MATH. SCAND. 81 (1997), 86–100

TOEPLITZ ALGEBRAS AND INFINITE SIMPLE C*-ALGEBRAS ASSOCIATED WITH REDUCED GROUP C*-ALGEBRAS

SHUANG ZHANG*

Abstract

Assume that Γ is the free product of an arbitrary number of finite cyclic groups and any free group, and the generators of Γ are $g_1, g_2, \ldots$. Let Γ_+ consist of the unit e of Γ and all those reduced words of the form $g_{i_1}^{n_1} g_{i_2}^{n_2} \ldots g_{i_k}^{n_k}$ where $n_1, n_2, \ldots, n_k$ are positive integers, and let R_+ be the projection onto the subspace $l^2(\Gamma_+)$ of $l^2(\Gamma)$. We prove that the C*-algebra $C_r^*(\Gamma, R_+)$ generated by the reduced group C*-algebra $C_r^*\Gamma$ and R_+ has either one or two non-trivial closed ideals which are stable and of real rank zero. This construction results some purely infinite simple C*-algebras.

1. Introduction.

Let $\mathcal{F}_n$ be the free group on n generators $(1 \le n \le +\infty)$, i.e., the free product $\underbrace{Z * Z * \ldots * Z * \ldots}_{n}$ of n copies of the group Z of all integers, and let Γ_t be the free product of some finite cyclic groups $Z_{n_i} := Z/n_iZ$, i.e.,

$$\Gamma_t(m) := \underbrace{Z_{n_1} * Z_{n_2} * \ldots * Z_{n_k} * \ldots}_{m}, \quad \text{where} \quad 2 \le n_i < +\infty, 2 \le m \le \infty.$$

The groups considered in this article are the free product

$$\Gamma := \Gamma_t * \mathcal{F}_n, \quad \text{or} \quad \mathcal{F}_\infty,$$

The unit of Γ is denoted by e, and the generators of Γ are denoted by $\{g_1, g_2, \ldots, g_k, \ldots\}$. Each element of Γ is a *reduced* word $w := g_{i_1}^{m_1} g_{i_2}^{m_2} \ldots g_{i_k}^{m_k}$ of finite length $l(w) := \sum_{j=1}^{k} |m_j|$; the word "reduced" means that all factors of the forms gg^{-1} and $g^{-1}g$ are canceled out. For any finite cyclic group Z_{n_i} we make a convention that each element in Z_{n_i} is uniquely expressed by g_i^m for some integer m with $0 \le m \le n_i - 1$. In this way each element in Γ is uniquely represented by a reduced word of finite length.

* Partially supported by NSF grant DMS - 9225076 .
Received July 27, 1995..

Let $\{f_g : g \in \Gamma\}$ be a standard orthonormal basis of the Hilbert space $l^2(\Gamma)$ of all complex valued, square-summable sequences indexed by Γ. Let $U : \Gamma \longrightarrow \mathscr{L}(l^2(\Gamma))$ be the left regular representation of Γ on $\mathscr{L}(l^2(\Gamma))$, defined by $U(g)f_h := f_{g^{-1}h}$ for $f_h \in l^2(\Gamma)$, where $\mathscr{L}(\mathscr{H})$ denotes the algebra of all bounded operators on a Hilbert space $\mathscr{H}$. Then $U(g)$ is a unitary operator in $\mathscr{L}(l^2(\Gamma))$ for any $g \in \Gamma$. The reduced group C*-algebra $C_r^*\Gamma$ is the norm closure of the group ring $\mathsf{C}[\Gamma]$ consisting of all linear combinations: $\sum_{i=1}^n \alpha_i U(h_i)$, $h_i \in \Gamma$, $\alpha_i \in \mathsf{C}$, and $n \in \mathsf{N}$.

The purpose of this article is to investigate the structure of the C*-algebra generated by $C_r^*\Gamma$ and the projection R_+ onto the subspace $l^2(\Gamma_+)$, denoted by $C_r^*(\Gamma, R_+)$, and to investigate the structure of the Toeplitz algebra $\mathscr{T}_+$ generated by $\{R_+U(g)R_+ : h \in \Gamma_+\}$, where Γ_+ consists of e and all those reduced words of the form $g_{i_1}^{m_1} g_{i_2}^{m_2} ... g_{i_k}^{m_k}$ ($m_i \in \mathsf{N}$). Briefly speaking, $\mathscr{T}_+$ is generated by some isometries and unitaries on $l^2(\Gamma_+)$, as the reader will see later. The structures of $\mathscr{T}_+$ and $C_r^*(\Gamma, R_+)$ depend on the generators of Γ. One of main results, Theorem 3.1, asserts that if $\Gamma = \mathsf{Z}_{n_0} * \mathscr{F}_n$ where $2 \leq n_0 < +\infty$ and $1 \leq n < +\infty$, then $C_r^*(\Gamma, R_+)$ contains exactly two nontrivial closed ideals, both are stable and of real rank zero; one is $\mathscr{I}_{R_+}$ generated by R_+, and the other is $\mathscr{I}_0$ which is *-isomorphic to the algebra $\mathscr{K}$ consisting of all compact operators on $l^2(\Gamma)$. Furthermore, the quotient algebra $\mathscr{I}_{R_+}/\mathscr{I}_0$ is *-isomorphic to $\mathcal{O}_{n_0 n} \otimes \mathscr{K}$, where $\mathcal{O}_{n_0 n}$ is the Cuntz algebra generated by $n_0 n$ isometries. This compares with the case when $\Gamma = \mathscr{F}_n$ ($2 \leq n < +\infty$) that we studied in [48], for which $C_r^*(\Gamma, R_+)$ has two stable, nontrivial, closed ideals of real rank zero, whose quotient is *-isomorphic to $\mathcal{O}_n \otimes \mathscr{K}$. For all the following cases,

$$\Gamma = \mathscr{F}_\infty,\ \mathsf{Z}_{n_0} * \mathscr{F}_\infty,\ \underbrace{(\mathsf{Z}_{n_1} * \mathsf{Z}_{n_2} * ...)}_{m} * \mathscr{F}_n$$

(where $2 \leq m \leq +\infty$ and $1 \leq n \leq +\infty$), the other main results, Theorem 4.1, concludes that the C*-algebra $C_r^*(\Gamma, R_+)$ contains only one nontrivial closed ideal $\mathscr{I}_{R_+}$ generated by R_+ which is a non-unital, purely infinite, simple C*-algebra. Thus, by our earlier result in [44], $\mathscr{I}_{R_+} \cong \mathscr{I}_{R_+} \otimes \mathscr{K}$. Here a C*-algebra is said to be simple, if $\{0\}$ and itself are the only closed ideals. We remind the reader that a unital simple C*-algebra is purely infinite if and only if for any nonzero element X there exist two elements Y and Z such that $YXZ = I$ ([16] and [29]), and that an arbitrary simple C*-algebra is purely infinite and simple if and only if each nonzero projection is infinite and it has real rank zero ([45,1.2]).

The classification of separable, purely infinite, simple C*-algebras has been under attack in recent years ([21], [37] ,[38]). With this classification problem in mind, we have lately put some efforts in dealing with the C*-algebras of the form $C_r^*(\Gamma, P)$, since for many other choices of the projection P it is a purely infinite, simple C*-algebra (see [47]). Indeed, the class of C*-algebras generated by $C_r^*\Gamma$ and one projection P_Ω onto a subspace $l^2(\Omega)$, i.e., $\{C_r^*(\Gamma, P_\Omega) : \Omega \subset \Gamma\}$, contains some new types of C*-algebras. However, it remains a difficulty task to classify this class up to *-isomorphism.

