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APPROXIMATION AND CONVEX DECOMPOSITION BY EXTREMALS IN A C*-ALGEBRA

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0. Introduction.

For a unital C^*-algebra $\mathfrak{A}$ with unit ball $\mathfrak{A}_1$ the set $\mathfrak{E}$ of extreme points in $\mathfrak{A}_1$ consists of elements V in $\mathfrak{A}_1$, necessarily partial isometries, such that

$$(I - VV^*)\mathfrak{A}(I - V^*V) = 0,$$

cf. [15] or [21, 1.4.7]. In [8] we defined the set of quasi-invertible elements in $\mathfrak{A}$ as

$$\mathfrak{A}_q^{-1} = \mathfrak{A}^{-1}\mathfrak{E}\mathfrak{A}^{-1}$$

and showed that this is an appropriate concept for infinite C^*-algebras, replacing to a certain extent the group $\mathfrak{A}^{-1}$ of invertible elements. In particular we studied the class of extremally rich C^*-algebras, i.e. C^*-algebras $\mathfrak{A}$ for which $\mathfrak{A}_q^{-1}$ is dense, as an analogue of Rieffel's stable rank one algebras.
In this paper we use the distance function

$$\alpha_q(T) = \text{dist}\,(T, \mathfrak{A}_q^{-1}), \qquad T \in \mathfrak{A},$$

to extend the theory (from [16], [19] and [30]) of regular approximation and unitary decomposition, where now quasi-invertibles and extreme points replace invertibles and unitaries. In particular we complete the study of the λ-function in C^*-algebras, begun in [23].

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1. Quasi–Invertibility.

For the convenience of the reader we recall here the main facts from [8] about the set $\mathfrak{A}_q^{-1}$ of quasi-invertible elements. For this we need the function m on $\mathfrak{A}$, defined by

$$m(T) = \inf\{\|Tx\| \mid x \in \mathfrak{H},\ \ \|x\| = 1\}$$
$$= m(|T|) = \sup\{\varepsilon \geq 0 \mid \varepsilon I \leq |T|\},$$

where $\mathfrak{A} \subset \mathrm{B}(\mathfrak{H})$ is any faithful non-degenerate representation of $\mathfrak{A}$.

1.1. THEOREM ([8, Theorem 1.1]). *For an element T in $\mathfrak{A}$ the following conditions are equivalent:*

(i) $T \in \mathfrak{A}^{-1}\mathfrak{E}\mathfrak{A}^{-1}$.

(ii) *There is an orthogonal pair of closed ideals $\mathscr{I}$, $\mathscr{J}$ of $\mathfrak{A}$, such that $T + \mathscr{I}$ is left invertible in $\mathfrak{A}/\mathscr{I}$ and $T + \mathscr{J}$ is right invertible in $\mathfrak{A}/\mathscr{J}$.*

(iii) *There is an orthogonal pair of closed ideals $\mathscr{I}$, $\mathscr{J}$ of $\mathfrak{A}$ and $\varepsilon > 0$, such that $m(T + \mathscr{I}) \geq \varepsilon$ and $m(T^* + \mathscr{J}) \geq \varepsilon$ in $\mathfrak{A}/\mathscr{I}$ and $\mathfrak{A}/\mathscr{J}$, respectively.*

(iv) *There is an $\varepsilon > 0$ such that*

$$m(\pi(T)) \vee m(\pi(T^*)) \geq \varepsilon$$

for every irreducible representation $(\pi, \mathfrak{H})$ of $\mathfrak{A}$.

(v) *T (thus also T^*) has closed range, and the kernel projections of T and T^* are centrally orthogonal in $\mathfrak{A}$.*

(vi) *If $T = V|T|$ is the polar decomposition of T then $V \in \mathfrak{E}$ and 0 is an isolated point in* $\mathrm{sp}(|T|)$.

(vii) $T \in \mathfrak{E}\mathfrak{A}_+^{-1}$.

The somewhat simpler condition

(viii) *For each irreducible representation $(\pi, \mathfrak{H})$ of $\mathfrak{A}$, the operator $\pi(T)$ is either left or right invertible,*

is equivalent to (i)–(vii) in Theorem 1.1 if the primitive ideal space of $\mathfrak{A}$ is a Hausdorff space, [8, Prop. 1.2], but not in general, [8, Ex. 1.3]. Nevertheless we have

1.2. PROPOSITION. *If $T \in (\mathfrak{A}^{-1})^= \setminus \mathfrak{A}^{-1}$, then for some irreducible representation $(\pi, \mathfrak{H})$ of $\mathfrak{A}$, the operator $\pi(T)$ is neither left nor right invertible. In particular,*

$$\partial(\mathfrak{A}^{-1}) \subset \partial(\mathfrak{A}_q^{-1}).$$

PROOF. If $T \notin \mathfrak{A}^{-1}$ then either $T^*T \notin \mathfrak{A}^{-1}$ or $TT^* \notin \mathfrak{A}^{-1}$. In the first case there is a pure state φ of $\mathfrak{A}$ with $\varphi(T^*T) = 0$ (cf. [21, 4.3.10]), which implies that T is not invertible in the irreducible representation $(\pi, \mathfrak{H})$ associated with φ. In the second case we have $\varphi(TT^*) = 0$ for some φ, and then $\pi(T^*)$ is not invertible in $\mathsf{B}(\mathfrak{H})$; but then neither is $\pi(T)$.

Now choose T_n in $\mathfrak{A}^{-1}$ such that $T_n \to T$, and define $E_n = \pi(T_n^{-1})\|\pi(T_n^{-1})\|^{-1}$. Thus all the E_n's belong to the unit sphere of $\pi(\mathfrak{A})$, and since $\pi(T)$ is not invertible we must have $E_n\pi(T) \to 0$ and $\pi(T)E_n \to 0$. But then $\pi(T)$, being a topological zero-divisor, can be neither left nor right invertible in $\mathsf{B}(\mathfrak{H})$.

