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DUALITY OVER AUSLANDER-GORENSTEIN RINGS

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Dedicated to the memory of Maurice Auslander

§ 1. Introduction.

It is well known that a quasi-Frobenius ring R has the duality between finitely generated left and right modules given by the R-dual functor $-^* = \mathrm{Hom}_R(-, R)$. Also it is shown in [8] and [17] that a commutative Gorenstein ring R has the duality given by the functor $\mathrm{Ext}^n_R(-, R)$ for some n. In this note, we will study a similar duality over Auslander-Gorenstein rings, which contains the duality over quasi-Frobenius rings as a special case.

Auslander introduced a non-commutative version of commutative Gorenstein rings based on the work by Bass [3]. A twosided Noetherian ring R is called *n-Gorenstein* for $n \geq 1$ if in a minimal injective resolution $0 \to {}_RR \to E_0 \to \cdots \to E_n \to \cdots$ of a left regular module ${}_RR$, the flat dimension $\mathrm{fd}(E_i)$ of E_i is at most i for each i $(0 \leq i \leq n-1)$. Auslander gave the following useful characterization of n-Gorenstein rings, which also shows the left-right symmetry of the notion. (The proof appeared in [9, Theorem 3.7])

THEOREM (Auslander). *Let R be a (twosided) Noetherian ring. Then the following are equivalent:*

(1) *In a minimal injective resolution* $0 \to {}_RR \to E_0 \to \cdots \to E_n \to \cdots$ *of* ${}_RR$, $\mathrm{fd}(E_i) \leq i$ *for each* i $(0 \leq i \leq n-1)$;

(2) *For any finitely generated right R-module X_R and any integer $j \leq n$, we have* $\mathrm{Ext}^i_R(M, R) = 0$ *if ${}_RM$ is a submodule of* $\mathrm{Ext}^j_R(X, R)$ *and* $i < j$;

(3) *In a minimal injective resolution* $0 \to R_R \to E_0' \to \cdots \to E_n' \to \cdots$ *of* R_R, $\mathrm{fd}(E_i') \leq i$ *for each* i $(0 \leq i \leq n-1)$;

(4) *For any finitely generated left R-module ${}_RY$ and any integer $j \leq n$, we have* $\mathrm{Ext}^i_R(N, R) = 0$ *if N_R is a submodule of* $\mathrm{Ext}^j_R(Y, R)$ *and* $i < j$.

In the condition (2) (or (4)), a module X_R (or ${}_RY$) is said to satisfy the *Auslander condition*. A Noetherian ring R is called *Auslander-Gorenstein* if R has finite (left and right) self-injective dimension and every finitely generated

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R-module satisfies the Auslander condition. The class of such rings contains interesting and important examples like Weyl algebras [18], certain rings of differential operators on non-singular algebraic varieties [10] and Artin-Schelter's regular algebras of global dimension≤ 3 [1] and [14].

REMARK. If R is an n-Gorenstein ring and has self-injective dimension n on both sides, then in a minimal injective resolution $0 \to {}_RR \to E_0 \to \cdots \to E_n \to 0$, we have $\mathrm{fd}(E_n) = \mathrm{pd}(E_n) = n$ ([13, Proposition 1]), where $\mathrm{pd}(E_n)$ is the projective dimension of E_n. Thus R is Auslander-Gorenstein.

Levasseur and Smith [15] showed that if a positively graded algebra $A = K \oplus A_1 \oplus \cdots$ over a field K is generated in degree one and is Auslander-Gorenstein, then for a finitely generated Z-graded A-module M, the correspondence $M \to M^\vee = \mathrm{Ext}_R^{j(M)}(M,\ R)$ gives a bijection between left and right CM-modules of projective dimension $j(M)$. In particular, $M \cong M^{\vee\vee}$ holds. Here $j(M) = \min\{j|\ \mathrm{Ext}_R^j(M,\ R) \neq 0\}$ is the grade of M. In this note, we will show that this bijection exists for general Auslander-Gorenstein rings, and give a duality between certain classes of finitely generated modules. According to Björk [4], a finitely generated module M over an Auslander-Gorenstein ring R of self-injective dimension n is called *holonomic* if $\mathrm{Ext}_R^i(M,\ R) = 0$ for any $i \neq n$ and $\mathrm{Ext}_R^n(M,\ R) \neq 0$. Inspired by Björk's work [5], we will show that the functor $\mathrm{Ext}_R^n(-,\ R) : \mathrm{mod}(R) \to \mathrm{mod}(R^{\mathrm{op}})$ gives a duality between holonomic left and right modules. In case of $n = 0$, a ring R should be considered as a quasi-Frobenius ring and then all finitely generated nonzero modules are holonomic. Thus this duality is a generalized version of the duality for quasi-Frobenius rings given by the R-dual $-^*$. As a consequence, we obtain a bijection between simple submodules of E_n and $E_n{}'$, where $0 \to {}_RR \to E_0 \to \cdots \to E_n \to 0$ and $0 \to R_R \to E_0{}' \to \cdots \to E_n{}' \to 0$ are minimal injective resolutions for ${}_RR$ and R_R, respectively.