2. Toeplitz operators and the Toeplitz algebra.

The article is essentially in a self-contained form. Most of the references in the list are relevant materials but not needed. We start with the following analyses on the construction of Toeplitz operators.

2.1 For each $h \in \Gamma$ one defines a Toeplitz operator associated with R_+ by

$$T_h := R_+ U(h) R_+.$$

The Toeplitz algebra $\mathscr{T}_+$ associated with R_+ is generated by $\{T_h : h \in \Gamma_+\}$, and the corner algebra associated with R_+, denoted by $R_+ C_r^*(\Gamma, R_+) R_+$, is generated by $\{T_h : h \in \Gamma\}$.

To consider the structures of $\mathscr{T}_+$ and $C_r^*(\Gamma, R_+)$, we start with the construction of each Toeplitz operator T_h. Let us collect as follows some obvious facts derived immediately from definition.

(1) $U(h)^* = U(h^{-1})$ and $T_h^* = T_{h^{-1}}\ \forall h \in \Gamma$.

(2) $U(h_1 h_2) = U(h_2) U(h_1)$ for $h_1, h_2 \in \Gamma$.

(3) For $g \in \Gamma$ the projection $U(g)^* R_+ U(g)$ is onto the subspace $l^2(g\Gamma_+)$ and the projection $U(g) R_+ U(g)^*$ is onto the subspace $l^2(g^{-1}\Gamma_+)$.

(4) $T_g(f_h) = R_+ f_{g^{-1}h} = \begin{cases} f_{g^{-1}h} & \text{if } h \in g\Gamma_+ \cap \Gamma_+ \\ 0 & \text{if } h \notin g\Gamma_+ \cap \Gamma_+. \end{cases}$ Thus, T_g is a partial isometry in $\mathscr{L}(l^2(\Gamma_+))$ whose initial projection $T_g^* T_g$ is onto the subspace $l^2(g\Gamma_+ \cap \Gamma_+)$ and whose final projection $T_g T_g^*$ is onto the subspace $l^2(g^{-1}\Gamma_+ \cap \Gamma_+)$.

From now on the notation $\Gamma_+(h)$ is reserved for the subset of Γ_+ consisting all reduced words of the form hh_1, where hh_1 is an *irreducible* product in the sense that the last word of h of length one is not the inverse of the rst word of h_1. The notion R_h denotes the projection onto the subspace $l^2(\Gamma_+(h))$. Here we point out that $\Gamma_+(h) \neq h\Gamma_+$ does not hold in general.

2.2. Proposition.

i) $\Gamma_+ = \{e\} \cup \Gamma_+(g_1) \cup \Gamma_+(g_2) \cup \ldots \cup \Gamma_+(g_k) \cup \ldots.$

(ii) *If* $g_i \neq g_j$, *then* $g_i \Gamma_+(g_j) = \Gamma_+(g_i g_j)$.

(iii) *If the order* n_i *of* g_i *is infinite, then* $g_i \Gamma_+(g_i) = \Gamma_+(g_i^2)$.

iv) *If the order* n_i *of* g_i *is finite, then* $g_i \Gamma_+ = \Gamma_+$; *furthermore,*

$$g_i \Gamma_+(g_i^{n_i - 1}) = \Gamma_+ \setminus \Gamma_+(g_i) \text{ and } g_i(\Gamma_+ \setminus \Gamma_+(g_i^{n_i-1})) = \Gamma_+(g_i).$$

PROOF. These facts are readily checked.

2.3. PROPOSITION.

(i) *If a generator* g_i *of* Γ *is of finite order (i.e.,* $g_i^{n_i} = e$ *for some* $2 \leq n_i < +\infty$*), then* $U(g_i)R_+ = R_+U(g_i)$*. Consequently, if* $h_0 \in \Gamma_t$*, then* T_{h_0} *is a unitary operator on* $l^2(\Gamma_+)$*.*

(ii) *If* g_i *is of infinite order, then* $U(g_i)R_+ \neq R_+U(g_i)$*, and* T_{g_i} *is a co-isometry such that* $T_{g_i}^*T_{g_i} = R_{g_i}$ *and* $T_{g_i}T_{g_i}^* = R_+$*.*

PROOF. (i) Assume that g_i is of finite order. To show $R_+U(g_i) = U(g_i)R_+$, it is equivalent to show

$$g_i\Gamma_+ \subset \Gamma_+ \text{ and } g_i\Gamma \setminus \Gamma_+ \subset \Gamma \setminus \Gamma_+.$$

The first inclusion is trivial by the definition of Γ_+. Let h be any reduced word in $\Gamma \setminus \Gamma_+$. Then h contains a factor of the form g_j^{-1} for some generator g_j of infinite order. If h is an irreducible product $g_j^{-1}h_1$ for some reduced word h_1, then $g_ih \in \Gamma \setminus \Gamma_+$. If h is an irreducible product of the form $g_kh_1g_j^{-1}h_2$ for some reduced words h_1 and h_2 and a generator g_k, then g_ih is again in $\Gamma \setminus \Gamma_+$. Thus, the second inclusion also holds. If $h_0 \in \Gamma_t$, write $h_0 = g_{j_1}g_{j_2}\cdots g_{j_k}$ for some generators of Γ with finite order. Then $U(h_0) = U(g_{j_k})U(g_{j_{k-1}})...U(g_{j_1})$, and hence $T_{h_0} = R_+U(h_0)R_+ = T_{g_{j_k}}T_{g_{j_{k-1}}}\cdots T_{g_{j_1}}$, which is a unitary operator on $l^2(\Gamma_+)$.

(ii) If g_i is of infinite order, then $g_i^{-1}h \in \Gamma \setminus \Gamma_+$ and $g_i(g_i^{-1}h) = h \in \Gamma_+$ as long as h is a reduced word in Γ_+ starting with another generator different from g_i. Thus, $l^2(\Gamma \setminus \Gamma_+)$ is not a reduced subspace of $U(g_i)$. By definition $T_{g_i}^*T_{g_i}$ is the projection onto the subspace $l^2(g_i\Gamma_+)$. Since $g_i\Gamma_+ = \Gamma_+(g_i)$ in case g_i is of infinite order, one sees that $T_{g_i}^*T_{g_i} = R_{g_i}$. ›

2.4. PROPOSITION. *Let* $h \in \Gamma$*. Then* $T_h \neq 0$ *if and only if* h *can be uniquely written as an irreducible product* $h'h_0h''^{-1}$*, where* $h_0 \in \Gamma_t$*, and* $h', h'' \in \Gamma_+$ *such that the last words of* h' *and* h'' *with length one are some generators of infinite order whenever* $h' \neq e$*,* $h'' \neq e$*.*

PROOF. First, each reduced word $h \in \Gamma$ can be written uniquely as a product $g_{i_1}^{n_1}g_{i_2}^{n_2}...g_{i_k}^{n_k}$ for some generators $g_{i_1}, g_{i_2}, ..., g_{i_k}$, where $g_{i_j} \neq g_{i_{j+1}}$ for $1 \leq j \leq k-1$, and all n_j are integers. By our convention, $n_j > 0$ whenever g_{i_j} is a generator of finite order. Our attention will be on the powers of those generators of infinite order in the above product. If all $g_{i_1}, ..., g_{i_k}$ are of finite order, then $h = h_0$, and hence $T_h \neq 0$. Assume that there is at least one generator involved is of infinite order. Select all generators of infinite order in the ordered tuple $(g_{i_1}, g_{i_2}, ..., g_{i_k})$, with the order kept, and write them as an ordered tuple $(g_{i_{j_1}}, g_{i_{j_2}}, ..., g_{i_{j_l}})$ (where $j_1 < j_2 < ... < j_l$). Clearly, the assertion of this proposition is equivalent to the following: $T_h \neq 0$ if and only if the signs of the corresponding powers $n_{j_1}, n_{j_2}, \ldots, n_{j_l}$, as an ordered tuple, have patterns $+, +, \ldots, +$ (for the case $h = h'h_0$; i.e., $h'' = e$), or $-, -, \ldots, -$ (for the case $h = h_0 h''^{-1}$; i.e., $h' = e$), or $+, +, \ldots, +, -, \ldots, -$ (for the case $h = h'h_0h''^{-1}$; i.e., $h' \neq e$ and $h'' \neq e$). These patterns are exactly all possibilities for which $h^{-1}\Gamma_+ \cap \Gamma_+ \neq \emptyset$, that is, the final projection of T_h is non-zero.

