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GROUPS AS THE UNION OF PROPER SUBGROUPS

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1. Introduction.

In [1], J.H.E. Cohn defined $\sigma(G)$ to be the smallest integer n such that the group G is the set-theoretic union of n proper subgroups. A number of results were proved for soluble groups leading to the conjecture that if G is a finite noncyclic soluble group then $\sigma(G) = p^a + 1$, where p^a is the order of a particular chief factor of G. It was also conjectured that there is no group G for which $\sigma(G) = 7$. It is well known that $\sigma(G)$ can never be equal to 2 and 7 is the next integer not of the form $p^a + 1$.

In this note we prove both of these conjectures. The methods are mostly elementary and the only technical result that is required is a theorem of Gaschütz [2] which says that, in a primitive soluble group G, the unique minimal normal subgroup has order greater than the other chief factors of G.

Cohn [1] also showed that $\sigma(A_5) = 10$ and $\sigma(S_5) = 16$. It would be interesting to know what integers can occur as $\sigma(G)$. These smallest non-soluble cases suggest that there may be no groups with $\sigma(G) = 11, 13$ or 15. It might be of interest to investigate $\sigma(G)$ for families of simple groups.

2. Covering soluble groups with proper subgroups.

We begin by dealing with a special case which is essentially Lemma 17 of [1] although our proof is slightly simpler than that given there.

LEMMA 2.1. *Let G be a primitive soluble group with minimal normal subgroup N such that G/N is cyclic. Then* $\sigma(G) = |N| + 1$.

PROOF. There are $|N|$ cyclic core-free maximal subgroups M_i which complement N. If $G = X_1 \cup \ldots \cup X_\sigma$, then $M_i = \bigcup_{j=1}^{\sigma}(M_i \cap X_j)$ and, since M_i is cyclic, it follows that M_i is equal to one of the X_j. Also $M_1 \cup \ldots \cup M_n \neq G$ and so $\sigma \geq |N| + 1$.

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For the converse, note that $M_i \cap M_j \lhd \langle M_i, M_j \rangle = G$ and so $M_i \cap M_j = 1$, for all $i \neq j$. Hence

$$|M_1 \cup \ldots \cup M_n| = |N|(|G/N| - 1) = |G| - (|N| - 1).$$

It follows that $G = N \cup M_1 \cup \ldots \cup M_n$ and so $\sigma \leq |N| + 1$.

THEOREM 2.2. *Let G be a finite soluble group and let H/K be the smallest chief factor of G having more than one complement in G. Then $\sigma(G) = |H/K| + 1$.*

PROOF. We prove the result by induction on $|G|$. Let $|H/K| = p^a$.

(a) $\sigma(G) \leq p^a + 1$.

Choose a chief factor V/W of G of order p^a such that V/W has at least two complements in G and so that $|G/V|$ is minimal with these properties. Hence if S/T is a complemented chief factor of G with $S > V$ and $|S/T| \leq p^a$ then S/T has a unique complement $C \lhd G$ and so S/T is central in G and so has prime order.

Let M be a complement to V/W and let $Y = \text{core}\, M \geq W$. Then G/Y is a primitive soluble group with unique minimal normal subgroup $X/Y \overset{G}{\cong} V/W$. If $X \neq V$, then X/Y is central and so V/W is also central of order p. The complements to V/W are then normal and G has a homomorphic image isomorphic to $C_p \times C_p$. Since $C_p \times C_p$ is the union of its $p + 1$ subgroups of order p, we have $\sigma(G) \leq p + 1$. Therefore we may assume that $X = V$, $Y = W$ and so G/W is a primitive soluble group with unique minimal normal subgroup V/W.

By Gaschütz's result every chief factor of G/V has order less than p^a and so is either a Frattini chief factor or is central. Hence G/V is nilpotent and each prime q dividing $|G/V|$ is less than p^a. If G/V were not cyclic it would have a homomorphic image isomorphic to $C_q \times C_q$ and so would have a chief factor of order q with more than one complement. So G/V is cyclic and, by Lemma 3.1, $\sigma(G) \leq \sigma(G/W) = p^a + 1$.

(b) $p^a + 1 \leq \sigma(G)$.

