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MAXIMALITY PROPERTIES FOR ONE-DIMENSIONAL ANALYTICALLY IRREDUCIBLE LOCAL GORENSTEIN AND KUNZ RINGS

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Introduction.

Let S be a numerical semigroup (just a semigroup in the sequel), i.e. a subsemigroup of $\mathsf{N} = \{0, 1, 2, \ldots\}$ with finite complement to N, and let $g(S)$ be its Frobenius number, $g(S) = \max\{i \in \mathsf{Z} \mid i \notin S\}$. The semigroup S is called *symmetric* if for each $i \in \mathsf{Z}$ we have $i \in S$ or $g(S) - i \in S$. It follows easily that $g(S)$ is odd if S is symmetric. If S is symmetric and T is a semigroup that strictly contains S, then $g(S) - s \in T$ for some $s \in S$, hence $g(S) = g(S) - s + s \in T$, so that $g(T) < g(S)$. It is not hard to see that a semigroup U with $g(U)$ odd can be extended to a symmetric semigroup $S \supseteq U$ with $g(S) = g(U)$ (in general S is not unique). Hence, given an odd integer $g \geq -1$, the symmetric semigroups are exactly the maximal semigroups S with respect to inclusion, that satisfy $g(S) = g$. A *pseudosymmetric* semigroup S is a semigroup such that $g(S)$ is even and for any $i \in \mathsf{Z}$ we have $i \in S$ or $g(S) - i \in S$ or $i = g(S)/2$. Given an even natural number g, the pseudosymmetric semigroups are exactly those semigroups S which are maximal with respect to inclusion and satisfy $g(S) = g$, (cf. [B-D-F2, Lemma 2] or [B-D-F3, Lemma I.1.9]). These results can be formulated in terms of semigroup rings. If S is a semigroup and k a field, the semigroup ring $k[[S]]$ is the set of power series $\{\sum_{i=0}^{\infty} a_i X^i \mid a_i \in k, i \in S\}$. A semigroup ring is a Noetherian one-dimensional local ring A with integral closure $\bar{A} = k[[X]]$, which is finite over A. The conductor $\bar{A} := \{z \in k((X)) \mid z\bar{A} \subseteq A\}$ equals $X^{g+1}k[[X]]$, where $g = g(S)$. Thus, among the semigroup rings $k[[S]]$ with conductor $X^c k[[X]]$, the maximal ones with respect to inclusion are exactly those with S symmetric (if c is even) or pseudosymmetric (if c is odd). Semigroup rings are standard examples of one-dimensional analytically irreducible local rings, i.e. local rings (A, m) such that the m-adic completion $\hat{A}$ is a domain. An equivalent definition of a one-

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dimensional analytically irreducible local ring A is that the integral closure $(\bar{A}, \bar{m})$ is a finite A-module and a DVR (cf. [Ka]). It is a classical result by Kunz that if (A, m) is analytically irreducible, and furthermore $A/m \simeq \bar{A}/\bar{m}$, then A is Gorenstein if and only if the semigroup of values of nonzero elements in A, $v(A)$ is symmetric (cf. [Ku]). Thus, given a DVR $(V, \pi V)$ and an ideal $\pi^c V$ in V, we investigate which are the maximal subrings of V (with respect to inclusion) with conductor $\pi^c V$ in V, or as we will say, which are the rings maximal with fixed conductor.

In [F-G-H] rings between $k + X^c k[[X]]$ and $k[[X]]$ have been studied. Such rings A are local one-dimensional and Noetherian with integral closure $\bar{A} = k[[X]]$. It has been shown that, if c is even, the rings in this class maximal with fixed conductor $A : \bar{A} = X^c k[[X]]$ are exactly the Gorenstein rings. This has been generalized in [B-D-F1] to one-dimensional analytically irreducible rings (A, m) which contain a field isomorphic to $\bar{A}/\bar{m}$. It was also shown in this more general setting that if c is odd, the maximal rings with fixed conductor $\bar{m}^c$ are exactly those, which have a pseudosymmetric semigroup of values, the so called Kunz rings (cf. [B-D-F2, Proposition 17(a)] or [B-D-F3, Proposition II.1.12]).