2.5. PROPOSITION.

(i) *If $T_h \neq 0$, write $h = h'h_0h''^{-1}$ as in Proposition 2.4, then $T_h = T^*_{h''}T_{h_0}T_{h'}$, which is a partial isometry whose final projection is $T^*_{h''}T_{h''}$ and whose initial projection is $T^*_{h'}T_{h'}$; both are independent of $h_0 \in \Gamma_t$.*

(ii) *If $h_1 \in \Gamma_+$, write $h = g_{i_1}^{n_1} g_{i_2}^{n_2} \ldots g_{i_k}^{n_k}$ for some generators $g_{i_1}, g_{i_2}, \ldots, g_{i_k}$, then $T_{h_1} = (T_{g_{i_k}})^{n_k} \ldots (T_{g_{i_2}})^{n_2} (T_{g_{i_1}})^{n_1}$.*

PROOF. (i) Obviously, $T_{h'h_0h''^{-1}} = R_+ U(h'')^* U(h_0) U(h') R_+$. It is easily seen by definition that

$$R_+ U(h'')^* U(h_0)(I - R_+) U(h') R_+ = 0.$$

Thus, $T_h = T^*_{h''} T_{h_0} T_{h'}$; here we use the fact that $U(h_0)R_+ = R_+ U(h_0)$. It is obvious that T_h is a partial isometry.

(ii) For each generator g_i of Γ one has $R_+ U(g_i) R_+^\perp = 0$ (where $R_+^\perp = I - R_+$). Thus, with respect to the decomposition $R_+ \oplus R_+^\perp = I$ one can write $U(g_i)$ as a 2×2 matrix

$$\begin{pmatrix} T_{g_i} & 0 \\ R_+^\perp U(g_i) R_+ & R_+^\perp U(g_i) R_+^\perp \end{pmatrix}.$$

It follows that $T_{g_i g_j} = T_{g_j} T_{g_i}$, and hence (by induction)

$$T_{h_1} = T_{g_{i_1}^{n_1} g_{i_2}^{n_2} \ldots g_{i_k}^{n_k}} = (T_{g_{i_k}})^{n_k} \ldots (T_{g_{i_2}})^{n_2} (T_{g_{i_1}})^{n_1}.$$

2.6. PROPOSITION.

(i) *If $h \in \Gamma_+$, then h can be uniquely written as an irreducible product of the form $h = (h_1 g_{t_1})(h_2 g_{t_2}) \ldots (h_k g_{t_k}) h_0$, where all $g_{t_1}, g_{t_2}, \ldots, g_{t_{k+l}}$ are generators of Γ of infinite order, and $h_0, h_1, h_2, \ldots, h_{k+l}$ are elements of Γ_t.*

(ii) *$T_{h_j g_{t_j}} T^*_{h_j g_{t_j}} = R_+$ and $T^*_{h_j g_{t_j}} T_{h_j g_{t_j}} = R_{h_j g_{t_j}}$. Consequently, T_h is a co-isometry.*

PROOF. (i) is trivial. (ii) is also straightforward. In fact,

$$T_{h_j g_{t_j}} T^*_{h_j g_{t_j}} = T_{g_{t_j}} T_{h_j} T^*_{h_j} T^*_{g_{t_j}} = R_+,$$

$$T^*_{h_j g_{t_j}} T_{h_j g_{t_j}} = T^*_{h_j} T^*_{g_{t_j}} T_{g_{t_j}} T_{h_j} = T^*_{h_j} R_{g_{t_j}} T_{h_j} = R_{h_j g_{t_j}}.$$

Clearly, $T_h T_h^* = R_+$, and $T_h^* T_h = R_h$; i.e., T_h is a co-isometry.

2.7. COROLLARY. *The Toeplitz algebra* $\mathscr{T}_+$ *coincides with* $R_+C_r^*(\Gamma, R_+)R_+$; *both are generated by* $\{T_h : h \in \Gamma_+\}$, *and in turn, by* $\{T_{g_i}\}$.

PROOF. By the fact each nonzero Toeplitz operator T_h can be written as a product $T_{h''}^* T_{h_0} T_{h'}$, where h_0, h', h'' are in Γ_+, one sees that the corner algebra is also generated by $\{T_h : h \in \Gamma_+\}$ as $\mathscr{T}_+$ is. Furthermore, Proposition 2.5(ii) asserts that each element $T_h \in \{T_h : h \in \Gamma_+\}$ is a product of Toeplitz operators in $\{T_{g_i}\}$ where $\{g_i\}$ is the set of all generators of Γ.

2.8. REMARK. Proposition 2.3 and Corollary 2.7 combined tell the following:

(a) If $\Gamma = \mathscr{F}_\infty$, then $\mathscr{T}_+$ is generated by a sequence $\{T_{g_i}^*\}$ of isometries.

(b) If Γ is any other group among the ones considered in this article, then $\mathscr{T}_+$ is generated by some isometries and unitaries, these isometries are T_{g_j} where g_j are of infinite order and these unitaries are T_{g_i} where g_i is of finite order.

2.9. PROPOSITION *The closed ideal* $\mathscr{I}_{R_+}$ *of* $C_r^*(\Gamma, R_+)$ *generated by* R_+ *is nontrivial.*

PROOF. We proved in [48] that the projection P_Ω onto the subspace $l^2(\Omega)$ (where $\Omega \subset \Gamma$) generates a nontrivial closed ideal of $C_r^*(\Gamma, P_\Omega)$ if and only if there is no finite subset $\{h_1, h_2, ..., h_m\}$ of Γ such that $\bigcup_{k=1}^m h_k\Omega = \Gamma$. Clearly, there is no finite subset $\{h_1, h_2, ..., h_m\}$ of Γ such that $\bigcup_{k=1}^m h_k\Gamma_+ = \Gamma$. Thus, $\mathscr{I}_{R_+}$ is non-trivial.

3. The case $\Gamma = \mathsf{Z}_{n_0} * \mathscr{F}_n$.

In [48] we have dealt with the case $\Gamma = \mathscr{F}_n$ for $2 \le n < +\infty$. As a result, $C_r^*(\Gamma, R_+)$ contains exactly two nontrivial (closed) ideals, the ideal $\mathscr{K}(l^2(\Gamma))$ of all compact operators and the ideal $\mathscr{I}_{R_+}$ generated by R_+; both are stable C*-algebras with real rank zero and $\mathscr{I}_{R_+}/\mathscr{K}(l^2(\Gamma)) \cong \mathscr{O}_n \otimes \mathscr{K}$. The structures of $C_r^*(\Gamma, R_+)$ for other cases turn out to be different. We will apply two different techniques to deal with the following separate cases :

(i) $\Gamma = \mathsf{Z}_{n_0} * \mathscr{F}_n$, where $2 \le n_0 < +\infty$ and $1 \le n < +\infty$.