Miyashita also gets a related result in [16, Theorem 3.5]. Let ${}_AT$ be a tilting module of projective dimension at most n over any ring A and $B = \mathrm{End}_A(M)$. Let

$$\mathscr{C}_j = \{{}_AM|\ \mathrm{Ext}_A^i(M,\ T) = 0 \text{ for all } i \neq j \text{ and } \mathrm{pd}(M) \leq j + n\},$$

$$\mathscr{D}_j = \{N_B|\ \mathrm{Ext}_B^i(N,\ T) = 0 \text{ for all } i \neq j \text{ and } \mathrm{pd}(N) \leq j + n\}$$

Here $\mathrm{pd}(X)$ means the projective dimension of a module X. Then the functor $\mathrm{Ext}_R^j(-,\ T) : \mathscr{C}_j \to \mathscr{D}_j$ gives a duality. However, in our case, there are holonomic modules of infinite projective dimension.

§ 2. Preliminaries.

Throughout this note, rings R are always twosided Noetherian rings. For a module M, we denote its projective, injective and flat dimensions by $\mathrm{pd}(M)$, $\mathrm{id}(M)$ and $\mathrm{fd}(M)$, respectively.

Let M be a finitely generated R-module. We define the *grade* $j(M)$ of M by

$$j(M) = \min\{j \geq 0 \mid \mathrm{Ext}_R^j(M, R) \neq 0\}.$$

Then $j(M) \leq \mathrm{pd}(M)$ holds.

We begin with the following useful lemma. The proof follows from the definition of homology groups.

Lemma 1. *Let R be a Noetherian ring and M a finitely generated R-module. For any $j \geq 1$, let $P_j \xrightarrow{f_j} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$ be a projective resolution with each P_i finitely generated and X_j the cokernel of the map $f_j^* : P_{j-1}{}^* \to P_j{}^*$ ($*$ is the R-dual).*

If $\mathrm{Ext}_R^i(M, R) = 0$ for any i $(0 < i < j)$, then we have the following exact sequence

$$0 \to \mathrm{Ext}_R^j(X_j, R) \to M \xrightarrow{\varepsilon_M} M^{**} \to \mathrm{Ext}_R^{j+1}(X_j, R) \to 0,$$

where ε_M is the evaluation map.

Auslander and Reiten use in [2] the exact sequence $0 \to \mathrm{Ext}_\Lambda^1(\mathrm{Tr}C, \Lambda) \to C \to C^{**} \to \mathrm{Ext}_\Lambda^2(\mathrm{Tr}C, \Lambda) \to 0$ for an artin algebra Λ and a finitely generated Λ-module C.

As a consequence, we have two corollaries.

Corollary 2. *Let ${}_RM$ be a finitely generated left R-module with $j(M) \geq 1$. Then there exists a finitely generated right R-module X_R satisfying $M \cong \mathrm{Ext}_R^{j(M)}(X, R)$ and $pd(X) \leq j(M)$.*

Corollary 3. *Let R be a Noetherian ring and M a finitely generated left module. Then we have $\mathrm{Hom}_R(M, E) = 0$ for any injective left module E with $\mathrm{fd}(E) < j(M)$.*

Proof. By Corollary 2, there is a finitely generated right R-module X_R with $M \cong \mathrm{Ext}_R^{j(M)}(X, R)$. Hence if E is injective with $\mathrm{fd}(E) < j(M)$, we get

$$0 = \mathrm{Tor}_{j(M)}^R(X, E) \cong \mathrm{Hom}_R(\mathrm{Ext}_R^{j(M)}(X, R), E) \cong \mathrm{Hom}_R(M, E)$$

by [6, Chap. VI, Proposition 5.3].

