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CONVERGENCE RESULTS FOR THE SQUARE ROOT OF THE POISSON KERNEL

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Abstract.

In the disk, we prove that integrals of boundary functions against the square root of the Poisson kernel converge in regions which we call L^p weakly tangential. If $p > 1$ these regions are strictly larger than the weakly tangential regions used by Sjögren. We also investigate how sharp these results are.

In the bidisk, we prove that we have convergence in the product region $A \times B$, where A is a nontangential cone, and B is a weakly tangential region. In this case, the kernel will be a tensor product of powers of Poisson kernels, with the exponent larger than $1/2$ in the first variable, and the exponent equal to $1/2$ in the second variable.

1. Introduction.

Let $P(z, \varphi)$ be the standard Poisson kernel in the unit disk U, that is,

$$P(z, \varphi) = \frac{1}{2\pi} \frac{1 - |z|^2}{|z - e^{i\varphi}|^2}, z \in \mathsf{U}, \varphi \in \mathsf{R}/2\pi\mathsf{Z}.$$

A well-known Fatou type result states that the Poisson integral

$$Pf(z) = \int_{\mathsf{T}} P(z, \varphi) f(e^{i\varphi}) d\varphi$$

of a function $f \in L^1(\mathsf{T})$ converges to $f(e^{i\beta})$ if z tends to $e^{i\beta}$ nontangentially, for a.a. $e^{i\beta} \in \mathsf{T}$. Littlewood proved that this is the largest "natural" region of convergence, although some larger regions of convergence were obtained by Nagel and Stein. Let

$$P_\lambda f(z) = \int_{\mathsf{T}} P(z, \varphi)^{\lambda + 1/2} f(\varphi) d\varphi, \qquad \lambda \geq 0,$$

where $f \in L^p(\mathsf{T})$, $1 \leq p \leq \infty$. We know that $P_\lambda f(z)$ is a solution of the equation $L_z u = (\lambda^2 - 1/4)u$ where, if $z = x + iy$,

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$$L_z = \tfrac{1}{4}(1-|z|^2)^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

is the hyperbolic Laplacian.

But $P_\lambda f(z)$ does not generally converge to $f(e^{i\beta})$ as z tends to $e^{i\beta}$ for $\lambda \neq 1/2$, because then $P_\lambda 1(z)$ does not converge to 1. In fact,

$$P_\lambda 1 \sim (1-|z|)^{1/2-\lambda} \text{ if } \lambda > 0 \text{ and}$$

$$P_\lambda 1 \sim (1-|z|)^{1/2}\log\frac{1}{1-|z|} \text{ if } \lambda = 0.$$

Here, by the notation $f \sim g$ we mean that there exist two constants $0 < k \leq K < \infty$, not necessarily the same at each occurrence, such that $k \leq f/g \leq K$.

In order to get convergence to f we consider the operator

$$\mathscr{P}_\lambda f(z) = \frac{P_\lambda f(z)}{P_\lambda 1(z)},$$

which has the kernel

$$\frac{P(z,\varphi)^{\lambda+1/2}}{P_\lambda 1(z)}.$$

If $\lambda > 0$ and $f \in L^p(\mathsf{T}), 1 \leq p \leq \infty$, we know that $\mathscr{P}_\lambda f(z)$ converges to $f(e^{i\beta})$ as z tends to $e^{i\beta}$ nontangentially, because then the kernel of $\mathscr{P}_\lambda f(z)$ essentially has the same behaviour as $P(z,\varphi)$. Thus nontangential convergence is essentially the best we can expect for $\mathscr{P}_\lambda f(z), \lambda > 0$.

If $\lambda = 0$ and $f \in L^1(\mathsf{T}^2)$ as above, we know that $\mathscr{P}_\lambda f(z)$ converges to $f(e^{i\beta})$ as z tends to $e^{i\beta}$ weakly tangentially, in the sense that z stays in the region

$$\left\{z \in \mathsf{U} : |\arg(z)-\beta| \leq A(1-|z|)\log\frac{1}{1-|z|}, \quad \text{A arbitrary but fixed}\right\}.$$

This was proved by Sjögren [Sjö84] in the beginning of the eighties.

If $f \in L^p(\mathsf{T}), 1 \leq p \leq \infty$, however, we will prove that $\mathscr{P}_0 f(z)$ converges to $f(z)$ in a larger region A^p_β defined by

$$A^p_\beta = \left\{z \in \mathsf{U} : |\arg(z)-\beta| \leq A(1-|z|)\left(\log\frac{1}{1-|z|}\right)^p\right\}.$$

We say that A^p_β is the L^p weakly tangential regions.

The main part of the proof is to prove that the corresponding maximal operator M_{A^p} defined by

$$M_{A^p}f(\beta) = \sup_{\substack{z\in A^p_\beta \\ |z|>1/2}} \mathscr{P}_\lambda|f|(z)\,,\, f \in L^p(\mathrm{T})\,,$$

is of weak type (p,p). We will also prove that in a natural sense the L^p weakly tangential regions are the largest convergence regions for $\mathscr{P}_0 f(z)$.