(ii) $\Gamma = \mathscr{F}_\infty$.

(iii) $\Gamma = \mathsf{Z}_{n_0} * \mathscr{F}_\infty$.

(iv) $\Gamma = \underbrace{(\mathsf{Z}_{n_1} * \mathsf{Z}_{n_2} * ...)}_{m} * \mathscr{F}_n$, where $2 \le m \le +\infty$ and $1 \le n < +\infty$.

(v) $\Gamma = \underbrace{(\mathsf{Z}_{n_1} * \mathsf{Z}_{n_2} * ...)}_{m} * \mathscr{F}_\infty$.

In this section we consider the case (i), $\Gamma = \mathsf{Z}_{n_0} * \mathscr{F}_n$ where $1 \le n < +\infty$.

Assume that g_0 is the generator of Z_{n_0}, and $g_1, g_2, ..., g_n$ are the generators of $\mathscr{F}_n$.

The structure of $C_r^*(\Gamma, R_+)$ is summarized as in the following theorem:

3.1. THEOREM. *Assume $\Gamma = \mathsf{Z}_{n_0} * \mathscr{F}_n$ where $1 \leq n < +\infty$. Then the following hold:*

(i) *The Toeplitz algebra $\mathscr{T}_+$ contains only one nontrivial closed ideal $\mathscr{I}_0$ which is *-isomorphic to $\mathscr{K}$.*

(ii) *$\mathscr{T}_+/\mathscr{I}_0 \cong \mathscr{O}_{n_0n}$, where $\mathscr{O}_{n_0n}$ the Cuntz algebra with n_0n generators.*

(iii) *$C_r^*(\Gamma, R_+)$ contains a chain of exactly two nontrivial closed ideals which are stable and of real rank zero; one is $\mathscr{I}$ generated by Q_0 which is *-isomorphic to $\mathscr{K}$, and the other is $\mathscr{I}_{R_+}$ generated by R_+ which is *-isomorphic to $\mathscr{T}_+ \otimes \mathscr{K}$; furthermore, $\mathscr{I}_{R_+}/\mathscr{I} \cong \mathscr{O}_{n_0n}\otimes$ $\{\mathscr{K}\}$.*

(iv) $\mathrm{RR}(\mathscr{I}_{R_+}) = \mathrm{RR}(\mathscr{T}_+) = 0$.

Before proving Theorem 3.1 we state the following immediate corollary.

3.2. COROLLARY. *Assume that Γ is as in Theorem 3.1. Then the following short sequences are exact:*

$$0 \longrightarrow \mathscr{I}_{R_+} \longrightarrow C_r^*(\Gamma, R_+) \longrightarrow C_r^*\Gamma \longrightarrow 0,$$
$$0 \longrightarrow \mathscr{I}_0 \longrightarrow \mathscr{T}_+ \longrightarrow \mathscr{O}_{n_0n} \longrightarrow 0,$$
$$0 \longrightarrow \mathscr{I} \longrightarrow \mathscr{I}_{R_+} \longrightarrow \mathscr{O}_{n_0n} \otimes \mathscr{K} \longrightarrow 0.$$

We now turn to the proof of Theorem 3.1.

3.3. LEMMA. *Let Q_0 be the projection onto the subspace spanned by $f_e, f_{g_0}, ..., f_{g_0^{n_0-1}}$ and P_n be the projection onto the subspace $l^2(\Gamma_+(g_1) \cup \Gamma_+(g_2) \cup ... \cup \Gamma_+(g_n))$, where $1 \leq n < +\infty$. Then*

(i) $P_n = \sum_{j=1}^n T_{g_j}^* T_{g_j}$, $Q_0 \in \mathscr{T}_+$, $R_+ = P_n + \sum_{i=1}^{n_1-1} U(g_0^i)^* P_n U(g_0^i) + Q_0$,

(ii) *$\mathscr{T}_+$ is generated by $\{T_{g_0^i g_j} : 0 \leq i \leq n_0 - 1,\ 1 \leq j \leq n\} \cup \{T_{g_0}\}$, and*

(iii) *the closed ideal $\mathscr{I}_0$ of $\mathscr{T}_+$ generated by Q_0 is *-isomorphic to $\mathscr{K}$.*

PROOF. (i) First, $T_{g_k}^* T_{g_k} = R_{g_k}$ is a projection in $\mathscr{T}_+$ for any $1 \leq k \leq n$. Hence $P_n = \sum_{k=1}^n R_{g_k} \in \mathscr{T}_+$. Clearly, $R_+ - P_n = R_{g_0} + P(e)$, where $P(e)$ is the one dimensional projection onto the subspace spanned by f_e (R_{g_0} and $P(e)$ may not in $\mathscr{T}_+$). It is easily seen that

$$\begin{aligned}\Gamma_+ &= (\cup_{k=1}^n \Gamma_+(g_k)) \cup \Gamma_+(g_0) \cup \{e\} \\ &= (\cup_{k=1}^n \Gamma_+(g_k)) \cup (\cup_{i=1}^{n_0-1} g_0^i \cup_{k=1}^n \Gamma_+(g_k)) \cup \{e, g_0, g_0^2, ..., g_0^{n_0-1}\}.\end{aligned}$$

Then $R_+ = P_n + \sum_{i=1}^{n_0-1} U(g_0^i)^* P_n U(g_0^i) + Q_0$. It follows that $Q_0 \in \mathscr{T}_+$. The conclusion (ii) follows from Proposition 2.5 and (iii) is obvious.

3.4. PROOF of Theorem 3.1. (i) has been proved in Lemma 3.3 (ii).

(ii) For any $X \in \mathscr{T}_+$ let $\tilde{X}$ denote the image of X in the quotient algebra $\mathscr{T}_+/\mathscr{I}_0$. Since $\mathscr{T}_+$ is generated by the following set

$$\{T_{g_0^k g_j} : 0 \le k \le n_0 - 1,\ 1 \le j \le n\} \cup \{T_{g_0}\},$$

of course $\mathscr{T}_+/\mathscr{I}_0$ is generated by the set

$$\{\tilde{T}_{g_0^k g_j} : 0 \le k \le n_0 - 1,\ 1 \le j \le n\} \cup \{\tilde{T}_{g_0}\}.$$

We further claim, for the special case $\Gamma = \mathsf{Z}_{n_0} * \mathscr{F}_n$ only, that $\mathscr{T}_+/\mathscr{I}_0$ is generated by the following set of $n_0 n$ isometries:

$$\{\tilde{T}_{g_0^k g_j} : 0 \le k \le n_0 - 1,\ 1 \le j \le n\}.$$

In fact, one has

$$\begin{aligned}
&\sum_{k=0}^{n_0-1}\sum_{j=1}^{n} T^*_{g_0^k g_j} T_{g_0^{k-1} g_j}\\
&= \sum_{k=0}^{n_0-1}\sum_{j=1}^{n} T^*_{g_0} T^*_{g_0^{k-1} g_j} T_{g_0^{k-1} g_j}\\
&= T^*_{g_0}(R_+ - Q_0),
\end{aligned}$$

the last equality is due to the following:

$$R_+ - Q_0 = \sum_{k=0}^{n_0-1}\sum_{j=1}^{n} T^*_{g_0^k g_j} T_{g_0^k g_j} = \sum_{k=0}^{n_0-1}\sum_{j=1}^{n} T^*_{g_0^{k-1} g_j} T_{g_0^{k-1} g_j};$$

note here $g_0^{-1} = g_0^{n_0-1}$. Thus, $T^*_{g_0}(R_+ - Q_0)$ is in the *-algebra generated by

$$\{T_{g_0^k g_j} : 0 \le k \le n_0 - 1,\ 1 \le j \le n\}.$$

It follows that $\tilde{T}_{g_0}$ is in the *-algebra generated by $\{\tilde{T}_{g_0^k g_j} : 0 \le k \le n_0 - 1,\ 1 \le j \le n\}$. Therefore, $\mathscr{T}_+/\mathscr{I}_0$ is generated by

$$\{\tilde{T}_{g_0^k g_j} : 0 \le k \le n_0 - 1,\ 1 \le j \le n\},$$

and hence, is *-isomorphic to $\mathscr{O}_{n_0 n}$ ([17]).