Now we assume that R is an Auslander-Gorenstein ring and M is a finitely generated R-module with $\mathrm{Ext}_R^i(M, R) = 0$ for any $i \neq j(M)$. Let

$P_j \xrightarrow{f_j} P_{j-1} \xrightarrow{f_{j-1}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$ be a projective resolution with each P_i finitely generated and $j = j(M)$. Then we have an exact sequence

$$0 \to P_0{}^* \xrightarrow{f_1{}^*} \cdots \to P_{j-1}{}^* \xrightarrow{f_j{}^*} \mathrm{Im}(f_j)^* \to \mathrm{Ext}^1_R(\mathrm{Im}(f_{j-1}),\ R) \to 0.$$

Here we see $\mathrm{Ext}^1_R(\mathrm{Im}(f_{j-1}),\ R) \cong \mathrm{Ext}^j_R(M,\ R)$ and by the Auslander condition, $\mathrm{Ext}^i_R(\mathrm{Ext}^j_R(M,\ R),\ R) = 0$ for any $i < j$. Moreover, $\mathrm{Im}(f_j)^*$ is finitely generated torsionless and $\mathrm{Im}(f_1{}^*) \cong P_0{}^*$. Hence the exact sequence above yields the following commutative diagram with exact rows

$$\begin{array}{ccccccc} P_1 & \xrightarrow{f_1} & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ {\scriptstyle \varepsilon_1}\downarrow & & {\scriptstyle \varepsilon_2}\downarrow & & {\scriptstyle \sigma_M}\downarrow & & \\ P_1{}^{**} & \xrightarrow{f_1{}^{**}} & P_0{}^{**} & \longrightarrow & \mathrm{Ext}^j_R(\mathrm{Ext}^j_R(M,\ R),\ R) & \longrightarrow & 0, \end{array}$$

where ε_1 and ε_2 are the evaluation maps and they are isomorphisms. Hence we get the canonical isomorphism

$$\sigma_M : M \xrightarrow{\sim} \mathrm{Ext}^{j(M)}_R(\mathrm{Ext}^{j(M)}_R(M,\ R),\ R).$$

Consequently, by [5, Proposition 1.6 (2)], we have

THEOREM 4. *Let R be an Auslander-Gorenstein ring and M a finitely generated left R-module with* $\mathrm{Ext}^i_R(M,\ R) = 0$ *for any $i \neq j(M)$. Then there exists the canonical isomorphism*

$$\sigma_M : M \xrightarrow{\sim} \mathrm{Ext}^{j(M)}_R(\mathrm{Ext}^{j(M)}_R(M,\ R),\ R),$$

and the correspondence $M \to \mathrm{Ext}^j_R(M,\ R)$ gives a bijection between finitely generated left and right modules with $\mathrm{Ext}^i_R(M,\ R) = 0$ *for any $i \neq j = j(M)$.*

As a byproduct of the argument above,

COROLLARY 5. *Let R be an Auslander-Gorenstein ring. If M is a finitely generated R-module with $pd(M) = j(M) < \infty$, then we have* $\mathrm{pd}(\mathrm{Ext}^{j(M)}_R(M,\ R)) = \mathrm{pd}(M)$.

§ 3. Holonomic modules.

Let R be an Auslander-Gorenstein ring of self-injective dimension n. According to [4], a finitely generated R-module M is called *holonomic* if $j(M) = n$. In case $n = 0$, R should be considered as a quasi-Frobenius ring and then all nonzero finitely generated modules are holonomic. For $n \geq 1$, we have the following characterization of holonomic modules.

THEOREM 6. *Let R be an Auslander-Gorenstein ring of self-injective dimen-*

sion $n \geq 1$, $0 \to {}_RR \to E_0 \to \cdots \to E_n \to 0$ *a minimal injective resolution and* ${}_RM \neq 0$ *a finitely generated left R-module. Then the following are equivalent:*

(1) *M is holonomic;*

(2) $M \cong \mathrm{Ext}^n_R(\mathrm{Ext}^n_R(M, R), R)$;

(3) $M \cong \mathrm{Ext}^n_R(X, R)$ *for some finitely generated right module* X_R;

(4) $\mathrm{Hom}_R(M, E_0 \oplus \cdots \oplus E_{n-1}) = 0$.

PROOF. (1) $\Rightarrow$ (2) follows from Theorem 4.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (4) follows from Corollary 3.