In this paper we will generalize the results in [B-D-F2], [B-D-F3] to all one-dimensional analytically irreducible local rings that are *residually rational*, i.e. that satisfy $A/m \simeq \bar{A}/\bar{m}$ (cf. Theorem 1). ($A/m \simeq \bar{A}/\bar{m}$ is true e.g. if A/m is algebraically closed.) We will also show, that if we drop the condition residually rational, Gorenstein and Kunz rings are still maximal with a given conductor (cf. Propositions 2 and 4), but there are rings maximal with fixed conductor that are neither Gorenstein nor Kunz (cf. Corollary 14 and Example). A class of one-dimensional analytically irreducible rings (A, m) generalizing the semigroup rings are then studied. Any such generalized semigroup ring A, with $A/m = k$ and $\bar{A}/\bar{m} = K$, that is not a semigroup ring (i.e. with $k \neq K$), is not residually rational. In this case the semigroup of values $S = v(A)$ does not give much information on A. As a consequence of Proposition 8, we get for example that there is, for any semigroup S, a one-dimensional analytically irreducible Gorenstein ring with S as semigroup of values. We get however that a ring A of this class, with $k \neq K$, is Gorenstein if and only if $\dim_k K = 2$ and A is maximal with fixed conductor (cf. Theorem 10) and it is Kunz if and only if $\dim_k K = 3$ and A is maximal with fixed conductor (cf. Theorem 12). All the generalized semigroup rings with $\dim_k K > 3$ and maximal with fixed conductor are characterized in Proposition 13. We get also that the Gorenstein rings of this class are characterized by a nice "symmetric" property, that generalizes the mentioned result of Kunz (cf. [Ku]) in the residually rational case (cf. Corollary 11). A similar symmetric property characterizes the Gorenstein rings in a more general situation (cf. Theorem 15).

The residually rational case.

Let $(V, \pi V)$ be a DVR, let $C = \pi^c V$ be an ideal in V, and let $O_c = \{A \mid (A, m)$ Noetherian ring, $\bar{A} = V, A : V = C, A/m \simeq V/\pi V\}$, ordered by inclusion. Let H be a canonical ideal of A, i.e. a fractional ideal of A such that for any fractional A- ideal J we have $H : (H : J) = J$. Such an ideal exists, cf. [H-K, Satz 6.21]. If $z \neq 0$ is an element in the field of fractions of A, then zH is a canonical ideal if and only if H is a canonical ideal. Hence we can assume that H is a usual ideal. An element z of minimal value in H is then a minimal reduction of H, i.e. $zH^n = H^{n+1}$ if $n \gg 0$, cf. [B-F, Corollary 17]. Finally, replacing H with $z^{-1}H$, we achieve that we can assume (as we do from now on) that $A \subseteq H \subseteq \bar{A}$. We have $A \subseteq H$ since $1 \in H$ and H is an A-module. We have $H \subseteq \bar{A}$ since every element in H has a non-negative value. A fractional ideal H is a canonical ideal if and only if $v(H) = v(A) \cup \{x \in \mathbf{N} \mid x \notin v(A), g - x \notin v(A)\}$, cf. [J, Satz 5]. If $I \subseteq J$ are fractional ideals, we have $l_A(J/I) = l_A(H : I/H : J)$. The ring A is Gorenstein if and only if $A = H$. As in [B-D-F2], [B-D-F3] we call A a Kunz ring if $v(A)$ is a pseudosymmetric semigroup. The ring A is Kunz if and only if $l_A(H/A) = 1$, cf. [B-F, Proposition 21]. Recall moreover that, since A is residually rational, then $l_A(J/I) = |v(J) \setminus v(I)|$, cf. [M, Proposition 1]. With these hypotheses and notation, we prove the following:

THEOREM 1. *A is Gorenstein if and only if A is maximal in O_c and c is even. A is Kunz if and only if A is maximal in O_c and c is odd.*

PROOF. If A is Gorenstein, then $v(A)$ is symmetric, $g = g(v(A))$ is odd and $c = g + 1$ is even. Extending A with a new element x means extending $v(A)$, since A is residually rational. Since $\mathbf{Z} \setminus v(A) = g - v(A)$, this means that $v(x) = g - s$ for some $s \in v(A)$. This gives also an element of value g in $A[x]$ (multiply x with an element of value s in A). Thus $C = A : V$ is strictly contained in $A[x] : V$ and A is maximal in O_c. If A is Kunz, then $v(A)$ is pseudosymmetric, $g = g(v(A))$ is even and c is odd. Since in this case $\mathbf{Z} = v(A) \cup (g - v(A)) \cup \{g/2\}$, the same argument applies. Now for the other implication. So suppose that A is neither Gorenstein nor Kunz. We shall show that A can be extended to a ring with the same conductor. Let H be a canonical ideal of A, $A \subseteq H \subseteq \bar{A}$. We know that $v(H) \setminus v(A)$ is symmetric around $g/2$. If A is neither Gorenstein nor Kunz, then $|v(H) \setminus v(A)| \geq 2$, hence there is an $h \in v(H) \setminus v(A)$ with $h < g/2$. Choose h minimal and let $v(x) = h, x \in H$. Then x belongs to a minimal system of generators for H, since every element of H of smaller value than h belongs to A. Let H be generated by $1, x, h_2, ..., h_t$. Let $L = A + Ah_2 + \cdots + Ah_t + xmA$. Then $l_A(H/L) = 1$ since $v(H) \setminus v(L) = \{h\}$. Thus $l_A(H : L/H : H) = l_A(H : L/A) = 1$, so $H : L = A + Ay$ for some y. We know that $v(H : L) \setminus v(H)$ consists of one value, we claim that this value is $g - h$, i.e. $v(y) = g - h$. We can't have $v(y) = g$, since no element of value g multiplies 1 into H. In the same way we see that $v(y) \notin g - v(A)$. Thus $v(y) = g - h'$ for some h', $\notin v(A) \cup (g - v(A))$. If $h' > h$, then there exists $z \in H, v(z) = h'$. If $v(y) = g - h'$, then $v(zy) = g$ which is a contradiction to $zy \in H$, since $g \notin v(H)$. Thus $v(y) = g - h > g/2$. Since $v(y^2) > g$ we have $y^2 \in A$, so $A + Ay = A[y]$, a ring. We also get that $A[y] : V = C$ since $C = A : V \subseteq A[y] : V$ and $g \notin v(A[y])$.