Choose $N \lhd G$ maximal with respect to $\sigma(G/N) = \sigma(G)$. Let K/N be a minimal normal subgroup of G/N. Then either G/K is cyclic or $\sigma(G/K) > \sigma(G)$.

Let $G = X_1 \cup \ldots \cup X_\sigma$, where the X_i are maximal subgroups of G containing N. Not all the X_i contain K, for otherwise we would have $\sigma(G/K) \leq \sigma(G)$. Therefore K/N has a complement M.

If K/N had a unique complement, say X_1, then $X_2, \ldots, X_\sigma$ must all contain K. But this is contrary to Lemma 5 of [1] or Lemma 2.1 of [3]. Therefore K/N has at least two complements.

If K/N has normal complements, then $|K/N| = p$ and G has a factor group isomorphic to $C_p \times C_p$. Hence $\sigma(G) = p + 1 = |K/N| + 1$.

If K/N has non-normal complements, then it has $|K/N|$ complements $M_1, \ldots, M_k$, say. Since $M_i = \bigcup_{j=1}^{\sigma}(M_i \cap X_\sigma)$ and $M/N \cong G/K$ is either cyclic or satisfies $\sigma(M/N) > \sigma$, we must have $M_i = X_j$, for some j. Therefore $|K/N| < \sigma$. In both cases, K/N is a chief factor with at least two complements and $\sigma(G) \geq |K/N| + 1$.

3. Groups covered by seven proper subgroups.

Our aim in this section is to show that any group which is the union of seven proper subgroups can be given as the union of fewer than seven subgroups; that is, there is no group G with $\sigma(G) = 7$.

The first step in the proof will be to give restrictions on the indices of the covering subgroups. The first result required for that is Lemma 3.3 of [3]. We state that result below and also prove a variation which we will need.

LEMMA 3.1. [3, Lemma 3.3] *Let M be a proper subgroup of the finite group G and let $H_1, \ldots, H_k$ be subgroups with $|G : H_i| = \beta_i$ and $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_k$. If $G = M \cup H_1 \cup \ldots \cup H_k$, then $\beta_1 \leq k$.*

Furthermore if $\beta_1 = k$, then $\beta_1 = \ldots = \beta_k = k$ and $H_i \cap H_j \leq M$, for all $i \neq j$.

LEMMA 3.2. *Let N be a proper normal subgroup of the finite group G. Let $U_1, \ldots, U_k$ be proper subgroups of G containing N and $V_1, \ldots, V_k$ be subgroups such that $V_iN = G$ with $|G : V_i| = \beta_i$ and $\beta_1 \leq \ldots \leq \beta_k$. If $G = U_1 \cup \ldots \cup U_h \cup V_1 \cup \ldots \cup V_k$, then $\beta_1 \leq k$.*

Furthermore, if $\beta_1 = k$, then $\beta_1 = \ldots = \beta_k = k$ and $V_i \cap V_j \leq U_1 \cup \ldots \cup U_h$, for all $i \neq j$.

PROOF. Suppose that $|U_1 \cup \ldots \cup U_h| = \gamma|G|$. Since $V_iN = G$, we have $V_i/V_i \cap N \cong G/N$ and so

$$|V_i \cap (U_1 \cup \ldots \cup U_h)| = \gamma|V_i| = \frac{\gamma}{\beta_i}|G|.$$

Therefore

$$|V_i \backslash (U_1 \cap \ldots \cap U_h)| = \left(\frac{1-\gamma}{\beta_i}\right)|G|.$$

But $|V_1 \cup \ldots \cup V_k \backslash (U_1 \cup \ldots \cup U_h)| = (1-\gamma)|G|$, and so

$$(1-\gamma)|G| \leq (1-\gamma)|G|\left(\frac{1}{\beta_1}+\ldots+\frac{1}{\beta_k}\right).$$

Hence $1 \leq \frac{1}{\beta_1}+\ldots+\frac{1}{\beta_k} \leq \frac{k}{\beta_1}$ and so $\beta_1 \leq k$.

If $\beta_1 = k$, then $\beta_1 = \ldots = \beta_k = k$ and the sets $V_i \backslash (U_1 \cup \ldots \cup U_h)$ are pairwise disjoint. That is, $V_i \cap V_j \leq U_1 \cup \ldots \cup U_h$.