It follows from condition (iv) in Theorem 1.1 that $T \notin \mathfrak{A}_q^{-1}$, but since $\mathfrak{A}^{-1} \subset \mathfrak{A}_q^{-1}$ we can conclude that $T \in (\mathfrak{A}_q^{-1})^= \setminus \mathfrak{A}_q^{-1}$, as claimed.

Two functions, m_q and α_q, will be important for the following results. If $T \in \mathfrak{A}_q^{-1}$ we define

$$\begin{aligned} m_q(T) &= \max\{\varepsilon > 0 \mid]0, \varepsilon[\cap \operatorname{sp}(|T|) = \varnothing\} \\ &= \inf\{m(\pi(T)) \vee m(\pi(T^*)) \mid (\pi, \mathfrak{H}) \text{ irreducible}\}, \end{aligned}$$

cf. (vi) and (iv) in Theorem 1.1. If $T \notin \mathfrak{A}_q^{-1}$ we set $m_q(T) = 0$. Most of the useful properties of m_q follow from the fact, [8, Prop. 1.5], that

$$m_q(T) = \text{dist } (T, \mathfrak{A} \setminus \mathfrak{A}_q^{-1}).$$

The other function, α_q, is defined straightforwardly as a distance function:

$$\alpha_q(T) = \text{dist } (T, \mathfrak{A}_q^{-1}).$$

Thus the two functions both measure the distance of an element to the boundary $\partial(\mathfrak{A}_q^{-1})$ of $\mathfrak{A}_q^{-1}$ – one from the inside, the other from the outside – and in many formulae $m_q(T)$ replaces $-\alpha_q(T)$, when T moves from $\mathfrak{A} \setminus \mathfrak{A}_q^{-1}$ into $\mathfrak{A}_q^{-1}$. Striking examples are Theorems 2.3 & 3.7.

If $T \in \mathfrak{A}_q^{-1}$ we define the *canonical quasi-inverse* as the unique element T^{-1} in $\mathfrak{A}$, such that $T^{-1}T$ and TT^{-1} are the projections on the support and the range of T, respectively. Thus, in particular, $TT^{-1}T = T$ and $T^{-1}TT^{-1} = T^{-1}$; equations that are also sometimes linked with the terminology quasi-inverse, although our notion is much stronger. If $T = V|T|$ is the polar decomposition of T, so that $V \in \mathfrak{E}$, then

$$T^{-1} = (|T| + I - V^*V)^{-1}V^* = V^*(|T^*| + I - VV^*)^{-1},$$

with "honest" inverses in the last two expressions, cf. condition (vii) in Theorem 1.1. Note that with this definition $T^{-1} \in \mathfrak{A}_q^{-1}$ with $(T^{-1})^{-1} = T$ and $(T^*)^{-1} = (T^{-1})^*$.

It is well known that a left invertible operator may have several left inverses. Indeed, the uniqueness of the (two-sided) inverse of an invertible elements rests on the fact that each left inverse must equal every right inverse. We do not, therefore, expect elements in $\mathfrak{A}_q^{-1}$ to have unique quasi-inverses. Since $\mathfrak{A}_q^{-1}$ is not a group we can also not expect formulae like

$$(*) \qquad (TS)^{-1} = S^{-1}T^{-1}$$

for the canonical quasi-inverses. Even in the case where, say, $S \in \mathfrak{A}_q^{-1}$ and $T \in \mathfrak{A}^{-1}$ (so that $TS \in \mathfrak{A}_q^{-1}$), the formula $(*)$ need not hold. Take e.g. S to be the unilateral shift on ℓ^2 in the C^*-algebra $\mathfrak{A} = \mathrm{B}(\ell^2)$; and let T be a positive, invertible operator, not commuting with SS^*. Then for quasi-inverses we have $S^{-1} = S^*$, but

$$(TS)^{-1} = (S^*T^2S)^{-1}S^*T \neq S^*T^{-1}.$$

With the same operators, form $R = ST$. Then $R^2 \in \mathfrak{A}_q^{-1}$, but $(R^2)^{-1} \neq (R^{-1})^2$.

In topological respects, however, the canonical quasi-inverse behaves nicely: If $T_n \to T$ inside $\mathfrak{A}_q^{-1}$, then $T_n^{-1} \to T^{-1}$.

Although, as mentioned, $\mathfrak{A}_q^{-1}$ is not a group, it is power stable:

1.3. Proposition. *If $V \in \mathfrak{E}$, then $V^n \in \mathfrak{E}$ for every n in $\mathbf{N}$. Similarly, if $T \in \mathfrak{A}_q^{-1}$, then $T^n \in \mathfrak{A}_q^{-1}$ with $m_q(T^n) \geq (m_q(T))^n$.*

Proof. If $V \in \mathfrak{E}$ and $W = V^n$ then since

$$I - W^*W = \sum_{k=0}^{n-1} V^{k*}(I - V^*V)V^k$$

(and similarly for WW^*), it follows that

$$(I - VV^*)\mathfrak{A}(I - V^*V) = 0 \Rightarrow (I - WW^*)\mathfrak{A}(I - W^*W) = 0,$$

so that $W \in \mathfrak{E}$. If $T \in \mathfrak{A}_q^{-1}$ with $\varepsilon = m_q(T)$, then for every irreducible representation $(\pi, \mathfrak{H})$ of $\mathfrak{A}$ we have

$$\begin{aligned} m(\pi(T^n)) &\geq m(\pi(T))m(\pi(T^{n-1})) \geq \dots \\ &\geq (m(\pi(T)))^n, \end{aligned}$$

and similarly for T^{*n}. By Theorem 1.1 (iv) it follows that $T^n \in \mathfrak{A}_q^{-1}$ with $m_q(T^n) \geq \varepsilon^n$.

For each element T in $\mathfrak{A}$ we define the *quasi-spectrum* of T as

$$\mathrm{qsp}(T) = \{\lambda \in \mathbf{C} \mid \lambda I - T \notin \mathfrak{A}_q^{-1}\}.$$

Since $\mathfrak{A}_q^{-1}$ is an open set containing $\mathfrak{A}^{-1}$, it is clear that $\mathrm{qsp}(T)$ is a closed subset of $\mathbf{C}$ contained in the ordinary spectrum, $\mathrm{sp}(T)$, of T.