(iii) Clearly, the closed ideal $\mathscr{I}$ of $C_r^*(\Gamma, R_+)$ generated by Q_0 is stably isomorphic to $\mathscr{I}_0$. Thus, $\mathscr{I} \cong \mathscr{K}$. The closed ideal $\mathscr{I}_{R_+}$ generated by $\mathscr{R}_+$ is stably isomorphic to $\mathscr{T}_+$ (i.e., $\mathscr{I}_{R_+} \otimes \mathscr{K} \cong \mathscr{T}_+ \otimes \mathscr{K}$). Since $\mathscr{T}_+/\mathscr{I}_0 \cong \mathscr{O}_{n_0 n}$, one can show that $\mathscr{I}_{R_+}/\mathscr{I} \cong \mathscr{O}_{n_0 n} \otimes \mathscr{K}$ by using exactly the same argument as in the proof of [48, Lemma 3.3]. Furthermore, again by exactly the same arguments as in the proof of [47, 3.5 and 3.7], one can show that

$\mathscr{I}_{R_+} \cong \mathscr{I}_{R_+} \otimes \mathscr{K}$. It is clear that $\mathscr{I}_{R_+}$ and $\mathscr{I}$ are the only two nontrivial closed ideals of $C_r^*(\Gamma, R_+)$.

(iv) Using a general lifting result ([7,3.14] and [45,2.4]), we conclude that

$$\mathrm{RR}(\mathscr{T}_+) = 0 \quad \text{and} \quad \mathrm{RR}(\mathscr{I}_{R_+}) = 0,$$

based on the fact $\mathrm{RR}(\mathscr{O}_{n_0 n}) = 0$ ([44]).

4. The other cases.

In this section we will investigate, by using a different technique, the structure of $C_r^*(\Gamma, R_+)$ for the remaining cases, i.e., Γ is any of the following groups

$$\mathscr{F}_\infty, \ \mathbf{Z}_{n_0} * \mathscr{F}_\infty, \ \underbrace{(\mathbf{Z}_{n_1} * \mathbf{Z}_{n_2} * \dots)}_{m} * \mathscr{F}_n, \ \underbrace{(\mathbf{Z}_{n_1} * \mathbf{Z}_{n_2} * \dots)}_{m} * \mathscr{F}_\infty;$$

where $2 \le m \le +\infty$ and $1 \le n < +\infty$.

4.1. THEOREM. *Assume that* $\Gamma = \mathscr{F}_\infty$, *or* $\mathbf{Z}_{n_0} * \mathscr{F}_\infty$, *or* $\underbrace{(\mathbf{Z}_{n_1} * \mathbf{Z}_{n_2} * \dots)}_{m} * \mathscr{F}_n$, *or* $\underbrace{(\mathbf{Z}_{n_1} * \mathbf{Z}_{n_2} * \dots)}_{m} * \mathscr{F}_\infty$, *where* $m \ge 2$ *but* $1 \le n < +\infty$ *is arbitrary. Then*

(i) $\mathscr{T}_+$ *is a purely infinite simple* C^**-algebra* (*and hence* $\mathrm{RR}(\mathscr{T}_+) = 0$), *and*

(ii) $C_r^*(\Gamma, R_+)$ *contains only one nontrivial closed ideal* $\mathscr{I}_{R_+}$ *which is generated by* R_+ *and* *-*isomorphic to* $\mathscr{T}_+ \otimes \mathscr{K}$ (*and hence, is a stable, purely infinite, simple* C^**-algebra*).

4.2. COROLLARY. *Assume that* Γ *is as in Theorem* 4.1. *Then the* *-*isomorphism* $\mathscr{I}_{R_+} \cong \mathscr{T}_+ \otimes \mathscr{K}$ *induces a short sequence:*

$$0 \longrightarrow \mathscr{T}_+ \otimes \mathscr{K} \longrightarrow C_r^*(\Gamma, R_+) \longrightarrow C_r^* \Gamma \longrightarrow 0.$$

Two reduced words $h_1, h_2 \in \Gamma$ are said to be comparable, if either $h_1 \in \Gamma(h_2)$, denoted by $h_1 \prec h_2$, or $h_2 \in \Gamma(h_1)$, denoted by $h_2 \prec h_1$ (cf. [47,4.1]). If neither $h_1 \prec h_2$ nor $h_2 \prec h_1$, we say that h_1 and h_2 are incomparable. Obviously, h_1 and h_2 are incomparable if and only if $\Gamma(h_1) \cap \Gamma(h_2) = \emptyset$.

To prove Theorem 4.1, we first prove two key lemmas.

4.3. LEMMA. *Assume that* Γ *is as in the statement of Theorem* 4.1. *If* $h_1, h_2, \dots, h_{n'}$ *are distinct reduced words in* Γ_+ *and* $k_1, k_2, \dots, k_{n''}$ *are distinct reduced words in* $\Gamma \setminus \Gamma_+$, *then there exists a reduced word* $h \in \Gamma_+$ *satisfying the following conditions:*

(i) $h_1 h, h_2 h, \dots, h_{n'} h \in \Gamma_+$,

(ii) $k_1 h, k_2 h, \dots, k_{n''} h \in \Gamma \setminus \Gamma_+$,

(iii) $h_1 h, h_2 h, \dots, h_{n'} h$ *are mutually incomparable, and*

(iv) *all* $h, h_1h, h_2h, ..., h_{n'}h$ *end with a generator of infinite order.*

PROOF. *Case 1.* $\Gamma = \underbrace{(\mathbb{Z}_{n_1} * \mathbb{Z}_{n_2} * ...)}_{m} * \mathscr{F}_n$, or $\underbrace{(\mathbb{Z}_{n_1} * \mathbb{Z}_{n_2} * ...)}_{m} * \mathscr{F}_\infty$.

Assume that g_1 is of infinite order, g_2 and g_3 are any two distinct generators of finite order. If $n_0 \in \mathbb{N}$ is chosen to be large enough, all $h_1(g_2g_3)^{n_0}g_1$, $h_2(g_2g_3)^{n_0}g_1$, ..., and $h_{n'}(g_2g_3)^{n_0}g_1$ are in Γ_+ and end with g_1, and all $k_1(g_2g_3)^{n_0}g_1$, $k_2(g_2g_3)^{n_0}g_1$, ..., and $k_{n''}(g_2g_3)^{n_0}g_1$ are still in $\Gamma \setminus \Gamma_+$, since each k_i contains at least one factor of the form g_j^{-1} for some generator g_j of infinite order that cannot be canceled with $(g_2g_3)^{n_0}g_1$ for any n_0.

