to the given one. Then the following conditions are equivalent:*

(i) *The Hilbert A-modules $\{\mathcal{M}, \langle .,. \rangle_1\}$ and $\{\mathcal{M}, \langle .,. \rangle_2\}$ are isomorphic as Hilbert C*-modules.*

(ii) *The Hilbert A-modules $\{\mathcal{M}, \langle .,. \rangle_1\}$ and $\{\mathcal{M}, \langle .,. \rangle_2\}$ are isometrically isomorphic as Banach A-modules.*

(iii) *The C*-algebras $\mathrm{K}_A^{(1)}(\mathcal{M})$ and $\mathrm{K}_A^{(2)}(\mathcal{M})$ of all "compact" bounded A-linear operators on both these Hilbert C*-modules, respectively, are $*$-isomorphic.*

(iv) *The unital C*-algebras $\mathrm{End}_A^{(1,*)}(\mathcal{M})$ and $\mathrm{End}_A^{(2,*)}(\mathcal{M})$ of all adjointable bounded A-linear operators on both these Hilbert C*-modules, respectively, are $*$-isomorphic.*

Further equivalent conditions in terms of positive invertible quasi-multipliers of $\mathrm{K}_A^{(1)}(\mathcal{M})$ can be found in [6].

PROOF. The equivalence of (i) and (ii) was shown by E. C. Lance [10], and the equivalence of (i) and (iii) turns out from a result for C*-algebras of L. G. Brown [2, Thm. 4.2, Prop. 4.4] in combination with [6, Prop. 2.3]. Referring to G. G. Kasparov [9, Thm. 1] the implication (iii)$\rightarrow$(iv) yields naturally.

Now, suppose the unital C*-algebras $\mathrm{End}_A^{(1,*)}(\mathcal{M})$ and $\mathrm{End}_A^{(2,*)}(\mathcal{M})$ are $*$-isomorphic. Denote this $*$-isomorphism by ω. One quickly checks that the formula

$$x \in \mathcal{M} \rightarrow \langle x, x \rangle_{1,Op.} = \theta_{x,x}^{(1)} \in \mathrm{K}_A^{(1)}(\mathcal{M})$$

defines a $\mathrm{K}_A^{(1)}(\mathcal{M})$-valued inner product on the Hilbert A-module $\mathcal{M}$ regarding it as a right $\mathrm{K}_A^{(1)}(\mathcal{M})$-module. Moreover, the set $\{K(x) : x \in \mathcal{M}, K \in \mathrm{K}_A^{(1)}(\mathcal{M})\}$ is norm-dense inside $\mathcal{M}$ since the limit equality

$$x = \|.\|_{\mathscr{M}} - \lim_{n \to \infty} (\theta^{(1)}_{x,x}(\theta^{(1)}_{x,x} + n^{-1})^{-1})(x)$$

holds for every $x \in \mathscr{M}$.

As a first step we consider the intersection of the two C*-subalgebras and two-sided ideals $\omega(\mathrm{K}^{(1)}_A(\mathscr{M}))$ and $\mathrm{K}^{(2)}_A(\mathscr{M})$ inside the unital C*-algebra $\mathrm{End}^{(2,*)}_A(\mathscr{M})$. The intersection of them is a C*-subalgebra and two-sided ideal of $\mathrm{End}^{(2,*)}_A(\mathscr{M})$ again. It contains the operators

$$\theta^{(2)}_{x,y} \cdot \omega(\theta^{(1)}_{z,t}) = \theta^{(2)}_{\omega(\theta^{(1)}_{z,t})^*(x),y} = \theta^{(2)}_{\omega(\theta^{(1)}_{t,z})(x),y}$$

for every $x, y, z, t \in \mathscr{M}$. Since the set of all finite linear combinations of special operators $\{\theta^{(1)}_{z,t} : z, t \in \mathscr{M}\}$ is norm-dense inside $\mathrm{K}^{(1)}_A(\mathscr{M})$ by definition the intersection of $\omega(\mathrm{K}^{(1)}_A(\mathscr{M}))$ and $\mathrm{K}^{(2)}_A(\mathscr{M})$ contains the set

$$\{\theta^{(2)}_{\omega(K^{(1)})(x),y} : K^{(1)} \in \mathrm{K}^{(1)}_A(\mathscr{M}), x, y \in \mathscr{M}\}.$$