1.4. THEOREM. *For every T in $\mathfrak{A}$ the quasi-spectrum* $\mathrm{qsp}(T)$ *is a non-empty compact subset of* $\mathrm{sp}(T)$ *containing the boundary of* $\mathrm{sp}(T)$ *in its boundary. Moreover,* $\mathrm{qsp}(T^*) = \overline{\mathrm{qsp}(T)}$, *and if f is a complex function, holomorphic on some open neighbourhood of* $\mathrm{sp}(T)$, *then*

$$f(\mathrm{qsp}(T)) \subset \mathrm{qsp}(f(T)). \qquad (*)$$

PROOF. If $\lambda \in \partial(\mathrm{sp}(T))$ then $\lambda I - T \in \partial(\mathfrak{A}^{-1})$, whence $\lambda I - T \in \partial(\mathfrak{A}_q^{-1})$ by Proposition 1.2, so that $\lambda \in \mathrm{qsp}(T)$. Of the remaining claims, only the holomorphic function calculus is non-obvious.

If $f(\lambda) \notin \mathrm{qsp}(f(T))$ for some λ in $\mathbf{C}$, there is by Theorem 1.1 a pair $\mathscr{I}$, $\mathscr{J}$ of closed, orthogonal ideals of $\mathfrak{A}$, such that if A denotes the canonical quasi-inverse of $f(\lambda)I - f(T)$, then

$$A(f(\lambda) - f(T)) - I \in \mathscr{I}, \ (f(\lambda) - f(T))A - I \in \mathscr{J}.$$

Let Γ denote a suitable choice of a finite number of (smooth or piecewise-linear) closed, oriented curves in the domain of f, encircling $\mathrm{sp}(T)$, so that

$$f(T) = (2\pi i)^{-1} \int_\Gamma (zI - T)^{-1} f(z) dz.$$

Note that we may assume that λ lies off Γ, since otherwise

$$\lambda \in \mathbf{C} \setminus \mathrm{sp}(T) \subset \mathbf{C} \setminus \mathrm{qsp}(T).$$

Define therefore the element

$$B = (2\pi i)^{-1} \int_\Gamma (zI - T)^{-1} (z - \lambda)^{-1} f(z) dz$$

and check that it satisfies the equations

$$(\lambda - T)B = B(\lambda - T) = f(\lambda)I - f(T).$$

It follows that

$$(\lambda - T)BA - I \in \mathscr{J},$$
$$AB(\lambda - T) - I \in \mathscr{I},$$

so that $\lambda I - T \in \mathfrak{A}_q^{-1}$ by Theorem 1.1 (ii), whence $\lambda \notin \mathrm{qsp}(T)$.

1.5. REMARKS. We can not, in general, expect the spectral formula, i.e. equality in the formula $(*)$ above, to hold for the quasi-spectrum. To see this, let S be a non-unitary isometry, e.g. the unilateral shift on ℓ^2, and define

$$T = \begin{pmatrix} S - I & 0 \\ 0 & I - S^* \end{pmatrix} \quad \text{in} \quad \mathfrak{A} = \mathsf{B}(\ell^2 \oplus \ell^2)$$

(or a suitably large separable C^*-subalgebra, like the C^*-algebra generated by T and the compact operators on $\ell^2 \oplus \ell^2$). We have

$$I - T = \begin{pmatrix} 2I - S & 0 \\ 0 & S^* \end{pmatrix}, \quad I + T = \begin{pmatrix} S & 0 \\ 0 & 2I - S^* \end{pmatrix},$$

$$I - T^2 = \begin{pmatrix} S & 0 \\ 0 & S^* \end{pmatrix} \begin{pmatrix} 2I - S & 0 \\ 0 & 2I - S^* \end{pmatrix}.$$

Both $2I - S$ and $2I - S^*$ are invertible (in $\mathsf{B}(\ell^2)$), so $I - T$ is right invertible and $I + T$ is left invertible in $\mathfrak{A}$. However, $I - T^2$ is neither left nor right invertible in $\mathfrak{A}$, because this is so for the first factor, while the second is invertible. It follows that even with $f(z) = z^2$ we can have $f(\mathrm{qsp}(T)) \neq \mathrm{qsp}(f(T))$.

If T is invertible, so is T^{-1}, and by Theorem 1.4 we see that

$$(*) \qquad \mathrm{qsp}(T^{-1}) = \{\lambda^{-1} \mid \lambda \in \mathrm{qsp}(T)\}.$$

However, if T is only quasi-invertible, and if T^{-1} denotes the canonical quasi-inverse, then the formula $(*)$ may fail. It suffices to set

$$T = \begin{pmatrix} S & 0 \\ 0 & 2I - S \end{pmatrix} \quad \text{in} \quad \mathfrak{A} = \mathsf{B}(\ell^2 \oplus \ell^2),$$

where S as above denotes the unilateral shift. Here the canonical quasi-inverses of T and of $2I - T$ are

$$T^{-1} = \begin{pmatrix} S^* & 0 \\ 0 & (2I - S)^{-1} \end{pmatrix}, \quad (2I - T)^{-1} = \begin{pmatrix} (2I - S)^{-1} & 0 \\ 0 & S^* \end{pmatrix},$$

so that $2 \notin \mathrm{qsp}(T)$. However, $2^{-1} \in \mathrm{qsp}(T^{-1})$, because

$$\tfrac{1}{2}I - T^{-1} = \begin{pmatrix} \tfrac{1}{2}I - S^* & 0 \\ 0 & -\tfrac{1}{4}S \sum_{n=0}^{\infty} (\tfrac{1}{2}S)^n \end{pmatrix},$$

which is neither left nor right invertible.