We will finally prove a convergence result in the bidisk, considering the operator

$$\mathscr{P}_{\lambda_1,0}f(z_1,z_2) = \frac{P_{\lambda_1,0}f(z_1,z_2)}{P_{\lambda_1,0}1(z_1,z_2)}\,,$$

where

$$P_{\lambda_1,0}f(z_1,z_2) = \int_{\mathrm{T}} P(z_1,\varphi_1)^{\lambda_1+1/2}P(z_2,\varphi_2)^{1/2}f(e^{i\varphi_1},e^{i\varphi_2})d\varphi_1 d\varphi_2,$$

$\lambda_1 > 0$ and $f \in L^1(\mathrm{T}^2)$. In this case the convergence region is a product region $A \times B$ where A is a nontangential cone and B is a weakly tangential region.

With a slight abuse of notation, we will identify $e^{i\beta} \in \mathrm{T}$ and $(e^{i\beta_1},e^{i\beta_2}) \in \mathrm{T}^2$ with $\beta \in \mathrm{R}$, $0 \le \beta \le 2\pi$, and $(\beta_1,\beta_2) \in \mathrm{R}^2$, $0 \le \beta_1,\beta_2 \le 2\pi$, respectively. C and c will denote various constants.

The structure of this article is as follows: In section two we give the convergence result, and the proof of it, for the square root of the Poisson kernel in the unit disk U, and give a statement about how strong the theorem is. In section three we give and prove the convergence result for $\mathscr{P}_{\lambda_1,0}f(z_1,z_2)$.

2. The one dimensional result.

In [Sjö84] Sjögren shows that $\mathscr{P}_0 f(z) = \frac{P_0 f(z)}{P_0 1(z)}$, $f \in L^1(\mathrm{T})$ converges a.e. to the function f as z tends weakly tangentially to a point β on the boundary of the unit disk U. We will extend this result by proving the following theorem:

THEOREM 2.1. *Let A^p_β be an L^p weakly tangential region, that is*

$$A^p_\beta = \left\{ z \in \mathrm{U} \colon |\arg(z) - \beta| \le A(1-|z|)\left(\log\frac{1}{1-|z|}\right)^p \right\},$$

where A is an arbitrary positive real number. If $f \in L^p(\mathrm{T})$, $1 < p < \infty$, then $\mathscr{P}_0 f(z)$ converges to $f(\beta)$ for a.a. $\beta \in \mathrm{T}$ *as z tends to β in A^p_β.*

Theorem 2.1 is established by standard methods using the following maximal function estimate:

THEOREM 2.2. *Let*

$$M_{A^p}f(\beta) = \sup_{\substack{z\in A^p_\beta \\ |z|>1/2}} \mathscr{P}_0|f|(z)\,,\, f \in L^p(\mathsf{T})\,,\, 1 < p < \infty$$

Then M_{A^p} is of weak type (p,p).

In order to prove theorem 2.2 we will need the following lemma which is an extension of a lemma given in [Sjö83].

LEMMA 2.3. *Assume that the operators $T_k, k = 1, 2, \ldots$ are defined in T^n by*

$$T_kf(x) = \sup_{s\in I_k} K_s * |f|(x),$$

where the K_s are nonnegative and integrable in T^n, and K_s and the index sets I_k are such that T_kf are measurable for any measurable function f. Let, for each $i = 1, \ldots, n$, a decreasing sequence $\{\gamma_{ki}\}_{k=1}^\infty$ be given, and assume that the operators T_k are of weak type (p,p) with constant at most C_0 for some $p, 1 \le p < \infty$. Also assume that

$$\text{supp } K_s \subset \{x = (x_1, \ldots, x_n) \in \mathsf{T}^n; |x_i| \le \gamma_{ki}, i = 1, \ldots, n\}, s \in I_k,$$

and, denoting

$$K_s^*(x) = \sup\{K_s(x+y); |y_i| \le \gamma_{k+N,i}\ ,\ i = 1, \ldots, n\}$$

for $s \in I_k$ and some natural number N,

$$\int K_s^*(x)\,dx \le C_0, \quad s \in \cup_k I_k. \tag{2.1}$$

Then the operator

$$Tf(x) = \sup_k T_kf(x)$$

is of weak type (p,p) with constant depending only on $C_0, N, n,$and p.

The proof of this lemma is given in the authors thesis [JOR]. It is almost analogous to the proof of the original lemma so we will not give it here.