The maximality of Gorenstein and Kunz rings.

We suppose in the sequel that (A, m) is a local analytically irreducible one-dimensional ring with integral closure $\bar{A}$, and that $C = A : \bar{A}$. Then A is Gorenstein if and only if $l_A(\bar{A}/A) = l_A(A/C)$. We define A to be Kunz if $l_A(\bar{A}/A) = l_A(A/C) + 1$ (cf. [B-F]). This extends the definitions above for the residually rational case.

PROPOSITION 2. *If A is Gorenstein, then A is maximal with fixed conductor (i.e. for any ring B strictly between A and $\bar{A}$ we have that $B : \bar{A}$ is strictly larger than $A : \bar{A}$).*

PROOF. We can assume that A is not a DVR. If A is Gorenstein, then any fractional ideal is divisorial, cf. [B, Theorem 6.3]. This gives in particular that A has a unique minimal overring, namely $A : m$. Now $A : m = m : m$ since if $A : m$ strictly contains $m : m$, then there is an x such that $xm \subseteq A, xm \not\subseteq m$, so $xm = A$ and m is a principal ideal, which contradicts the hypothesis that A is not a DVR. Hence $A : m$ is the unique minimal overring of A. Let zA be a minimal reduction of m. Then $(A : m) : \bar{A} = A : (m\bar{A}) = A : (z\bar{A}) = z^{-1}(A : \bar{A})$ (cf. [B-F, Proposition 16]), which is strictly larger than $A : \bar{A}$. Hence a Gorenstein ring is maximal with fixed conductor.

To prove the corresponding fact for Kunz rings we need a lemma.

LEMMA 3. *Suppose that (A, m) and (B, n) are local artinian rings with $m = A \cap n$, with A strictly contained in B, and with $A/m \simeq B/n$. Then $l_A(A) < l_B(B)$.*

PROOF. If $l_B(B) = 1$, then $B = B/n$ and, since $A/m \simeq B/n$, there is for each $b \in B$ an $a \in A$ such that $b - a \in m$, hence $b \in A$, so $A = B$ and the claim is vacously true. Now suppose $l_B(B) > 1$. Take an element $x \in B$ with

$xn = 0$. By induction $l_A(A/A \cap Bx) < l_B(B/Bx)$. Since $l_B(Bx) = 1$ it is enough to show that $l_A(A \cap Bx) \leq 1$. If $bx \in A \cap Bx$ then, since for some $a \in A$ we have $b - a \in m$, we get $(b-a)x \in mx = 0$, hence $bx = ax$ and $A \cap Bx = Ax$. Since $xm = 0$ we have $l_A(Ax) \leq 1$.

PROPOSITION 4. *If A is Kunz, then A is maximal with fixed conductor (i.e. for any ring B strictly between A and $\bar{A}$ we have that $B : \bar{A}$ is strictly larger than $A : \bar{A}$).*

PROOF. If A is Kunz with $A/m = k$, then $l_A(A : m/A) = 2$ (cf. [B-F, Proposition 21]) and $A : m/m$ is either a field extension of k of degree 3, or $A : m/m \simeq k[X]/(X^3)$, or $A : m/m \simeq k[X, Y]/(X, Y)^2$ (cf. [B-F, Proposition 27]). Notice that in the last two cases the residue field of $A : m/m$ is k. In the first case, if $(A, m) \subseteq (B, n) \subseteq (\bar{A}, M)$, then $B = A$ or $B = \bar{A}$, cf. [B-F, Proposition 27]. Otherwise, it suffices to assume that B is a minimal overring of A. Then $B \subseteq A : m$, cf. [F-O,Théorème 2.2]. It is clear that $l_A(\bar{A}/A) > l_B(\bar{A}/B)$. Suppose that $A : \bar{A} = B : \bar{A} = C$. Then Lemma 3, used on A/C and B/C, gives $l_A(A/C) = l_{A/C}(A/C) < l_{B/C}(B/C) = l_B(B/C)$. Hence, since A is Kunz, then $1 = l_A(\bar{A}/A) - l_A(A/C)$ gives $l_B(\bar{A}/B) - l_B(B/C) < 0$ which is a contradiction.