Since a holomorphic function is an open map we see from the (ordinary) spectral mapping theorem that $\partial(\mathrm{sp}(f(T))) \subset f(\partial(\mathrm{sp}(T)))$. It follows from Theorem 1.4 that

$$\partial(\mathrm{sp}(f(T))) \subset f(\partial(\mathrm{qsp}(T))),$$

and it can be shown, using [8, Theorem 1.8] that

$$\partial(\mathrm{qsp}(f(T))) \subset f(\partial\,(\mathrm{qsp}(T))).$$

1.6. PROPOSITION. *For S and T in $\mathfrak{A}$ we have*

$$\operatorname{qsp}(ST) \setminus \{0\} = \operatorname{qsp}(TS) \setminus \{0\}.$$

PROOF. Assume that $\lambda \neq 0$ and $\lambda \notin \operatorname{qsp}(ST)$. Thus $\lambda I - ST = A \in \mathfrak{A}_q^{-1}$, and we can define the canonical quasi-inverse A^{-1}. Put

$$B = \lambda^{-1}(I + TA^{-1}S).$$

Let $\mathscr{I}$ and $\mathscr{J}$ be the closed orthogonal ideals modulo which A is left and right invertible, respectively. Direct computation shows that

$$B(\lambda - TS) - I \in \mathscr{I}, \; (\lambda - TS)B - I \in \mathscr{J},$$

whence $\lambda I - TS \in \mathfrak{A}_q^{-1}$ with $m_q(\lambda I - TS) \geq \|B\|^{-1}$. Thus $\lambda \notin \operatorname{qsp}(TS)$. It follows that $\operatorname{qsp}(TS) \setminus \{0\} \subset \operatorname{qsp}(ST) \setminus \{0\}$, and by symmetry we get equality.

1.7. PROPOSITION. *If $V \in \mathfrak{E}$, but V is not unitary, then its quasi-spectrum is the unit circle, whereas its spectrum is the whole unit disk.*

PROOF. Suppose that $\lambda \notin \operatorname{sp}(V)$, but $|\lambda| < 1$. Thus $V - \lambda I = A \in \mathfrak{A}^{-1}$. If V is an isometry, this gives $V^*A = I - \lambda V^* \in \mathfrak{A}^{-1}$, since $\|\lambda V^*\| < 1$. Consequently V^* and V are invertible, i.e. V is unitary. Similarly V becomes unitary if it is a co-isometry, because now $AV^* = I - \lambda V^*$. Since these arguments hold also for quotient images of V, we conclude that $\pi(V)$ is unitary in every irreducible representation $(\pi, \mathfrak{H})$ of $\mathfrak{A}$. But then V itself is unitary, contradicting our original assumption. It follows that $\lambda \in \operatorname{sp}(V)$ for all λ with $|\lambda| < 1$, hence for the whole unit disk.

From Theorem 1.4 we see that the quasi-spectrum of V is contained in the unit disk and contains the unit circle – its boundary. But if $|\lambda| < 1$ then $V - \lambda I \in \mathfrak{A}_q^{-1}$, because $m_q(V) = 1$. Thus $\operatorname{qsp}(V)$ is precisely the unit circle.

1.8. LEMMA. *Every normal element in $\mathfrak{A}_q^{-1}$ is invertible.*

PROOF. If $T \in \mathfrak{A}_q^{-1}$ with $T^*T = TT^*$, then $|T| = |T^*|$, and by condition (v) in Theorem 1.1 we see that T must be invertible.

1.9. COROLLARY. *If T is normal in $\mathfrak{A}$, then*

$$\operatorname{qsp}(T) = \operatorname{sp}(T).$$

2. Distance to the Extremals.

In this section and the next we show how the theory of regular approximation and unitary decomposition for elements in a unital C^*-algebra $\mathfrak{A}$ extends. Surprisingly enough, every single result from the old theory has an exact analogue, where $\mathfrak{E}$ and $\mathfrak{A}_q^{-1}$ replace $\mathfrak{U}$ (the unitary group) and $\mathfrak{A}^{-1}$.

The first results generalize a theorem proved by Olsen [18, 2.3], to the effect that the spectrum of each product U^*T, with T in $\mathfrak{A}$ and U in $\mathfrak{U}$, contains a solid disk $\mathsf{D}(r)$ in C with center 0 and radius r not smaller than dist $(T, \mathfrak{A}^{-1})$. Actually, two generalizations are possible, each using the fact that $\mathfrak{U}\mathfrak{E}\mathfrak{U} = \mathfrak{E}$. The first, whose easy proof is left to the reader, uses the quasi-spectrum but retains the unitary group, and states that if $T \notin \mathfrak{A}_q^{-1}$ then

$$\mathsf{D}(\alpha_q(T)) \subset \operatorname{qsp}(U^*T) = \operatorname{qsp}(TU^*)$$

for every U in $\mathfrak{U}$. The second is slightly more delicate.

2.1. Lemma. *If $T \in \mathfrak{A} \setminus \mathfrak{A}_q^{-1}$ and $W \in \mathfrak{E}$, then*

$$\mathsf{D}(\alpha_q(T)) \subset \operatorname{sp}(W^*T) = \operatorname{sp}(TW^*).$$

Proof. If $W^*T \in \mathfrak{A}^{-1}$ then, in particular, T is left invertible, in contradiction with $T \notin \mathfrak{A}_q^{-1}$. Similarly, $TW^* \in \mathfrak{A}^{-1}$ would make T right invertible, which is also a contradiction. Thus $0 \in \operatorname{sp}(W^*T)$ and $0 \in \operatorname{sp}(TW^*)$.

Assume now that $\lambda \neq 0$ and $\lambda \notin \operatorname{sp}(W^*T)$. Then $\lambda \notin \operatorname{sp}(TW^*)$, as well. Thus $G = \lambda I - W^*T$ and $H = \lambda I - TW^*$ are invertible elements in $\mathfrak{A}$, whence $m(G^*) \geq \varepsilon$ and $m(H) \geq \varepsilon$ for some $\varepsilon > 0$. Let $(\pi, \mathfrak{H})$ be an irreducible representation of $\mathfrak{A}$. If $\pi(W)$ is a co-isometry we calculate

$$\begin{aligned} m(\pi(\bar\lambda W^* - T^*)) &= m(\pi((\bar\lambda I - T^*W)W^*)) \\ &= m(\pi(G^*W^*)) \geq \varepsilon m(\pi(W^*)) = \varepsilon. \end{aligned}$$

If $\pi(W)$ is an isometry we similarly have

$$\begin{aligned} m(\pi(\lambda W - T)) &= m(\pi((\lambda I - TW^*)W)) \\ &= m(\pi(HW)) \geq \varepsilon m(\pi(W)) = \varepsilon. \end{aligned}$$

From condition (iv) in Theorem 1.1 it follows that $\lambda W - T \in \mathfrak{A}_q^{-1}$. Since $\|W\| = 1$ this immediately shows that $|\lambda| \geq \alpha_q(T)$. We conclude that $\lambda \in \operatorname{sp}(W^*T)$ whenever $|\lambda| \leq \alpha_q(T)$.