We will further find a reduced word h' in Γ_+ such that $h := (g_2g_3)^{n_0}g_1h'$ is an irreducible product satisfying all the conditions (i), (ii), (iii) and (iv). Here we notice that $h_1(g_2g_3)^{n_0}g_1h'$, $h_2(g_2g_3)^{n_0}g_1h'$, ..., $h_{n'}(g_2g_3)^{n_0}g_1h'$ remain in Γ_+ and $k_1(g_2g_3)^{n_0}g_1h'$, $k_2h_2(g_2g_3)^{n_0}g_1h'$, ..., $k_{n''}(g_2g_3)^{n_0}g_1h'$ remain in $\Gamma \setminus \Gamma_+$ for any irreducible product of the form $(g_2g_3)^{n_0}g_1h'$.

The lemma is trivial in case $n' = 1$, since $h := (g_2g_3)^{n_0}g_1$ is as wanted. Consider the case $n' = 2$ (we will need the arguments later for the general situation). If $h_1(g_2g_3)^{n_0}g_1$ and $h_2(g_2g_3)^{n_0}g_1$ are incomparable, set $h := (g_2g_3)^{n_0}g_1$, as wanted. If $h_1(g_2g_3)^{n_0}g_1$ is comparable with $h_2(g_2g_3)^{n_0}g_1$, then

$$\text{either} \quad h_1(g_2g_3)^{n_0}g_1 \prec h_2(g_2g_3)^{n_0}g_1 \quad \text{or} \quad h_2(g_2g_3)^{n_0}g_1 \prec h_1(g_2g_3)^{n_0}g_1.$$

We need only to consider one case, say $h_1(g_2g_3)^{n_0}g_1 \prec h_2(g_2g_3)^{n_0}g_1$, since a symmetric argument applies to the other. Write the following irreducible product:

$$h_1(g_2g_3)^{n_0}g_1 = h_2(g_2g_3)^{n_0}g_1g_{i_1}^{n_1}g_{i_2}^{n_2}...g_{i_t}^{n_t}(g_2g_3)^{n_0}g_1,$$

where $g_{i_1}, ..., g_{i_t}$ are some generators of Γ. Let g_i be a generator of Γ such that $g_i \neq g_{i_1}$, then $h_1(g_2g_3)^{n_0}g_1g_ig_1$ and $h_2(g_2g_3)^{n_0}g_1g_ig_1$ are incomparable. Set $h := (g_2g_3)^{n_0}g_1g_ig_1$, as desired. In any case, for h_1 and h_2 we can choose a reduced word $h \in \Gamma_+$ that ends with g_1 and satisfies all the conditions (i), (ii), (iii) and (iv).

We now consider the general situation by induction on n' for each fixed n''. Applying the above arguments for the case $n' = 2$ to $h_{n'}(g_2g_3)^{n_0}g_1$ and $h_1(g_2g_3)^{n_0}g_1$, one gets a reduced word h_1' such that $h_1(g_2g_3)^{n_0}g_1h_1'$ and $h_{n'}(g_2g_3)^{n_0}g_1h_1'$ are incomparable, and h_1' ends with g_1. Applying the same argument to $h_{n'}(g_2g_3)^{n_0}g_1h_1'$ and $h_2(g_2g_3)^{n_0}g_1h_1'$, one gets a reduced word h_2' such that h_2' ends with g_1, and $h_{n'}(g_2g_3)^{n_0}g_1h_1'h_2'$ and $h_2(g_2g_3)^{n_0}g_1h_1'h_2'$ are incomparable. Furthermore,

$$h_{n'}(g_2g_3)^{n_0}g_1h_1'h_2' \quad \text{and} \quad h_1(g_2g_3)^{n_0}g_1h_1'h_2'$$

remain incomparable, since the product $(g_2g_3)^{n_0}g_1h'_1h'_2$ is irreducible. Repeat this process $n'-1$ times, one gets an irreducible product $h'_0 := h'_1h'_2...h'_{n'-1}$ such that h'_0 ends with g_1, and $h_{n'}(g_2g_3)^{n_0}g_1h'_0$ is incomparable with each of

$$h_1(g_2g_3)^{n_0}g_1h'_0, \quad h_2(g_2g_3)^{n_0}g_1h'_0, \quad ..., \quad h_{n'-1}(g_2g_3)^{n_0}g_1h'_0.$$

By the inductive assumption, there exists a reduced word h_0 such that h_0 ends with g_1, and

$$h_1(g_2g_3)^{n_0}g_1h'_0h_0, \quad h_2(g_2g_3)^{n_0}g_1h'_0h_0, \quad ..., \quad h_{n'-1}(g_2g_3)^{n_0}g_1h'_0h_0$$

are mutually incomparable. Since $(g_2g_3)^{n_0}g_1h'_0h_0$ is irreducible, $h_{n'}(g_2g_3)^{n_0}g_1h'_0h_0$ remains incomparable with each $h_i(g_2g_3)^{n_0}g_1h'_0h_0$ for $1 \le i \le n'-1$. Set $h := (g_2g_3)^{n_0}g_1h'_0h_0$. Then h satisfies all the conditions (i), (ii), (iii), and (iv).

Case 2. $\Gamma = \mathscr{F}_\infty$, or $\mathbf{Z}_{n_0} * \mathscr{F}_\infty$.

In this case one can take a generator g of Γ with an infinite order such that g and g^{-1} is not a factor of any h_i and k_j for $1 \le i \le n'$ and $1 \le j \le n''$. Then $h_ig^{n_0}$ and $k_jg^{n_0}$ are irreducible products for any $n_0 \ge 1$, $1 \le i \le n'$, and $1 \le j \le n''$. Using the same arguments as in the above case 1 (just replace g_2g_3 by g everywhere), one can find a reduced word $h' \in \Gamma_+$ such that $h := g^{n_0}h'$ satisfies all the conditions (i), (ii), (iii), and (iv).

4.4. Lemma. *Assume that Γ is as in the statement of Theorem 4.1. If $X = \sum_{j=1}^m \alpha_j T_{k(j1)}T_{k(j2)}...T_{k(jm_j)} \in \mathscr{T}_+$, where $\{k(jl) : 1 \le j \le m,\ 1 \le l \le m_j\}$ is a subset of Γ and $T_{k(j1)}T_{k(j2)}...T_{k(jm_j)} \ne 0$ for $1 \le j \le m$, then for any $\epsilon > 0$ there exists a projection $Q \in \mathscr{T}_+$ satisfying that following conditions:*

(i) $\|XQ\| \ge \|X\| - \epsilon$.

(ii) *XQX^* generates a finite dimensional *-subalgebra of $X\mathscr{T}_+X^*$.*

Proof. Since the proof is almost exactly the same as the one for [47,5.1], we only sketch the main ideas as follows and leave the details to the reader.

To get such a projection Q, we start with a vector $\xi = \sum_{j=1}^{m_0} \beta_i f_{h_i}$ where $h_i \in \Gamma_+$ such that

$$\|\xi\| \le 1 \quad \text{and} \quad \|X(\xi)\| > \|X\| - \epsilon.$$

Observe that $T_{k(j1)}T_{k(j2)}...T_{k(jm_j)}(f_{h_i}) \ne 0$ if and only if $T_{k(jl)}...T_{k(jm_j)}(f_{h_i}) \ne 0$ for $1 \le l \le m_j$, again if and only if $k(jl)^{-1}...k(jm_j)^{-1}h_i \in \Gamma_+$ for $1 \le l \le m_j$. If $T_{k(j1)}T_{k(j2)}...T_{k(jm_j)}(f_{h_i}) \ne 0$, then $T_{k(j1)}T_{k(j2)}...T_{k(jm_j)}(f_{h_i}) = f_{k_j^{-1}h_i}$, where k_j denotes the reduced word obtained by simplifying the product $k(jm_j)...k(j2)k(j1)$. Write

$$X(\xi) = \sum_{k_j^{-1}h_i} (\sum_l \alpha_{j_l}\beta_{i_l}) f_{k_j^{-1}h_i}.$$

Then

$$\|X(\xi)\|^2 = \sum_{k_j^{-1}h_i} (\sum_l \alpha_{j_l}\beta_{i_l})^2,$$

where the sum $\sum_{k_j^{-1}h_i}$ is indexed by all different resulting reduced words from the products $k_j^{-1}h_i$ for which $T_{k(j1)}T_{k(j2)}...T_{k(jm_j)}(f_{h_i}) \neq 0$ (i.e., all those products $k_{j'}^{-1}h_{i'}$ satisfying $k_{j'}^{-1}h_{i'} = k_j^{-1}h_i$ give only one term which is indexed by $k_j^{-1}h_i$), and the sum $\sum_l \alpha_{j_l}\beta_{i_l}$ is indexed by pairs (i_l, j_l) such that $k_{j_l}^{-1}h_{i_l} = k_j^{-1}h_i$.