On the other hand there are analytically irreducible rings which are maximal with fixed conductor, which are neither Gorenstein nor Kunz (cf. the following Corollary 14 and Example).

Generalized semigroup rings.

Let $A = \{\sum_{i=0}^{\infty} a_i X^i, a_i \in k_i\}$, where k_i are subfields of a field L or $k_i = \{0\}$ and $k_i \neq \{0\}$ for almost all i. We assume further that for all i,j we have $k_i k_j \subseteq k_{i+j}$. (For simplicity we set $k_i = \{0\}$ if $i < 0$.) In particular all k_i are vector spaces over k_0 and we will denote $[k_i : k_0]$ by just $\dim k_i$. Then $S = \{i \mid k_i \neq \{0\}\}$ is a semigroup and A is a ring, and we call such rings generalized semigroup rings. Let $K = \cup k_i$, and let $N = N(A) = \sup\{i \mid k_i \neq K\}$.

PROPOSITION 5. *Suppose that the generalized semigroup ring A is Noetherian. Then we have* $\dim K < \infty$ *and* $N < \infty$.

PROOF. Let V_n be the vector space generated by $\cup_{i=1}^{n-1} k_i k_{n-i}$ over k_0. Let $\{b_1^{(1)}, b_2^{(1)}, \ldots\}$ be a k_0-basis for k_1. Then $\{Xb_1^{(1)}, Xb_2^{(1)}, \ldots\}$ is part of a minimal system of generators for $m = \sum_{i \geq 1} X^i k_i$. Let $\{b_1^{(2)}, b_2^{(2)}, \ldots\}$ be elements of k_2 that complete a k_0-basis of V_2 to a k_0-basis of k_2. Then $\{Xb_1^{(1)}, Xb_2^{(1)}, \ldots, X^2 b_1^{(2)}, X^2 b_2^{(2)}, \ldots\}$ is part of a minimal system of generators for m. Continuing like this we see that $\dim m/m^2 = \infty$ if $\dim K = \infty$, and A

is not Noetherian in this case. Let $g = g(S)$. (Note that $N \geq g$.) If $\dim K < \infty$, we must have $k_h = K$ for some h. If $i > g$ we have $i \in S$, hence $k_{h+i} = K$ for all $i > g$, so $N \leq h + g + 1$.

The following lemma is easy.

LEMMA 6. *Suppose that A is a generalized semigroup ring with* $\dim K < \infty$. *Then:*

(1) $\bar{A} = K[[X]]$ *and* $v(A) = \{i \mid k_i \neq \{0\}\}$.
(2) $l_A(\bar{A}/A) = \sum_{i=0}^{N}(\dim K - \dim k_i)$.
(3) $C = A : \bar{A} = X^{N+1}K[[X]]$.
(4) $l_A(A/C) = \sum_{i=0}^{N} \dim k_i$.

PROPOSITION 7. *The generalized semigroup ring A is Noetherian if and only if* $\dim K < \infty$.

PROOF. By Proposition 5, if A is Noetherian, $\dim K < \infty$. That the condition $\dim K < \infty$ is also sufficient for A to be Noetherian follows from Eakin's theorem and (2) above.

From now on, we suppose that the generalized semigroup ring A is Noetherian (i.e. that $\dim K < \infty$). We will now investigate when A is Gorenstein and Kunz, respectively.

PROPOSITION 8. *For any semigroup S of Frobenius number g, any $N \geq 2g + 1$ and any $t \geq 1$, there is a generalized semigroup ring A of type t with $v(A) = S$ and $N(A) = N$.*

PROOF. Let S be any semigroup. Let $g = g(S)$ and $N \geq 2g + 1$. Let $k \subset K$ be fields with $\dim K = t + 1$ and let $A = \sum_{i=0}^{\infty} X^i k_i$, where $k_i = \{0\}$ if $i \notin S$, $k_i = k$ if $i \in S$ and $N - i \in S, k_i = K$ if $i \in S$ and $N - i \notin S$. We claim that $A : m = A + X^N K[[X]]$, hence that $l_A(A : m/A) = t$. Since $X^N K[[X]] \subseteq A : m$, it is enough to show that, if $x = \sum_{i=0}^{\infty} X^i a_i \in A : m$, then for any $i < N$ we have $X^i a_i \in A$. Let j be the smallest integer such that $a_j \neq 0$, i.e. let $v(x) = j$. Suppose that $j < N$. If $j \notin S$, then $j \leq g$, hence $N - j \geq g + 1$, thus $N - j \in S$ and $k_{N-j} = K$ since $N - (N - j) = j \notin S$. Let $b \in K \setminus k$. Since $y = X^{N-j} a_j^{-1} b \in m$, we have $xy = X^N b + \cdots \notin A$, a contradiction. So we can suppose $j \in S$. If $N - j \in S$, then $X^{N-j} \in m$, thus $a_j \in k$ (otherwise $X^j X^{N-j} a_j = X^N a_j \notin A$). Hence in this case $X^j a_j \in A$. If $N - j \notin S$, since $X^j K \subseteq A$, we have also $X^j a_j \in A$. Since $x \in A : m$ and $X^j a_j \in A \subseteq A : m$, we have $z = x - X^j a_j \in A : m$. Replacing the element x with z and arguing as above we get that, if $v(z) = j_1, X^{j_1} a_{j_1} \in A$ as requested.