2.2. Proposition *If $T \in \mathfrak{A} \setminus \mathfrak{A}_q^{-1}$, then*

$$\bigcap_{W \in \mathfrak{E}} \operatorname{sp}(W^*T) = \mathsf{D}(\alpha_q(T)).$$

Proof. The inclusion $\supset$ follows from Lemma 2.1. To prove the other, note that since $\mathfrak{E}$ is rotation-invariant, so is the intersection of all spectra of W^*T, $W \in \mathfrak{E}$. Thus it suffices to take $\beta > \alpha_q(T)$ and show that $-\beta \notin \operatorname{sp}(W^*T)$ for some W in $\mathfrak{E}$. Toward this end, choose A in $\mathfrak{A}_q^{-1}$, of the form $A = W|A|$ with W in $\mathfrak{E}$, such that $\|T - A\| < \beta$. Then $W^*T + \beta I \in \mathfrak{A}^{-1}$ by [8, Proposition 1.7], as desired.

2.3. Theorem cf. [22, 10] & [30, 2.7]. *Consider an element* T *in* $\mathfrak{A}$. *If* $T \in \mathfrak{A}_q^{-1}$ *then*

$$\operatorname{dist}(T, \mathfrak{E}) = \max\{1 - m_q(T),\ \|T\| - 1\}.$$

If $T \notin \mathfrak{A}_q^{-1}$ then

$$\operatorname{dist}(T, \mathfrak{E}) = \max\{1 + \alpha_q(T),\ \|T\| - 1\}.$$

Proof. If $T \in \mathfrak{A}_q^{-1}$ it has a polar decomposition $T = V|T|$ with V in $\mathfrak{E}$. By elementary spectral theory we get

$$\|T - V\| = \||T| - V^*V\| = \max\{\|T\| - 1\,,\ 1 - m_q(T)\}.$$

In the converse direction we always have

$$\|T - W\| \geq \|T\| - 1$$

for any W in $\mathfrak{E}$. Moreover,

$$m_q(T) = m_q(W - (W - T)) \geq 1 - \|W - T\|,$$

whence also $\|T - W\| \geq 1 - m_q(T)$, completing the proof of the first case.

If $T \notin \mathfrak{A}_q^{-1}$ we still have $\|T - W\| \geq \|T\| - 1$ for every W in $\mathfrak{E}$. Moreover, if $\|T - W\| < \beta$ we first note that $\beta \geq 1$, because $T \notin \mathfrak{A}_q^{-1}$. Then, applying m_q as a distance function again, we calculate

$$m_q((\beta - 1)W + T) \geq m_q(\beta W) - \|W - T\| = \beta - \|W - T\| > 0,$$

whence $(\beta - 1)W + T \in \mathfrak{A}_q^{-1}$. But this implies that $\beta - 1 \geq \alpha_q(T)$, i.e. $\beta \geq 1 + \alpha_q(T)$. We have thus established the inequality

$$\max\{1 + \alpha_q(T)\,,\ \|T\| - 1\} \leq \operatorname{dist}(T, \mathfrak{E}).$$

To obtain the reverse inequality, take $\delta > \alpha_q(T)$. By [8, Theorem 2.2] there is an extreme partial isometry U in $\mathfrak{E}$, such that $UE_\delta = VE_\delta$, where $T = V|T|$ is the polar decomposition and E_δ denotes the spectral projection of $|T|$ corresponding to the interval $]\delta, \infty[$, whereas F_δ is the corresponding projection for $|T^*|$. Thus

$$\begin{aligned}\|T - U\| &= \|(V|T| - U)(E_\delta + I - E_\delta)\| \\ &= \|F_\delta V(|T| - I)E_\delta + (I - F_\delta)(V|T| - U)(I - E_\delta)\|,\end{aligned}$$

where we use also that $F_\delta U = F_\delta V$. The two summands in the last line have orthogonal supports and orthogonal ranges, so the norm of the sum is the maximum of the norms of the summands. Clearly

$$\|F_\delta V(|T| - I)E_\delta\| \leq (\|T\| - 1) \vee (1 - \delta).$$

For the other summand, note that $|T|(I - E_\delta) \le \delta I$. Thus

$$\|(I - F_\delta)(V|T| - U)(I - E_\delta)\| \le \delta + 1.$$

Combining the estimates we get

$$(*) \qquad \|T - U\| \le \max\{\|T\| - 1\,,\ 1 + \delta\},$$

and since δ is arbitrarily close to $\alpha_q(T)$, the desired inequality follows.

2.4. Corollary. *If* $\|T\| < 2$ *then*

$$\operatorname{dist}(T, \mathfrak{E}) < 1 \Leftrightarrow T \in \mathfrak{A}_q^{-1}.$$

2.5. Remark. From the proof of Theorem 2.3 it follows that when $T \in \mathfrak{A}_q^{-1}$ it has an extremal *approximant*, i.e. there is a V in $\mathfrak{E}$ (arising from the polar decomposition $T = V|T|$) such that

$$\|T - V\| = \operatorname{dist}(T, \mathfrak{E}).$$

When $T \notin \mathfrak{A}_q^{-1}$ there is, in general, no extremal approximant. The existence of approximants is closely related to the possibility of (extremal) polar decompositions. Thus, if $\mathfrak{A}$ is a von Neumann algebra, each T in $\mathfrak{A}$ has a decomposition $T = W|T|$ with W in $\mathfrak{E}$ (but probably $\ker W \neq \ker T$); and then W is an extremal approximant for T in $\mathfrak{E}$, see [23, Theorem 4.2].