Apply Lemma 4.3 to the following set

$$\mathscr{W}_0 := \{h_i : 1 \leq i \leq m_0\} \cup (\cup_{j=1}^m \{k(jl)^{-1}...k(jm_j)^{-1}h_i : 1 \leq l \leq m_j\}).$$

Some elements of $\mathscr{W}_0$ are in Γ_+ and some are not. We get a reduced word $h \in \Gamma_+$ satisfying the following conditions:

(a) $\{h_i h : 1 \leq i \leq m_0\} \subset \Gamma_+$ and any two elements in this set are incomparable,

(b) $k(jl)^{-1}...k(jm_j)^{-1}h_i h \in \Gamma_+$ if and only $k(jl)^{-1}...k(jm_j)^{-1}h_i \in \Gamma_+$,

(c) all elements in $\{h, h_1 h, h_2 h, ..., h_{m_0}h\}$ end with the same generator g_1 of infinite order, and

(d) all elements in $\{k_j^{-1}h_i h : 1 \leq i \leq m_0, 1 \leq j \leq m\} \cap \Gamma_+$ are mutually incomparable.

Set $\xi' = \sum_{i=1}^{m_0} \beta_i f_{h_i h}$; then $\|\xi'\| = \|\xi\|$. Observe that $k_j^{-1}h_i = k_{j_1}^{-1}h_{i_1}$ if and only if $k_j^{-1}h_i h = k_{j_1}^{-1}h_{i_1}h$. Then the above condition (b) warrants

$$\|X(\xi')\| = \|X(\xi)\|.$$

By the condition (a) above one sees that $R_{h_1 h}, R_{h_2 h}, ..., R_{h_{m_0}h}$ are mutually orthogonal projections in $\mathscr{T}_+$, and furthermore, all these projections are equivalent to R_{g_1} by the condition (c) above, and in turn, equivalent to R_+. It is obvious that $R_{h_i h}$ is a subprojection of the initial projection of $T_{k(j1)}T_{k(j2)}...T_{k(jm_j)}$ whenever $T_{k(j1)}T_{k(j2)}...T_{k(jm_j)}(f_{h_i}) \neq 0$. Set

$$Q = R_{h_1 h} \oplus R_{h_2 h} \oplus ... \oplus R_{h_{m_0}h}.$$

Then $Q \in \mathscr{T}_+$ satisfies the two conditions (i) and (ii) required. In fact,

$$\|XQ\| \geq \|X(\xi')\| \geq \|X\| - \epsilon,$$

and the condition (d) above implies that XQX^* is in a finite dimensional *-subalgebra of $X\mathscr{T}_+X^*$. For more details the reader is referred to the proof of [47,5.1].

4.5. PROOF OF THEOREM 4.1. (i) To show $\mathscr{T}_+ := R_+C_r^*(\Gamma, R_+)R_+$ is a purely infinite simple C*-algebra, we prove by definition [16] that the norm closure of $A\mathscr{T}_+A$ contains a projection equivalent to R_+ for each nonzero positive element $A \in \mathscr{T}_+$. We use the same argument as in the proof of [47,5.2]; here is a sketch of the main ideas.

Without loss of generality, we assume that $\|A\| = 1$. Let $\epsilon \in (0, \frac{1}{4})$. By the construction of $\mathscr{T}_+$ there exists an element $X = \sum_{j=1}^{m} \alpha_i T_{k(j1)} T_{k(j2)} \dots T_{k(jm_j)}$ such that

$$\|A - X\| < \frac{\epsilon}{3}.$$

By Lemma 4.4 there exists projection $Q \in \mathscr{T}_+$ satisfying:

(i) $\|XQ\| \geq \|X\| - \epsilon/3$; and

(ii) XQX^* generates a finite dimensional C*-subalgebra of the hereditary C*-subalgebra $(X\mathscr{T}_+X^*)^-$. Now the following estimates are in order:

$$\|XQX^* - AQA^*\| \leq \|(X - A)QX^*\| + \|AQ(X^* - A^*)\| < \epsilon.$$

Take the largest eigenvalue μ of XQX^* with the corresponding spectral projection $P' \in \mathscr{T}_+$. Then

$$\mu = \|XQX^*\| > \left(\|X\| - \frac{\epsilon}{3}\right)^2 > \frac{25}{36}, \quad \text{and}$$

$$\|\mu P' - P'AQA^*P'\| = \|P'XQX^*P' - P'AQA^*P'\| < \epsilon.$$

It follows that

$$\|P' - \frac{1}{\mu} P'AQA^*P'\| < \frac{\epsilon}{\mu} < \frac{36\epsilon}{25} < 1.$$

Then $P'AQA^*P'$ is an invertible element in $P'\mathscr{T}_+P'$. Set

$$W := (P'AQA^*P')^{-\frac{1}{2}}(AQA^*)^{\frac{1}{2}}.$$

Then $W \in \mathscr{T}_+$, and $WW^* = P'$. It follows that W^*W is a projection in the norm closure of $A\mathscr{T}_+A^*$ which equivalent to P'. By the construction of Q one sees that R_+ is equivalent to a subprojection of P'.

(ii) We now show that $\mathscr{I}_{R_+}$ is *-isomorphic to $\mathscr{T}_+ \otimes \mathscr{K}$. Clearly, $\mathscr{I}_{R_+}$ is generated by $\mathscr{T}_+$, and thus $\mathscr{I}_{R_+} \otimes \mathscr{K} \cong \mathscr{T}_+ \otimes \mathscr{K}$ by [6,2.8]. Since $\mathscr{T}_+$ is a purely infinite, simple C*-algebra, the stabilization $\mathscr{T}_+ \otimes \mathscr{K}$ is also a purely infinite, simple C*-algebra. Thus, $\mathscr{I}_{R_+}$ is a purely infinite simple C*-algebra. Observe that $\mathscr{I}_{R_+}$ is a non-unital separable C*-algebra. Then $\mathscr{I}_{R_+}$ must be stable by our result in [44], asserting that a σ-unital, purely infinite, simple C*-algebra is either unital or stable. Therefore, $\mathscr{I}_{R_+} \cong \mathscr{T}_+ \otimes \mathscr{K}$

Now $C_r^*(\Gamma, R_+)/\mathscr{I}_{R_+} \cong C_r^*\Gamma$ which is a simple C*-algebra by a result in

[33]. We conclude that $\mathscr{I}_{R_+}$ is the only nontrivial closed ideal of $C_r^*(\Gamma, R_+)$. We conclude this note with the following problem:

4.6. PROBLEM. *Calculate K_0, K_1, and* Ext *of $\mathscr{T}_+$ in case*

$$\Gamma = \mathscr{F}_\infty,\ \mathsf{Z}_{n_0} * \mathscr{F}_\infty,\ \underbrace{(\mathsf{Z}_{n_1} * \mathsf{Z}_{n_2} * \dots)}_{m} * \mathscr{F}_n,\ \underbrace{(\mathsf{Z}_{n_1} * \mathsf{Z}_{n_2} * \dots)}_{m} * \mathscr{F}_\infty,$$

where $2 \leq m \leq +\infty$ and $1 \leq n < +\infty$.