From the previous proposition we get in particular that, for any semigroup S, there is a one-dimensional analytically irreducible Gorenstein ring with S as semigroup of values. To get further results on Gorenstein or Kunz generalized semigroup rings we need the following lemma.

LEMMA 9. *Let A be a generalized semigroup ring with $v(A) = S, g = g(S)$, and $N(A) = N$. Suppose that A is Gorenstein or Kunz. Then:*

(1) *$N = g$ if and only if A is a semigroup ring (*i.e. *$k_i = K$ or $k_i = \{0\}$ for all i).*

(2) *If $N > 0$, then* $\dim K \leq 2$.

PROOF. For (1), consider the semigroup ring $A' = \sum X^i k_i'$, where $k_i' = K$ if $k_i \neq \{0\}$ and $k_i' = \{0\}$ otherwise. Suppose that $N = g$. If A is not a semigroup ring, then $A' \neq A$. Thus, since $C = A : \bar{A} = C' = A' : \bar{A} = X^{N+1}K[[X]]$, we have $l_A(A'/C') > l_A(A/C)$ and, since $\bar{A} = \bar{A'} = K[[X]]$, we have $l_A(\bar{A'}/A') < l_A(\bar{A}/A)$. Hence $l_A(\bar{A}/A) - l_A(A/C) \geq 2$, a contradiction. Conversely, if A is a semigroup ring, then $N = g$.

(2) Define $A_2 = \sum k_i' X^i$, where $k_i' = \{0\}$ if $k_i = \{0\}, k_i' = \mathbb{Q}$ if $k_i \neq K$ and $k_i \neq \{0\}$, and finally $k_i' = \mathbb{Q}(\sqrt{2})$ if $k_i = K$. Then A_2 is a generalized semigroup ring with the same semigroup of values as A and with $N(A_2) = N(A)$. We get $l_A(\bar{A}/A) \geq l_A(\bar{A}_2/A_2), l_A(A/C) \leq l_A(A_2/C_2)$, where $C_2 = A_2 : \bar{A}_2$. If $\dim K \geq 3$, then $l_A(\bar{A}/A) \geq l_A(\bar{A}_2/A_2) + 1$ and $l_A(A/C) \leq l_A(A_2/C_2) - 1$. Since $l_A(\bar{A}_2/A_2) - l_A(A_2/C_2) \geq 0$, we have $l_A(\bar{A}/A) - l_A(A/C) \geq 2$, hence A is neither Gorenstein nor Kunz.

THEOREM 10. *Let A be a generalized semigroup ring with $v(A) = S$.*

(1) *If A is Gorenstein then dim* $\dim K \leq 2$.

(2) *If $k_0 = K$, then A is Gorenstein if and only if S is symmetric.*

(3) *If* $\dim K = 2$, *then the following are equivalent:*

i) *A is Gorenstein.*

ii) *A is maximal with fixed conductor.*

iii) $\dim k_i + \dim k_{N-i} = 2$ *for any i.*

PROOF. (1) According to Lemma 9 we need only consider the case $N \leq 0$. Then $A = k_0 + XK[[X]]$ and Lemma 6 gives that A is Gorenstein if and only if $\dim K \leq 2$.

(2) If $k_0 = K$, then A is a semigroup ring, so A is Gorenstein if and only if S is symmetric, cf. [Ku].

(3) Let $\dim K = 2$. If A is Gorenstein, then by Proposition 2, A is maximal with fixed conductor, so i) $\Rightarrow$ ii).

ii) $\Rightarrow$ iii) Let A be maximal with fixed conductor $C = X^{N+1}K[[X]]$. Notice that $k_N = k_0$. Indeed $k_N \neq K$ since the conductor is C, and, if $k_N = \{0\}$, then A can be properly extended to the ring $B = A + X^N k_0$, that has the same conductor $C = B : \bar{A}$, a contradiction. Thus, since A is a ring and $\dim K = 2$, we have $\dim k_i + \dim k_{N-i} \leq 2$ for any i. Suppose $\dim k_i + \dim k_{N-i} < 2$ for some i and let i be the smallest integer such that $\dim k_i = 0$ and $\dim k_{N-i} < 2$. Consider $A' = A + X^{N-i}K$. To get a contradiction, we prove that A' is a ring. This is enough since $A' : \bar{A} = A : \bar{A}$. In order to show that A' is a ring, it is enough to show that $k_{N-i+j} = K$ for any $j \in S \cup \{N - i\}$. If $k_{N-i+j} \neq K$, then, by the minimality of i, $N - (N - i + j) = i - j \in S$. Thus, if $j \in S$, we get $i - j + j = i \in S$, a contradiction, and if $j = N - i$, we get $i - (N - i) = -N \in S$, which is also a contradiction.