The following result shows when one of the bounds in the formula for $\operatorname{dist}(T, \mathfrak{E})$ can be obtained. It will later play a key rôle for the main result in the next section, Theorem 3.3.

2.6. Proposition. *If* $T \in \mathfrak{A}$ *with* $\|T\| > \alpha_q(T) + 2$, *then* $\|T - U\| = \|T\| - 1$ for some U in $\mathfrak{E}$.

Proof. Choose δ such that $\|T\| - 2 \ge \delta > \alpha_q(T)$. With $T = V|T|$ the polar decomposition and E_δ, F_δ as in the proof of Theorem 2.3 we can, again by [8, Theorem 2.2] find U in $\mathfrak{E}$ such that $UE_\delta = F_\delta U = VE_\delta$. From the proof (see $(*)$) of Theorem 2.3 it follows that

$$\|T - U\| \le \max\{\|T\| - 1\,,\ 1 + \delta\} = \|T\| - 1.$$

By Theorem 2.3 we therefore have the equality $\|T - U\| = \|T\| - 1$.

3. Convex Decompositions and the λ-Function.

3.1. THEOREM. *If $T \in \mathfrak{A}_q^{-1}$, $\|T\| \leq 1$, and $\frac{1}{2} \leq \lambda \leq \frac{1}{2}(1 + m_q(T))$, there are extreme partial isometries U, W in $\mathfrak{E}$, such that*

$$T = \lambda U + (1 - \lambda) W.$$

If $\lambda > \frac{1}{2}(1 + m_q(T))$ we can not even have a convex combination

$$T = \lambda U + (1 - \lambda) B,$$

with U in $\mathfrak{E}$ and B in $\mathfrak{A}_1$.

PROOF. By Theorem 1.1 we have a polar decomposition $T = V|T|$ with V in $\mathfrak{E}$ and $]0, m_q(T)[\ \cap \mathrm{sp}(|T|) = \emptyset$. Thus with

$$H = |T| + I - V^* V$$

we have an invertible element with $\mathrm{sp}(H) \subset [m_q(T), 1]$. By [16, Lemma 6] there are unitaries U_1, U_2 in the C^*-algebra generated by H, such that $H = \lambda U_1 + (1 - \lambda) U_2$, whenever $\frac{1}{2} \leq \lambda \leq \frac{1}{2}(1 + m_q(T))$. Setting $U = VU_1$ and $W = VU_2$ we obtain the desired decomposition

$$T = \lambda U + (1 - \lambda) W.$$

If we had a convex combination $T = \lambda U + (1 - \lambda) B$ with U in $\mathfrak{E}$ and $\|B\| \leq 1$, then

$$\|T - U\| = (1 - \lambda)\|B - U\| \leq 2(1 - \lambda).$$

By Theorem 2.3 this means that

$$1 - m_q(T) \leq 2(1 - \lambda),$$

i.e. $\lambda \leq \frac{1}{2}(1 + m_q(T))$.

3.2. PROPOSITION. *If $W \in \mathfrak{E}$ and $B \in \mathfrak{A}_1$, then for all $\alpha, \beta, \gamma, \delta$ in R_+ with $\alpha > \beta$, $\alpha + \beta = \gamma + \delta$, $\gamma, \delta \in [\beta, \alpha]$, there are extreme partial isometries U, V in $\mathfrak{E}$ such that*

$$\alpha W + \beta B = \gamma U + \delta V.$$

PROOF. Since $m_q(W) = 1$ it follows that the element $T = \alpha W + \beta B \in \mathfrak{A}_q^{-1}$, with $m_q(T) \geq \alpha - \beta$. Thus, assuming that $\delta \leq \gamma$, we can apply Theorem 3.1 to $(\alpha + \beta)^{-1} T$, taking $\lambda = (\alpha + \beta)^{-1}\gamma$, which is legitimate since

$$\lambda = (\alpha + \beta)^{-1}\gamma \leq (\alpha + \beta)^{-1}\alpha = \tfrac{1}{2}(1 + (\alpha + \beta)^{-1}(\alpha - \beta)).$$

We get extreme partial isometries U, V in $\mathfrak{E}$ such that

$$(\alpha+\beta)^{-1}T = (\alpha+\beta)^{-1}\gamma U + (\alpha+\beta)^{-1}\delta V,$$

and multiplying with $\alpha+\beta$ $(=\gamma+\delta)$ this reads

$$\alpha W + \beta B = \gamma U + \delta V,$$

as desired.

3.3. THEOREM. *If $T \in \mathfrak{A}_1$ but $\alpha_q(T) < 1$, there are for each convex combination $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, such that $\lambda_k < \frac{1}{2}(1-\alpha_q(T))$ for all k, extreme partial isometries $U_1, U_2, \ldots, U_n$ in $\mathfrak{E}$ such that*

$$T = \lambda_1 U_1 + \lambda_2 U_2 + \cdots + \lambda_n U_n.$$

PROOF. Assume first that $\lambda_1 > \lambda_k$ for all $k \neq 1$, and put $\beta = \lambda_1^{-1}$. If $\|T\| < 1$ we have

$$\operatorname{dist}(\beta T, \mathfrak{E}) = \max\{\beta\|T\| - 1\,,\ \beta\alpha_q(T) + 1\} < \beta - 1$$

by Theorem 2.3, since $\beta(1-\alpha_q(T)) > 2$. We can therefore find W in $\mathfrak{E}$, such that $\|\beta T - W\| \leq \beta - 1$. If $\|T\| = 1$ we have

$$\|\beta T\| = \beta > \alpha_q(\beta T) + 2$$

for the same reason as above. By Proposition 2.6 we can therefore find W in $\mathfrak{E}$ such that $\|\beta T - W\| = \beta - 1$.