Our main proof will consider a number of cases each of which reduces to considering certain subgroups of $S_6 \times S_6$. We deal with the particular groups which arise first.

LEMMA 3.3. *The only noncyclic subgroups of S_6 which are not the union of fewer than 7 proper subgroups are:*

$$S_6, A_6, S_5, A_5 \text{ and } F,$$

where $F \cong (C_3 \times C_3) \rtimes C_4$ is a group of order 36 with a minimal normal subgroup of order 9.

PROOF. This follows easily from Theorem 2.2.

LEMMA 3.4. *None of the following groups is the union of 7 proper subgroups:*

$$\begin{array}{ll} A_5, S_5, A_6, S_6, & \\ C_p \times A_5, C_p \times A_6, & (p = 2, 3 \text{ or } 5) \\ C_p \times S_5, C_p \times S_6, & (p = 3 \text{ or } 5) \\ C_3 \times F, & \\ A_5 \times A_5, A_5 \times S_5. & \end{array}$$

PROOF. Cohn [1] has shown that $\sigma(A_5) = 10, \sigma(S_5) = 16$.

Suppose that $A_6 = X_1 \cup \ldots \cup X_7$ with each X_i maximal in A_6. There are six maximal subgroups M of A_6 which are isomorphic to A_5. Since $M = (M \cap X_1) \cup \ldots \cup (M \cap X_7)$ and $\sigma(M) > 7$, it follows that M is one of the X_i. So the remaining elements of A_6 must all be contained in the seventh covering subgroup. But the conjugacy class containing (123)(456) generates the whole of A_6.

Similarly for S_6; this contains six maximal subgroups isomorphic to S_5 which must be six of the covering subgroups. The conjugacy class of 6-cycles generates S_6.

Now let $G = C \times H$, where C is cyclic of order 2, 3 or 5 and H is A_5, S_5, A_6, S_6 or F. Suppose $G = X_1 \cup \ldots \cup X_7$. One of the X_i, say X_1, does not contain C. Then $X_1 C = G$ and $X_1 \lhd G$ with $G/X_1 \cong C$. Since $G/H \cong C$ and G has no factor group isomorphic to $C \times C$, we have $X_1 = H$. The remaining X_i's must contain C and so we get a covering of $G/C \cong H$ by six proper subgroups, contrary to Lemma 4.3.

Finally, let $G = H \times K$ with $H \cong A_5$ and $K \cong A_5$ or S_5. Suppose that M is a maximal subgroup of G such that $MH = MK = G$. Then $M/M \cap K \cong H$ and $M/M \cap H \cong K$. Since $M/M \cap K$ is simple we have either $(M \cap K)(M \cap H) = M$ or $M \cap H \leq M \cap K$ so that $M \cap H = 1$. The former leads to $M = (M \cap K) \times (M \cap H) \cong H \times K = G$ which is clearly impossible. Therefore $M \cap H = 1$ and $|G : M| = |K| \geq 60$.

If $G = X_1 \cup X_2 \cup \ldots \cup X_7$, with each X_i maximal, $|G : X_i| = \alpha_i$ and $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_7$, then Lemma 3.1 shows that $\alpha_2 \leq 6$ and a further counting argument shows that $\alpha_3 < 60$. Therefore X_1, X_2, X_3 each contain either H or K. Since $G/K \cong A_5$, there are no subgroups of index less than 5 containing K. By Lemma 3.2, at most two of the X_i contain H. If exactly two of X_1, X_2, X_3 contain H, then the third contains K and, by Lemma 3.2, it has index 5. Also $\alpha_4 = \ldots = \alpha_7 = 5$ and so five of the X_i contain K. But then Lemma 3.2 implies that the remaining two have index two which is impossible.

So we may assume that two of X_1, X_2, X_3 contain K. If the third also contained K then we would have $\alpha_4 \leq 4$ which is impossible. So the third contains H and has index 2 or 5. If it has index 5 then $\alpha_4 = \ldots = \alpha_7 = 5$ and we again have five of the X_i containing K. Therefore X_1 has index 2 and X_1, X_2, X_3 contain $X_1 \cap K$. Since $\alpha_4 \geq 5$, it follows from Lemma 3.2 that all the X_i contain $X_1 \cap K$ and so $G/X_1 \cap K \cong C_2 \times A_5$ is the union of seven proper subgroups, contrary to the earlier case.