iii) $\Rightarrow$ i) Since $\dim k_i + \dim k_{N-i} = 2$ for all i, we get by Lemma 6 that $l_A(A/C) = \sum_{i=0}^{N} \dim k_i = \frac{1}{2}\sum_{i=0}^{N}(\dim k_i + \dim k_{N-i}) = \frac{1}{2}2(N+1) = N+1$. On the other hand, again by Lemma 6, $l_A(\bar{A}/A) = \sum_{i=0}^{N}(\dim K - \dim k_i) = (N+1)\dim K - \sum_{i=0}^{N} \dim k_i = (N+1)2 - (N+1) = N+1$, and so A is Gorenstein.

COROLLARY 11. *A generalized semigroup ring A is Gorenstein if and only if there exists an integer M such that for each integer i, $\dim k_i + \dim k_{M-i} = \dim K$.*

PROOF. If A is Gorenstein, then, by Theorem 10, $\dim K = 1$ or 2. If $k_0 = K$ we have a semigroup ring with $S = v(A)$ symmetric. In this case we have $\dim k_i = 1$ if and only if $k_i \neq \{0\}$, i.e. if and only if $i \in S$. Since S is symmetric, we have that $i \in S$ if and only if $g - i \notin S$, where $g = g(S)$. Thus, setting $M = g$, we get $\dim k_i + \dim k_{M-i} = 1 = \dim K$ for any i. If $\dim K = 2$, we showed in Theorem 10 (3), ii)$\Rightarrow$iii), that $\dim k_i + \dim k_{N-i} = 2$ for any i. Conversely, suppose there exists an M such that $\dim k_i + \dim k_{M-i} = \dim K$ for any i. Thus $\dim k_0 + \dim k_M = 1 + \dim k_M = \dim K$. Since $k_M \subseteq K$, $\dim k_M$ divides $\dim K$ and so the previous equality gives $\dim K = 1$ or 2. If $\dim K = 1$, then A is a semigroup ring with $S = v(A)$ such that $i \in S$ if and only if $M - i \notin S$. So S is a symmetric semigroup with $g(S) = M$ and A is Gorenstein. If $\dim K = 2$, then $k_M = k_0$ and $k_i = K$ if $i > M$. Thus $M = N(A)$ and, by Theorem 10 (3), A is Gorenstein.

REMARK. Let A be a generalized semigroup ring with $\dim K = 2$. Then A can be extended to a generalized semigroup ring B, maximal with fixed conductor. By Theorem 10, B is Gorenstein and of course $v(B) \supseteq v(A)$. Notice however that B is not unique. Consider for example $A = \mathbb{Q} + X^2\mathbb{Q} + X^4\mathbb{Q}(\sqrt{2})[X]]$. Then A can be extended both to $B_1 = \mathbb{Q} + X\mathbb{Q} + X^2\mathbb{Q} + X^3\mathbb{Q} + X^4\mathbb{Q}(\sqrt{2})[[X]]$ and to $B_2 = \mathbb{Q} + X^2\mathbb{Q}(\sqrt{2}) + X^3\mathbb{Q} + X^4\mathbb{Q}(\sqrt{2})[[X]]$. Notice moreover that, if $N \geq 2g+1$ (where $N = N(A)$ and $g = g(v(A))$), then there is a unique Gorenstein overring B of A with $v(B) = v(A) = S$ and with $B : \bar{A} = X^{N+1}K[[X]]$. Indeed B is the ring constructed as in the proof of Proposition 8. i.e. $B = \sum X^i k_i'$, where, if $i < N/2, k_i' = k_i$ and, if $i \geq N/2, \{i \mid k_i = k_0\} = \{i \mid N - i \in S\}$.

THEOREM 12. *Let A be a generalized semigroup ring with $v(A) = S$.*
(1) *If A is Kunz, then* $\dim K = 1$ *or 3.*
(2) *If $k_0 = K$, then A is Kunz if and only if S is pseudosymmetric.*
(3) *If* $\dim K = 3$, *then the following are equivalent*
i) *A is Kunz.*
ii) *A is maximal with fixed conductor.*
iii) $A = k_0 + XK[[X]]$.

PROOF. (1) If $N > 0$ we need, by Lemma 9 (2), only show that $\dim K = 2$ is not possible. If $\dim K = 2$, we have by Lemma 6 that $l_A(\bar{A}/A) - l_A(A/C) = (N+1)\dim K - 2\sum_{i=0}^{N} \dim k_i$ which is even, so A is not Kunz. If $N = 0$ we have that $l_A(\bar{A}/A) - l_A(A/C) = \dim K - 2 = 1$ implies $\dim K = 3$.