Because of the limit equality

$$\begin{aligned} x &= \|.\|_{\mathscr{M}} - \lim_{n \to \infty} \omega(\theta^{(1)}_{x,x}(\theta^{(1)}_{x,x} + n^{-1})^{-1})(x) \\ &= \|.\|_{\mathscr{M}} - \lim_{n \to \infty} \omega(\theta^{(1)}_{x,x})\omega((\theta_{x,x}(1) + n^{-1})^{-1})(x) \end{aligned}$$

the set $\{\omega(K^{(1)})(x) : K^{(1)} \in \mathrm{K}^{(1)}_A(\mathscr{M}), x \in \mathscr{M}\}$ is norm-dense inside $\mathscr{M}$. Consequently, the intersection of $\omega(\mathrm{K}^{(1)}_A(\mathscr{M}))$ and $\mathrm{K}^{(2)}_A(\mathscr{M})$ inside the unital C*-algebra $\mathrm{End}^{(2,*)}_A(\mathscr{M})$ contains the set of "compact" operators $\{\theta^{(2)}_{x,y} : x, y \in \mathscr{M}\}$ generating one of the intersecting sets, $\mathrm{K}^{(2)}_A(\mathscr{M})$, completely, and the inclusion relation $\mathrm{K}^{(2)}_A(\mathscr{M}) \subseteq \omega(\mathrm{K}^{(1)}_A(\mathscr{M}))$ holds.

Secondly, by the symmetry of the situation and of the arguments the inclusion relation $\mathrm{K}^{(1)}_A(\mathscr{M}) \subseteq \omega^{-1}(\mathrm{K}^{(2)}_A(\mathscr{M}))$ holds, too, inside the unital C*-algebra $\mathrm{End}^{(1,*)}_A(\mathscr{M})$. Both inclusions together prove that ω realizes a $*$-isomorphism of the C*-algebras $\mathrm{K}^{(1)}_A(\mathscr{M})$ and $\mathrm{K}^{(2)}_A(\mathscr{M})$ automatically, what implies (iii) and hence, (i).

Whether the $*$-isomorphism of the C*-algebras of "compact" bounded A-linear operators of two different Hilbert A-modules $\mathscr{M}$ and $\mathscr{N}$ over some C*-algebras A implies their isomorphism as Hilbert C*-modules, or not? The answer is negative, even in the quite well-behaved cases. Counterexamples appear because of nontrivial K_0-groups of A, for instance. Let A be the hyperfinite type II_1 W*-factor. Set $\mathscr{M} = A$ and $\mathscr{N} = A^2$ with the usual A-valued inner products. Both these Hilbert A-modules are self-dual and finitely generated. Obviously, $\mathrm{K}_A(\mathscr{M})$ and $\mathrm{K}_A(\mathscr{N})$ are $*$-isomorphic to A as C*-algebras. Nevertheless, $\mathscr{M}$ and $\mathscr{N}$ are not isomorphic as Banach A-modules because of the non-existence of non-unitary isometries for the identity

caused by the existence of a faithful trace functional on A. The K_0-group of A equals R, i. e., it is non-trivial, and $A \cong A \otimes M_2(C)$.
In general, one could search for some special unital C*-algebra A with non-trivial K_0-group, a natural number $n \geq 1$ and two projections $p, q \in M_n(A)$ such that for every $N \geq n$ the finitely generated Hilbert A-modules $A^N p$ and $A^N q$ are non-isomorphic (i. e., $[p] \neq [q] \in K_0(A)$), but the C*-algebras $pM_n(A)p$ and $qM_n(A)q$ are $*$-isomorphic.

Closing, we pose the problem whether for a Banach C*-module $\mathscr{M}$ over a C*-algebra A carrying two A-valued inner products $\langle .,. \rangle_1$, $\langle .,. \rangle_2$ which induce norms on $\mathscr{M}$ equivalent to the given one the appropriate Banach algebras of all (not necessarily adjointable) bounded A-linear operators on $\mathscr{M}$ are *isometrically* multiplicatively isomorphic, or not, especially in the case of non-isomorphic Hilbert structures. Those properties of all these kinds of operator algebras which are preserved switching from one A-valued inner product on $\mathscr{M}$ to another have to be investigated in the future extending results for the "compact" case of [4,6].

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