To prepare for the main body of our proof we consider a number of other very special situations which can not arise in a group G with $\sigma(G) = 7$. These situations will occur repeatedly in the proof.

LEMMA 3.5. *Suppose that G is a group with* $\sigma(G) = 7$.

(i) *If* $X \leq G$ *with* $|G : X| \leq 4$, *then* $X \triangleleft G$ *and* G/X *is cyclic.*

(ii) *G has no factor group ismorphic to* $C_p \times C_p$ *with* $p = 2, 3$ *or* 5.

(iii) *G does not have a subgroup D of index 25 contained in six maximal subgroups of index 5.*

(iv) *G does not have three maximal subgroups* X, Y, Z *with* $|G : X| = 2, |G : Y| = |G : Z| = 5$ *and* $Y \cap Z \leq X$.

(v) *G does not have 6 maximal subgroups* $H_0, H_1, \ldots, H_5$ *of index 5 satisfying* $H_0 H_i = G$, *for all* $i \geq 1$, *and* $H_i \cap H_j \leq H_0$, *for all* $i > j \geq 1$.

PROOF. (i) $G/\mathrm{core}\, X$ is isomorphic to a subgroup of S_4. If $G/\mathrm{core} X$ is not cyclic, then $\sigma(G/\mathrm{core} X) < 7$.

(ii) This is clear since $\sigma(C_p \times C_p) = p + 1$.

(iii) Suppose that $M_1, \ldots, M_6$ have index 5 in G so that $|M_i : D| = 5$. Then $M_i \cap M_j = D$ and we see that $|M_1 \cup \ldots \cup M_6| = |D| + 6|M_i \backslash D| = |D| + 6 \times 4|D| = 25|D| = |G|$ and so $G = M_1 \cup \ldots \cup M_6$ and $\sigma(G) \leq 6$.

(iv) Suppose that G does have three maximal subgroups X, Y, Z with the properties stated. Then $Y \cap Z \leq Y \cap X$ and $Y \cap X$ has index 2 in Y. Therefore $Y \cap Z$ has index 2 or 4 in Y (and index 10 or 20 in G).

If $|G : Y \cap Z| = 10$, then $Y \cap Z = Y \cap X \lhd Y$ and also $Y \cap Z = Z \cap X \lhd Z$ so that $Y \cap Z \lhd \langle Y, Z \rangle = G$. But then $G/Y \cap Z$ has order 10 so is either cyclic or dihedral. But the dihedral group of order 10 is the union of six proper subgroups and the cyclic group of order 10 has only one subgroup of index 5.

So we may assume that $|G : Y \cap Z| = 20$. Therefore $Y \cap Z$ has index 2 in both $Y \cap X$ and $Z \cap X$ and so is normal in $\langle Y \cap X, Z \cap X \rangle$. But $Y \cap X \neq Z \cap X$, for otherwise we would have $Y \cap Z = Y \cap X = Z \cap X$ having index 10 in G. So $Y \cap X < \langle Y \cap X, Z \cap X \rangle \leq X$. But $|X : Y \cap X| = 5$ and so $Y \cap X$ is maximal in X. Therefore $\langle Y \cap X, Z \cap X \rangle = X$ and we have $Y \cap Z \lhd X$.

Now $X/Y \cap Z$ is a group of order 10 and so contains a subgroup $K/Y \cap Z$ of index 2. Then $|G : K| = 4$ and, by (i), $K \lhd G$. Since Y is maximal, $KY = G$ and $|Y : Y \cap K| = 4$. It follows that $Y \cap Z = Y \cap K \lhd Y$ and, similarly, $Y \cap Z \lhd Z$. Therefore $Y \cap Z \lhd \langle Y, Z \rangle = G$ and $G/Y \cap Z$ is a group of order 20. Since $|G/K| = 4, G/K$ is cyclic and so $G/Y \cap Z$ is either cyclic or a split extension of C_5 by C_4. In the former case this is contrary to there being two subgroups of index 5. In the latter case $G/Y \cap Z$ is the union of 6 proper subgroups.