(2) If $k_0 = K$, then A is a semigroup ring, so A is Kunz if and only if S is pseudosymmetric, cf. [B-D-F2, Proposition 17(a)] or [B-D-F3, Proposition II.1.12].

(3) That i) $\Rightarrow$ ii) is proved in Proposition 4. Since 3 is a prime there are no fields between k_0 and K. Hence $k_N = \{0\}$ or $k_N = k_0$. If $N > 0$ and V is a 2-dimensional vector space over $k_0, k_0 \subseteq V \subseteq K$, then A is strictly contained in $A' = A + X^N V + X^{N+1}K[[X]]$ and $A' : \bar{A} = A : \bar{A} = X^{N+1}K[[X]]$, so A is not maximal with fixed conductor. Hence $A = k_0 + XK[[X]]$, so ii)$\Rightarrow$ iii). To prove iii) $\Rightarrow$ i) observe that, by Lemma 6, $l_A(\bar{A}/A) = \dim K - 1 = 2$ and $l_A(A/C) = 1$, so A is Kunz.

Let A be a generalized semigroup ring. We know that if $\dim K = 1$ (i.e. A is a semigroup ring), then A is maximal with fixed conductor if and only if A is Gorenstein or Kunz. By the results above, if $\dim K = 2$, then A is maximal with fixed conductor if and only if A is Gorenstein and, if $\dim K = 3$, then A is maximal with fixed conductor if and only if A is Kunz. If $\dim K > 3$, we have the following:

PROPOSITION 13. *Let A be a generalized semigroup ring with* $\dim K > 3$. *Then A is maximal with fixed conductor if and only if $A = k_0 + K[[X]]$ and there are no fields strictly between k_0 and K.*

PROOF. Let A be maximal with fixed conductor. Suppose $N > 0$. Since $\dim K > 3$ and $\dim k_N$ divides $\dim K$, we have $\dim k_N < \dim K - 1$. So, if V is a $(\dim K - 1)$-dimensional vector space over k_0 strictly between k_N and K, then A is strictly contained in $A' = A + X^N V + X^{N+1}K[[X]]$ and $A' : \bar{A} = A : \bar{A} = X^{N+1}K[[X]]$, so A is not maximal with fixed conductor. Hence $A = k_0 + XK[[X]]$. If there is a field k strictly between k_0 and K, then $k + K[[X]]$ is strictly between A and $K[[X]]$ and A is not maximal with fixed conductor. Conversely, let $A = k_0 + XK[[X]]$ and suppose that there are no fields strictly between k_0 and K. If B is a ring, $A \subseteq B \subseteq K[[X]]$ and $C = XK[[X]]$, then $k_0 \subseteq B/C \subseteq K$ and B/C is a field, so $B/C = k_0$ or $B/C = K$, so A is maximal with fixed conductor.

REMARK. For any n there exists a field extension Q_n of $\mathbb{Q}$ of degree n such that there exists no field between $\mathbb{Q}$ and Q_n. To see this, let $f \in \mathbb{Q}[X]$ be irreducible of degree n such that the Galois group of the splitting field over $\mathbb{Q}$ is S_n. Take an $S_{n-1} \subseteq S_n$. Then the fixed field L of S_{n-1} has degree n over $\mathbb{Q}$ and it is well known that S_{n-1} is a maximal subgroup of S_n so there are no fields between $\mathbb{Q}$ and $L = Q_n$.

COROLLARY 14. *There are generalized semigroup rings which are maximal with fixed conductor but neither Gorenstein nor Kunz.*

EXAMPLE. Let $A = \mathbb{Q} + X\mathbb{Q}(\sqrt[p]{2})[[X]]$, where p is a prime, $p > 3$. Then A is maximal with fixed conductor, but A is neither Gorenstein nor Kunz.

A further generalization.