In any case we have W in $\mathfrak{E}$ with $\|\beta T - W\| \leq \beta - 1$, and we define $B = (\beta-1)^{-1}(\beta T - W)$. Then $\|B\| \leq 1$, and

$$\beta T = W + (\beta - 1)B = W + \beta(\lambda_2 + \cdots + \lambda_n)B.$$

Since $\beta\lambda_2 < 1$ we have

$$W + \beta\lambda_2 B = V_2 + \beta\lambda_2 U_2$$

for some V_2, U_2 in $\mathfrak{E}$ by Proposition 3.2. Repeating the argument we find V_3, U_3 in $\mathfrak{E}$ such that

$$V_2 + \beta\lambda_3 B = V_3 + \beta\lambda_3 U_3\ ;$$

and after $n-1$ steps we have found extreme partial isometries $U_2, U_3, \ldots, U_n$, U_1 in $\mathfrak{E}$ (where $U_1 = V_n$ in the induction argument), such that

$$\beta T = \beta\lambda_2 U_2 + \beta\lambda_3 U_3 + \cdots + \beta\lambda_n U_n + U_1.$$

Multiplying with λ_1 we get the desired expression.

In the general case we can, of course, assume that $\lambda_1 \geq \lambda_k$ for all k. Put $\lambda_1' = \lambda_1 + \varepsilon$ and $\lambda_2' = \lambda_2 - \varepsilon$ for ε sufficiently small, so that $0 < \lambda_2'$ and

$\lambda'_1 < \frac{1}{2}(1-\alpha_q(T))$. Then we can use the argument above to obtain an extremal decomposition

$$T = \lambda'_1 U'_1 + \lambda'_2 U'_2 + \lambda_3 U_3 + \cdots + \lambda_n U_n.$$

Now use Proposition 3.2 on the element $\lambda'_1 U'_1 + \lambda'_2 U'_2$ to obtain extreme partial isometries U_1, U_2 in $\mathfrak{E}$, such that

$$\lambda'_1 U'_1 + \lambda'_2 U'_2 = (\lambda_1 + \varepsilon) U'_1 + (\lambda_2 - \varepsilon) U'_2 = \lambda_1 U_1 + \lambda_2 U_2.$$

Inserting this in the decomposition above we have the desired expression.

3.4. Corollary. *If $T \in \mathfrak{A}_1$ and $\alpha_q(T) < 1$ take the natural number n such that*

$$n - 1 \leq 2(1-\alpha_q(T))^{-1} < n.$$

Then for any λ such that $n^{-1} \leq \lambda < \frac{1}{2}(1-\alpha_q(T))$, there are extreme partial isometries $U_1, U_2, \ldots, U_n$ in $\mathfrak{E}$, such that

$$T = \lambda U_1 + \lambda U_2 + \cdots + \lambda U_{n-1} + (1-(n-1)\lambda) U_n.$$

3.5. Proposition. *If $T \in \mathfrak{A}_1 \setminus \mathfrak{A}_q^{-1}$ and*

$$T = \lambda U + (1-\lambda) B$$

with $0 < \lambda < 1$, $U \in \mathfrak{E}$ and $B \in \mathfrak{A}_1$, then $\alpha_q(T) < 1$ and moreover $\lambda \leq \frac{1}{2}(1-\alpha_q(T))$. Thus if

$$T = \lambda_1 U_1 + \lambda_2 U_2 + \cdots + \lambda_n U_n$$

for some convex combination $(\lambda_1, \ldots, \lambda_n)$ and $U_1, \ldots, U_n$ in $\mathfrak{E}$, then necessarily $n \geq 2\,(1-\alpha_q(T))^{-1}$.

Proof. If $T = \lambda U + (1-\lambda)B$, then

$$\|T - U\| = (1-\lambda)\|B - U\| \leq 2(1-\lambda).$$

By Theorem 2.3 this means that

$$1 + \alpha_q(T) \leq 2(1-\lambda).$$

This proves the first half of the proposition. For the second, note that since $\lambda_k \leq \frac{1}{2}(1-\alpha_q(T))$ we get $1 \leq \frac{1}{2}(1-\alpha_q(T))n$, as claimed.

3.6. Proposition. *If $\mathfrak{A}$ is a C^*-algebra such that $\alpha_q(T) > 0$ for some T in $\mathfrak{A}$, then $\alpha_q(T_0) = 1$ for some T_0 in $\mathfrak{A}$ with $\|T_0\| = 1$. In particular, $T_0 \notin \mathrm{conv}\,(\mathfrak{E})$.*

Proof. We may as well assume that $\|T\| = 1$. Let $T = V|T|$ be the polar decomposition of T in $\mathfrak{A}''$ and put $T_0 = Vf(|T|)$, where $f(t) = \alpha_q(T)^{-1}t \wedge 1$. Note that if $\gamma < 1$, the spectral projection of $|T_0|$ corresponding to the interval $]\gamma, \infty[$ is precisely E_δ (relative to $|T|$) with $\delta = \alpha_q(T)\gamma$. If therefore $\alpha_q(T_0) < 1$, we could by [8, Theorem 2.2] find U in $\mathfrak{E}$ such that

$$VE_\delta = UE_\delta = F_\delta U$$

for some δ with $\alpha_q(T)^{-1}\delta < 1$. But this contradicts [8, Theorem 2.2], when applied to T. Thus $\alpha_q(T_0) = 1$.

For any Banach space $\mathfrak{A}$ with unit ball $\mathfrak{A}_1$ and $\mathfrak{E}$ the set of extreme points of the convex set $\mathfrak{A}_1$, consider for each T in $\mathfrak{A}_1$ the possible convex combinations

$$T = \lambda V + (1 - \lambda)B,$$

where $V \in \mathfrak{E}$ and $B \in \mathfrak{A}_1$. The supremum of all λ's appearing in such decompositions is denoted by $\lambda(T)$ by Aron and Lohman, and this defines the λ-*function* $\lambda : \mathfrak{A}_1 \to [0, 1]$, see [2]. The space $\mathfrak{A}$ is said to have the λ-*property* if $\lambda(T) > 0$ for each T in $\mathfrak{A}_1$, and $\mathfrak{A}$ has the *uniform* λ-*property* if $\lambda(T) \geq \varepsilon > 0$ for all T in $\mathfrak{A}_1$.