(v) Suppose that G does have subgroups $H_0, \ldots, H_5$ with the properties stated. Since $H_0 H_i = G$, $H_0 \cap H_i$ has index 5 in H_i. Since $H_i \cap H_j \leq H_0 \cap H_i$, it follows that $H_i \cap H_j = H_0 \cap H_i$ and, similarly, $H_i \cap H_j = H_0 \cap H_j$. Therefore $H_i \cap H_j = D$ for all $i \neq j$ and we have a contradiction to (iii).

We now make use of these particular cases in proving the main result of this section.

THEOREM 3.6. *There is no group G with* $\sigma(G) = 7$.

PROOF. Suppose that there is a group G with $\sigma(G) = 7$. Then we can write G as the union of 7 maximal subgroups $X_1, \ldots, X_7$. We let $|G : X_i| = \alpha_i$, with $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_7$.

By Lemma 3.1, $\alpha_2 \leq 6$. Since the X_i are maximal subgroups, Lemma 3.5(i) shows that no α_i is equal to 4 and so $\alpha_2 = 3, 5$ or 6. We consider these three cases separately.

Case I: $\alpha_2 = 3$.

In this case $\alpha_1 = 2, X_1 \lhd G, X_2 \lhd G$ and $G/X_1 \cap X_2 \cong C_6$. Since $X_i(X_1 \cap X_2)$ is not contained in either X_1 or X_2 we have $X_i(X_1 \cap X_2) = G$, for all $i \geq 3$.

By Lemma 3.5(ii), $\alpha_i \geq 5$, for all $i \geq 3$. By Lemma 3.2, $\alpha_3 = \ldots = \alpha_7 = 5$ and $X_i \cap X_j \leq X_1 \cup X_2$, for all $i > j \geq 3$.

By Lemma 3.5(iv), we can not have $X_i \cap X_j \leq X_1$ and so $X_i \cap X_j \leq X_2$.

But then $X_i \cap X_j \leq X_i \cap X_2$ and $X_i \cap X_2$ has index 3 in X_i. Therefore $X_i \cap X_j = X_i \cap X_2 \lhd X_i$ and, similarly, $X_i \cap X_j \lhd X_j$. Hence $X_i \cap X_j \lhd \langle X_i, X_j \rangle = G$ and $|G/X_i \cap X_j| = 15$. But then $G/X_i \cap X_j$ is cyclic, contrary to it having two subgroups of index 5.

Case II: $\alpha_2 = 6$.

It follows from Lemma 3.1 that $\alpha_2 = \ldots = \alpha_7 = 6$.

Let $N_1 =$ core $X_1, N_2 =$ core X_2 and $N = N_1 \cap N_2 =$core $(X_1 \cap X_2)$. It follows from Lemma 3.2 that $X_i \geq N$, for all i, and so $\sigma(G/N) = 7$.

If $\alpha_1 = 2$ or 3, then G/N is isomorphic to a subgroup of $C_2 \times S_6$ or $C_3 \times S_6$. Since G/N_2 can not be covered by six proper subgroups and G has no factor group isomorphic to $C_2 \times C_2$, the only possibilities for G/N are: $S_6, S_5, C_2 \times A_6, C_3 \times S_6, C_3 \times A_6, C_3 \times A_5, C_3 \times F$. All of these possibilities are excluded by Lemma 3.4.

We may therefore assume that $\alpha_1 = 5$ or 6. If $X_i \geq N_1$, for all i, then $\sigma(G/N_1) = 7$. But G/N_1 is a subgroup of S_6 and we again have a contradiction to Lemmas 3.3 and 3.4. Therefore there is an $i \geq 2$ such that $X_i \ngeq N_1$. By Lemma 3.2, we must have $X_i \ngeq N_1$, for all $i \geq 2$, and $X_i \cap X_j \leq X_1$, for all $i > j \geq 2$.

Now $N_1 X_i = G$ and so $X_1 X_i = G$ and $X_1 \cap X_i$ has index $\alpha_1 (= 5$ or $6)$ in X_i. Hence $X_i \cap X_j = X_1 \cap X_i = X_1 \cap X_j$. It follows that $X_i \cap X_j = D$ for all $i > j \geq 1$.