REFERENCES

1. C. A. Akemann and T.-Y. Lee, *Some simple C*-algebras associated with free groups,* Indiana Univ. J. Math. 29 (1980), 505–511.
2. C. A. Akemann and P. A. Ostrand, *Computing norms in group C*-algebras,* Amer. J. Math. 98 (1976), 1015–1047.
3. J. Anderson, *A C*-algebra $\mathscr{A}$ for which Ext($\mathscr{A}$) is not a group,* Ann. of Math. 107 (1978), 455–458.
4. J. Anderson, B. Blackadar and U. Haagerup, *Minimal projections in the reduced group C*-algebra of* $\mathsf{Z}_n * \mathsf{Z}_m$, J. Operator Theory26 (1991), 3–23 .
5. W. Arveson, *C*-algebras associated with sets of semigroups of isometries,* Internal. J. Math. 2 (1991), 235–255 .
6. L. G. Brown, *Stable isomorphism of hereditary C*-algebras,* Pacific J. Math. 71 (1977), 335–348.
7. L. G. Brown and G. K. Pedersen, *C*-algebras of real rank zero,* J. Funct. Anal. 99 (1991), 131–149.
8. M.-D. Choi, *A simple C*-algebra generated by two finite-order unitaries,* Canad. J. Math. 31 (1979), 867–880.
9. M.-D. Choi, *The full C*-algebra of the free group on two generators,* Pacific J. Math. 87 (1980), 41–48.
10. M.-D. Choi and E. G. Effros, *Separable nuclear C*-algebras and injectivity,* Duke Math. J. 43 (1976), 309–322.
11. M. -D. Choi and E. G. Effros, *Nuclear C*-algebras and injectivity : The general case,* Indiana Univ. Math. J. 26 (1977), 443–446.
12. J. M. Cohen and A. Figà-Talamanca, *Idempotents in the reduced C*-algebra of a free group,* Proc. Amer. Math. Soc. 103 (1988), 779–782.
13. J. M. Cohen, *C*-algebras without idempotents,* J. Funct. Anal. 33 (1979), 211–216.
14. A. Connes, *Non-commutative differential geometry,* Inst. Hautes Études Sci. Publ. Math. 62 (1982), 257–360.
15. J. Cuntz, *K-theoretic amenability for discrete groups,* J. Reine Angew. Math. 344 (1983), 180–195.
16. J. Cuntz, *K-theory for certain C*-algebras,* Ann. of Math. 131 (1981), 181–197.
17. J. Cuntz, *Simple C*-algebras generated by isometries,* Comm. Math. Phys. 57 (1977), 173–185.
18. J. Cuntz and W. Krieger, *A class of C*-algebras and topological Markov chains,* Invent. Math. 56 (1980), 251–268 .
19. H. T. Dinh, *Discrete product systems and their C*-algebras,* J. Funct. Anal., to appear .
20. E. G. Effros, *Why the circle is connected: An introduction to quantized topology,* Math. Intelligencer 11 (1989), 27–34.
21. G. A. Elliott, *A classification of certain simple C*-algebras,* preprint.

22. M. Enomoto, H. Takehana and Y. Watatani, *C^*-algebras on free semigroups as extensions of Cuntz algebras,* Math. Japon. 24 (1980), 527–531.
23. A. Figa-Talamance and M. A. Picardello *Harmonic Analysis on Free Groups,* Lecture Notes in Pure and Appl. Math. 87, New York and Basel 1983.
24. D. Handelman and H-S Yin, *Toeplitz extensions and automorphisms affiliated with linear characters of groups,* Bull. London Math. Soc. 20 (1988), 54– 60.
25. U. Haagerup, *An example of a non nuclear C^*-algebra, which has the metric approximation property,* Invent. Math. 50 (1979), 279–293.
26. V. Jones, *Subfactors and Knots,* CBMS Regional Conference Series in Math. 80 (1988).
27. R. V. Kadison and J. R. Ringrose *Fundamentals of the Theory of Operator Algebras, Vol.* II Academic Press, 1986.
28. E. C. Lance, *K-theory for certain group C^*-algebras,* Acta Math. 151 (1983), 209–230.
29. H. Lin and S. Zhang, *On infinite simple C^*-algebras,* J. Funct. Anal. 100 (1991), 221–231.
30. G. J. Murphy, *Toeplitz operators and algebras,* Math. Z. 208 (1991), 355–362.
31. M. V. Pimsner, *KK-groups of crossed products by groups acting on trees,* Invent. Math. 86 (1986), 603–634.
32. M. V. Pimsner and D. Voiculescu, *K-groups of reduced crossed products by free groups,* J. Operator Theory 8 (1982), 131–156.
33. W. L. Paschke and N. Salinas, *C^*-algebras associated with free products of groups,* Pacific J. Math. 82 (1979), 211–221.
34. G. K. Pedersen, *Measure theory for C^*-algebras* II, Math. Scand. 22 (1968), 63–74 .
35. R. T. Powers, *Simplicity of the C^*-algebra associated with the free group on two generators,* Duke Math. J. 42 (1975), 151–156.
36. G. Price, *The C^*-algebras generated by pairs of semigroups of isometries satisfying certain commutation relations,* Pacific J. Math. 146 (1990), 315–330.
37. M. Rørdam, *Classification of inductive limits of Cuntz algebras,* preprint .
38. M. Rørdam, *Classification of Cuntz-Krieger algebras,* preprint.
39. E. Størmer, *Entropy of some automorphisms of the type* II_1*-factor of the free group in infinite number of generators,* Invent. Math. 110 (1992), 63–73.
40. T. Sunada, *Group C^*-algebras and the spectrum of a periodic Schrödinger operator on a manifold,* Can. J. Math. 44 (1992), 180–193.
41. D. Voiculescu, K. J. Dykema, and A. Nica *Free Random Variables,* CRM Monograph Series, Amer. Math. Soc. 1991.
42. S. Wassermann, *C^*-algebras associated with groups with Kazhdan's property* T, Ann. of Math. 134 (1991), 423–431 .
43. H.-S. Yin, *Two C^*-algebras related to the discrete Heisenberg group,* J. Operator Theory 19 (1988), 275–282.
44. S. Zhang, *A property of purely infinite simple C^*-algebras,* Proc. Amer. Math. Soc. 109 (1990), 717–720.
45. S. Zhang, *Certain C^*-algebras with real rank zero and their corona and multiplier algebras,* I, Pacific J. Math. 155 (1992), 169–197.
46. S. Zhang, *K_1-groups, quasidiagonality, and interpolation by multiplier projections,* Trans. Amer. Math. Soc. 325 (1991), 793–818 .
47. S. Zhang, *Purely infinite simple C^*-algebras arising from reduced group C^*-algebras,* preprint.
48. S. Zhang, *Toeplitz algebras and infinite simple C^*-algebras associated with reduced group C^*-algebras,* preprint .

DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF CINCINNATI
CINCINNATI, OHIO 45221-0025
U.S.A.
E-mail address: szhang@ucbeh.san.uc.edu