In [23] the second author began an investigation of the properties of the λ-function in C^*-algebras. A complete characterization was obtained for von Neumann algebras, finite C^*-algebras and prime C^*-algebras. With the aid of the class $\mathfrak{A}_q^{-1}$ of quasi-invertible elements the program can now be completed.

3.7. THEOREM. *The* λ-*function on the unit ball* $\mathfrak{A}_1$ *of a* C^*-*algebra* $\mathfrak{A}$ *is given by the following formulae*:

$$\lambda(T) = \tfrac{1}{2}(1 + m_q(T)) \qquad \text{if} \qquad T \in \mathfrak{A}_q^{-1}\ ;$$

$$\lambda(T) = \tfrac{1}{2}(1 - \alpha_q(T)) \qquad \text{if} \qquad T \notin \mathfrak{A}_q^{-1}.$$

In particular, $\mathfrak{A}$ *has the* λ-*property if and only if it has the uniform* λ-*property with* $\lambda(T) \geq \frac{1}{2}$ *for all* T *in* $\mathfrak{A}_1$, *whence*

$$T = \tfrac{1}{2}(1 - \varepsilon)U_1 + \tfrac{1}{2}(1 - \varepsilon)U_2 + \varepsilon U_3$$

for any given $\varepsilon > 0$ $(\varepsilon \leq \frac{1}{3})$ *and some* U_1, U_2, U_3 *in* $\mathfrak{E}$.

PROOF. The formulae for the λ-function follow from Theorem 3.1, Theorem 3.3 and Proposition 3.5. The last statement follows from Proposition 3.6 followed by an application with $n = 3$ of Corollary 3.4.

3.8. COROLLARY. *If* $T \in \mathfrak{A}_1$ *then*

$$T \in \mathfrak{A}_q^{-1} \Leftrightarrow \lambda(T) > \tfrac{1}{2}.$$

3.9. REMARK. If we replace $\mathfrak{A}$ with $P\mathfrak{A}Q$ for some projections P and Q in $\mathfrak{A}$, we obtain a Banach space whose unit ball has extreme points U, characterized (by Sakai, cf. [21, 1.4.8]) as being partial isometries such that

$$(P - UU^*)\mathfrak{A}(Q - U^*U) = 0.$$

Even though $\mathfrak{A}$ is unital, the set $\mathfrak{E}(P\mathfrak{A}Q)$ of extreme points in the unit ball of $P\mathfrak{A}Q$ may be empty. Assuming, however, that $\mathfrak{E}(P\mathfrak{A}Q) \neq \emptyset$, the results from Sections 1, 2 and 3 carry over with minor modifications. Thus, if $T \in P\mathfrak{A}Q$ with polar decomposition $T = V|T|$, then the spectral resolutions $\{E_\delta\}$ and $\{F_\delta\}$ for $|T|$ and $|T^*|$, used in the proof of Theorem 2.3, lie in the enveloping von Neumann algebras for $Q\mathfrak{A}Q$ and $P\mathfrak{A}P$, respectively. The statements in Propositions 2.1 and 2.2 remain true, even though $P\mathfrak{A}Q$ is not a *-algebra, because $W^*T \in Q\mathfrak{A}Q$ and $TU^* \in P\mathfrak{A}P$. (This is the reason why the formulae were stated with adjoints.) Of course, the distance functions m_q and α_q are defined on $P\mathfrak{A}Q$ relative to the set of quasi-invertible elements

$$(P\mathfrak{A}Q)_q^{-1} = (P\mathfrak{A}P)^{-1}\mathfrak{E}(P\mathfrak{A}Q)(Q\mathfrak{A}Q)^{-1}.$$

With this modification all the remaining results from Sections 2 and 3 carry over verbatim. As a sample we obtain a Russo-Dye theorem for such spaces (i.e. also for Hilbert bimodules, see the discussion in [8, §4]).

3.10. PROPOSITION. *Let P and Q be projections in a C^*-algebra $\mathfrak{A}$ such that $P\mathfrak{A}_1Q$ contains an extreme point. Then the open unit ball of $P\mathfrak{A}Q$ is contained in* conv $(\mathfrak{E}(P\mathfrak{A}Q))$.

PROOF. Direct application of Theorem 3.3, because $\alpha_q(T) \leq \|T\| < 1$ for every T in the open unit ball of $P\mathfrak{A}Q$.

3.11. REMARK. If $\mathfrak{A}$ is a unital C^*-algebra and $U_0 \in \mathfrak{E}$ with defect projections $P_0 = I - U_0U_0^*$ and $Q_0 = I - U_0^*U_0$, then every element T in the open unit ball of $\mathfrak{A}$ has a convex decomposition

$$T = \sum \lambda_i U_i$$

with U_i in $\mathfrak{E}$, such that $I - U_iU_i^* \sim P_0$ and $I - U_i^*U_i \sim Q_0$ for all i (Murray-von Neumann equivalence). This can be proved by examining the key proofs, notably those of theorems [8, 2.2], 2.3 and 3.3, together with elementary arguments well-known in semi-Fredholm index theory. Obviously the same conclusion does *not* hold if $\|T\| = 1$, even if $\mathfrak{A}_q^{-1}$ is dense in $\mathfrak{A}$ ($\mathfrak{A}$ is extremally rich). As with the main results of Sections 1, 2 and 3, this observation is also valid for spaces of the form $P\mathfrak{A}Q$, where now $P_0 = P - U_0U_0^*$ and $Q_0 = Q - U_0^*U_0$, cf. Remark 3.9.

As a possible generalization of our theory, we could therefore put special restrictions on the equivalence classes of the defect projections (or on the ideals they generate) of the extreme partial isometries or on the quasi-invertibles, and define a modified distance function α_* (as long as there is at least one extremal partial isometry satisfying the restrictions). All of the theory will go through with α_* substituted for α_q. Of course, if U is an extreme partial isometry not satisfying the restrictions we will get $\alpha_*(U) = 1$.

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