Now $N_1 \cap D = N_1 \cap X_i \lhd X_i$, for all $i \geq 2$, and so $N_1 \cap D \lhd \langle X_i, X_j \rangle = G$ and $|N_1/N_1 \cap D| = 6$. Therefore $N_1/N_1 \cap D$ has a subgroup $T/N_1 \cap D \lhd G/N_1 \cap D$ with $|N_1/T| = 2$. But then $X_i < TX_i < G$, contrary to X_i being maximal in G.

Case III: $\alpha_2 = 5$.

Again, let $N_1 =$core$X_1, N_2 =$coreX_2 and $N = N_1 \cap N_2 =$core$(X_1 \cap X_2)$.

(a) Suppose first that $X_i \ngeq N$, for some $i \geq 3$. By Lemma 3.2, we have $\alpha_i = 5$, for all $i \geq 2$, and $X_i \cap X_j \leq X_1 \cup X_2$, for all $i > j \geq 3$.

We also have $\alpha_1 = 2, 3$ or 5 and consider these three cases separately

(i) $\alpha_1 = 2$.

By Lemma 3.5(iv), we can not have $X_i \cap X_j \leq X_1$ and so $X_i \cap X_j \leq X_2$, for all $i > j \geq 3$. Since $X_i N = G$, we have $X_i X_2 = G$ and so $|X_i : X_i \cap X_2| = 5$. Hence $X_i \cap X_j = X_i \cap X_2 = X_j \cap X_2$ and so $X_i \cap X_j = D$, for all $i > j \geq 2$. This is contrary to Lemma 3.5(v).

(ii) $\alpha_1 = 3$.

If $X_i \cap X_j \leq X_1$, then $X_i \cap X_j = X_i \cap X_1 \lhd X_i$ and, similarly, $X_i \cap X_j \lhd X_j$ so that $X_i \cap X_j \lhd \langle X_i, X_j \rangle = G$ and $|G/X_i \cap X_j| = 15$. Then $G/X_i \cap X_j$ is cyclic contrary to it having two subgroups of index 5.

So $X_i \cap X_j \leq X_2$, for all $i > j \geq 3$. Again, we see that $X_i \cap X_j = D$, for all $i > j \geq 2$, and obtain a contradiction to Lemma 4.5(v).

(iii) $\alpha_1 = 5$.

Since $NX_i = G$, we have $X_1X_i = X_2X_i = G$ and so $|X_i : X_i \cap X_1| = |X_i : X_i \cap X_2| = 5$, for all $i \geq 3$.

Therefore $X_i \cap X_j = X_i \cap X_1$ or $X_i \cap X_2$.

Suppose that there is a $k \geq 3$ and i,j such that

$$X_i \cap X_k = X_k \cap X_1 = X_i \cap X_1 \text{ and } X_j \cap X_k = X_k \cap X_2 = X_j \cap X_2.$$

Now $X_i \cap X_j \leq X_1 \cup X_2$ and, without loss of generality, we may assume $X_i \cap X_j \leq X_2$ so that

$$X_i \cap X_j = X_i \cap X_2 = X_j \cap X_2.$$

But $X_j \cap X_2 = X_k \cap X_2$ and so $X_i \cap X_2 = X_k \cap X_2 \leq X_i \cap X_k = X_i \cap X_1$ giving $X_i \cap X_1 = X_i \cap X_2$. Hence $X_i \cap X_k \leq X_2$. It follows that $X_i \cap X_j \leq X_2$, for all $i > j \geq 3$, and we again have $X_i \cap X_j = D$, for all $i > j \geq 2$, contrary to Lemma 3.5(v).

(b) We may therefore assume that $X_i \geq N$, for all i, and so $\sigma(G/N) = 7$.

If $\alpha_1 = 2$ or 3, then $G/N_1 \cong C_2$ or C_3. If $\alpha_1 = 5$, then $G/N_1 \cong C_5, A_5$ or S_5. Since $G/N_2 \cong C_5$, A_5 or S_5, G/N must be $C \times A_5, C \times S_5, A_5 \times A_5$ or $A_5 \times S_5$. All these possibilities are excluded by Lemma 3.4.

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