(4) $\Rightarrow$ (1): First we have $M^* = \mathrm{Hom}_R(M, R) \subseteq \mathrm{Hom}_R(M, E_0) = 0$. For any i $(1 \leq i < n)$, the exact sequence $0 \to K_{i-1} \to E_{i-1} \to K_i \to 0$ with $\mathrm{E}(K_{i-1}) = E_{i-1}$ yields the exact sequence $\mathrm{Hom}_R(M, K_i) \to \mathrm{Ext}^1_R(M, K_{i-1}) \to 0$. From $\mathrm{Hom}_R(M, K_i) \subseteq \mathrm{Hom}_R(M, E_i) = 0$, we get $0 = \mathrm{Ext}^1_R(M, K_{i-1}) \cong \mathrm{Ext}^i_R(M, R)$. Hence $\mathrm{Ext}^n_R(M, R) \neq 0$ follows by [7, Theorem 2].

By [13, Proposition 4] and Theorem 6, we see

COROLLARY 7. (1) *Every holonomic module has finite composition length.*

(2) *If M is a holonomic left module over an Auslander-Gorenstein ring R, then M embeds in ${E_n}^{(t)}$ for some $t > 0$. Here E_n is the last injective term in a minimal injective resolution of ${}_RR$.*

PROOF. (1) follows from [13, Proposition 4] and Theorem 6.

(2) The socle, $\mathrm{Soc}(M)$, is essential in M and finitely generated by (1). Thus, let $\mathrm{Soc}(M) = S_1 \oplus \cdots \oplus S_t$ with each S_i simple. Then each S_i is holonomic by Theorem 6 (4) and so S_i embeds in E_n by [11, Theorem 2]. Consequently, we have $M \subseteq \mathrm{E}(M) \cong \mathrm{E}(S_1) \oplus \cdots \oplus \mathrm{E}(S_t) \hookrightarrow {E_n}^{(t)}$.

Let $\mathscr{H}_l$ (resp. $\mathscr{H}_r$) be the class of all holonomic left (resp. right) R-modules. Then, by Theorem 6 (4), we see that $\mathscr{H}_l$ is closed under extensions, submodules and factor modules. That is, $\mathscr{H}_l$ is a torsion class with respect to a Lambek torsion theory. Moreover, $\mathscr{H}_l$ is a non-empty class since E_n has a simple submodule S by [13, Theorem 6] and then S belongs to $\mathscr{H}_l$.

Assume $M \in \mathscr{H}_l$, then by Theorem 6 (3), we see $\mathrm{Ext}^n_R(M, R) \in \mathscr{H}_r$ and by Theorem 4, we have a canonical isomorphism

$$\sigma_M : M \xrightarrow{\sim} \mathrm{Ext}^n_R(\mathrm{Ext}^n_R(M, R), R).$$

Thus, the functor $F = \mathrm{Ext}^n_R(-, R) : \mathrm{mod}(R) \to \mathrm{mod}(R^{\mathrm{op}})$ induces a duality between $\mathscr{H}_l$ and $\mathscr{H}_r$, and F is exact on $\mathscr{H}_l$ (and $\mathscr{H}_r$). Moreover, M has projective dimension n or ∞ by [12, Theorem 2] and hence $\mathrm{pd}(\mathrm{Ext}^n_R(M, R)) = n$ or ∞ by Corollary 5. There actually exists a simple submodule S of E_n (hence $S \in \mathscr{H}_l$) with $\mathrm{pd}(S) = \infty$.

As we showed in [13, Proposition 1 and Theorem 6], the last injective term in a minimal injective resolution of an Auslander-Gorenstein ring has significant properties. In the following, $0 \to {}_RR \to E_0 \to \cdots \to E_n \to 0$ and $0 \to R_R \to E_0' \to \cdots \to E_n' \to 0$ stand for minimal injective resolutions of ${}_RR$ and R_R, respectively for an Auslander-Gorenstein ring R of self-injective dimension n. Then any simple submodule of E_n (or E_n') is holonomic, and if S is a simple submodule of E_n, then $\mathrm{Ext}_R^n(S, R)$ is a simple submodule of E_n' by Theorem 6.

Therefore we obtain

THEOREM 8. *Let R be an Auslander-Gorenstein ring of self-injective dimension n. Then, for a holonomic module M, we have the canonical isomorphism $M \xrightarrow{\sim} \mathrm{Ext}_R^n(\mathrm{Ext}_R^n(M, R), R)$. Moreover*

(1) *The correspondence $M \to \mathrm{Ext}_R^n(M, R)$ gives a bijection between holonomic left and right modules;*

(2) *In particular, holonomic left and right modules of projective dimension n correspond bijectively;*

(3) *Simple submodules of E_n and E_n' are holonomic and correspond bijectively.*

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