Let $k \subseteq K$ be fields with $\dim_k K < \infty$, let for $i \geq 1$ V_i be k-subspaces of K satisfying $V_iV_j \subseteq V_{i+j}$. (We allow $V_i = \{0\}$.) We will investigate when rings of the form $A = k + XV_1 + \cdots + X^nV_n + X^{n+1}K[[X]]$, $V_n \neq K$ are Gorenstein. Let $cX^s \in A$. If A is Gorenstein then $B = A/cX^sA$ is a graded artinian Gorenstein ring, $B = \oplus_{i=0}^{n+s}B_i$, with $B_0 = k$ and, for $i > 0$, $B_i = X^i(V_i/cV_{i-s})$, hence $\dim_k B_i = \dim_k V_i - \dim_k V_{i-s}$, (for simplicity we set $V_j = \{0\}$ if $j < 0$). We recall some well known facts about graded artinian Gorenstein rings, which are useful for us. The socle of B, $0 : \oplus_{i=1}^{n+s}B_i$, is generated by one element, hence is a one-dimensional k-space, say yB, where $\deg y = n + s$, and yB is the unique minimal nonzero ideal in B. Hence if $f \in B_i$ there is a $g \in B_{n+s-i}$ such that $fg = y$. This shows that $\Phi : B_i \times B_{n+s-i} \to k$, where $\Phi(f,f') = \alpha$ if $ff' = \alpha y$, is a perfect pairing (i.e. Φ is bilinear and $\Phi(f,f') = 0$ for all f only if $f' = 0$ and vice versa). Φ induces for all $f \in B_i$ a linear map $L_f : B_{n+s-i} \to k$ by $L_f(f') = \Phi(f,f')$, so there is a map $B_i \to B^*_{n+s-i} = \mathrm{Hom}_k(B_{n+s-i}, k)$ which is injective since Φ is a perfect pairing. This gives that $\dim_k B_i \leq \dim_k B^*_{n+s-i} = \dim_k B_{n+s-i}$. In the same way $\dim_k B_{n+s-i} \leq \dim_k B_i$, so $\dim_k B_i = \dim_k B_{n+s-i}$. We are now ready to characterize the rings of this form that are Gorenstein.

THEOREM 15. *Let $k \subseteq K$ be fields with* $\dim_k K < \infty$, *let V_i be k-subspaces of K satisfying $V_iV_j \subseteq V_{i+j}$, and let $A = k + XV_1 + \cdots + X^nV_n + X^{n+1}K[[X]]$, $V_n \neq K$. Then A is Gorenstein if and only if $\dim_k V_i + \dim_k V_{n-i} = \dim_k K$ for all i. In particular, if n is even,* $\dim_k K$ *must be even.*

PROOF. A trivial modification of Lemma 6 gives $l_A(A/C) = \sum_{i=0}^{n} \dim_k V_i$ and $l_A(\bar{A}/A) = (n+1)\dim_k K - \sum_{i=0}^{n} \dim_k V_i$. If $\dim_k V_i + \dim_k V_{n-i} = \dim_k K$ for all i we get $l_A(A/C) = l_A(\bar{A}/A)$ so A is Gorenstein. Now suppose A is Gorenstein. Since $KX^n \subseteq A : m$, and $l_A(A : m/A) = \dim_k(A : m/A) = 1$, we get $\dim_k V_n = \dim_k K - 1$. For $i \in v(A)$, let $cX^i \in A$, and let $B = A/cX^iA$. Then $\dim_k V_i - 1 = \dim_k B_i = \dim_k B_{n+i-i} = \dim_k V_n - \dim_k V_{n-i}$, hence $\dim_k V_i + \dim_k V_{n-i} = \dim_k V_n + 1 = \dim_k K$. Now suppose $i \notin v(A)$. We claim that $V_{n-i} = K$ and so $\dim_k V_i + \dim_k V_{n-i} = 0 + \dim_k K = \dim_k K$ also in this case. Let i be the smallest integer such that $i \notin v(A)$ and $V_{n-i} \neq K$. Consider $A' = A + X^{n-i}K$. Arguing as in the proof of Theorem 10, ii) $\Rightarrow$ iii), we get that A' is a ring. Since A is Gorenstein, by Proposition 2, A is maximal with fixed conductor, which is a contradiction, because A is strictly contained in A' and $A' : \bar{A} = A : \bar{A}$.

We conclude by giving some examples of Gorenstein rings of this form.

EXAMPLE.
$\mathbb{Q} + X(\mathbb{Q} + \mathbb{Q}\sqrt[k]{2}) + X^2(\mathbb{Q} + \mathbb{Q}\sqrt[k]{2} + \mathbb{Q}\sqrt[k]{4}) + \cdots +$
$X^{k-2}(\mathbb{Q} + \mathbb{Q}\sqrt[k]{2} + \cdots + \mathbb{Q}\sqrt[k]{2^{k-2}}) + X^{k-1}\mathbb{Q}(\sqrt[k]{2})[[X]]$,
$\mathbb{Q} + X\mathbb{Q} + .. + X^{k-1}\mathbb{Q} + X^k\mathbb{Q}(\sqrt{2}) + X^{k+1}V + \cdots + X^{2k}V +$
$X^{2k+1}\mathbb{Q}(\sqrt[4]{2})[[X]]$ with $\dim_{\mathbb{Q}} V = 3$ and $\mathbb{Q}(\sqrt{2}) \subseteq V$, and
$\mathbb{Q} + X\mathbb{Q} + \cdots + X^{k-1}\mathbb{Q} + X^kV + \cdots + X^{2k-1}V + X^{2k}\mathbb{Q}(\sqrt[4]{2})[[X]]$
with $\dim_{\mathbb{Q}} V = 3$ and $\mathbb{Q} \subseteq V$ are all Gorenstein.

NOTE. After submitting this paper we have noticed that some of our results, Corollary 11 and Theorem 15, also follows from [C-D-K].

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