PROOF OF THEOREM 2.2. Let $z = re^{i\theta}$ and suppose $0 \le f(\varphi) \in L^p(\mathsf{T})$. Also suppose $A = 1$ in the definition of A^p_β. We have that

$$\mathscr{P}_0 f(z) \sim \frac{1}{\log\frac{1}{1-r}} \int_{\mathbb{T}} \frac{f(\varphi)}{1-r+|\theta-\varphi|} d\varphi$$

$$= \frac{1}{\log\frac{1}{1-r}} \left(\frac{1}{1-r+|x|} \chi_{(1-r)\left(\log\frac{1}{1-r}\right)^p<|x|<\pi} + \frac{1}{1-r+|x|} \chi_{|x|<(1-r)\left(\log\frac{1}{1-r}\right)^p} \right) * f(\theta)$$

$$= I_1 * f(\theta) + I_2 * f(\theta).$$

Let

$$T^i f(\beta) = \sup_{\substack{z\in A^p_\beta \\ r>1/2}} I_i * f(\theta) \qquad i = 1, 2.$$

We have

$$M_{A^p} f(\beta) \leq T^1 f(\beta) + T^2 f(\beta)$$

so we want to prove that T^i, $i = 1, 2$, are of weak type (p, p).

By considering the values of I_1 on the sets $\{2^{k-1}(1-r)(\log\frac{1}{1-r})^p \leq |x| \leq 2^k(1-r)(\log\frac{1}{1-r})^p\}$, we get, after some elementary calculation, the estimate

$$I_1(x,r) \leq C \sum_{k=1}^{Q(p,r)} \frac{1}{\log\frac{1}{1-r}} \frac{1}{2^k(1-r)\left(\log\frac{1}{1-r}\right)^p} \chi_{|x|\leq 2^k(1-r)\left(\log\frac{1}{1-r}\right)^p},$$

where

$$Q(p,r) = O\left(-\log\left((1-r)\left(\log\frac{1}{1-r}\right)^p\right)\right) = O\left(\log\frac{1}{1-r}\right)$$

Because $z \in A^p_\beta$ implies

$$|\theta - \beta| \leq (1-r)\left(\log\frac{1}{1-r}\right)^p,$$

we also have

$$\chi_{|x|\leq 2^k(1-r)\left(\log\frac{1}{1-r}\right)^p} * f(\theta) \leq \chi_{|x|\leq 2^{k+1}(1-r)\left(\log\frac{1}{1-r}\right)^p} * f(\beta).$$

Thus

$$T^1 f(\beta) \leq C \sup_{\substack{z\in A^p_\beta \\ r>1/2}} \sum_{k=1}^{Q(p,r)} \frac{1}{\log\frac{1}{1-r}} \frac{1}{2^k(1-r)\left(\log\frac{1}{1-r}\right)^p} \chi_{|x|\leq 2^{k+1}(1-r)\left(\log\frac{1}{1-r}\right)^p} * f(\beta)$$

$$\leq C \sup_{\substack{r \\ 1/2<r<1}} \sum_{k=1}^{Q(p,r)} \frac{1}{\log\frac{1}{1-r}} Mf(\beta) \leq C\, Mf(\beta) ,$$

where Mf is the Hardy-Littlewood maximal function. Hence $T_1 f$ is of strong type (p,p).

By considering the values of I_2 on the sets $\{2^{k-1}(1-r) \leq |x| \leq 2^k(1-r)\}$, we easily get that

$$I_2(x,r) \leq C \sum_{k=0}^{N(p,r)} \frac{1}{\log\frac{1}{1-r}} \frac{2^{-k}}{1-r} \chi_{|x|\leq 2^k(1-r)} ,$$

where

$$N(p,r) \leq 1 + \frac{p}{\log 2} \log\left(\log\frac{1}{1-r}\right) = O\left(p \log\left(\log\frac{1}{1-r}\right)\right) .$$

Because of this estimation, we have

$$T^2 f(\beta) \leq C \left(\sup_{z\in A^p_\beta} \sum_{k=0}^{N(p,r)} \frac{1}{\log\frac{1}{1-r}} \frac{2^{-k}}{1-r} \chi_{|x|\leq 2^k(1-r)} \right) * f(\theta)$$

$$\leq (\tau = \theta - \beta) \leq \sup_{\substack{1/2<r<1 \\ |\tau|\leq(1-r)\left(\log\frac{1}{1-r}\right)^p}} \left(\sum_{k=0}^{N(p,r)} \frac{1}{\log\frac{1}{1-r}} \frac{2^{-k}}{1-r} \chi_{|x+\tau|\leq 2^k(1-r)} \right) * f(\beta) .$$

Let $B_j = \{r : 2^{-2^j} \leq 1 - r \leq 2^{-2^{j-1}}\}, j \in \mathbf{Z}_+$. Then we have

$$T^2 f(\beta) \leq C \sup_j \sup_{\substack{r\in B_j \\ |\tau|\leq(1-r)\left(\log\frac{1}{1-r}\right)^p}} \left(\sum_{k=0}^{[jp+1]} \frac{1}{2^j} \frac{2^{-k}}{1-r} \chi_{|x+\tau|\leq 2^k(1-r)} \right) * f(\beta) \leq$$

$$\leq C \sup_j T_j f(\beta) ,$$

where

$$T_j f(\beta) = \sup_{\substack{r\in B_j \\ |\tau|\le(1-r)2^{jp}}} \left(\sum_{k=0}^{[jp+1]} \frac{1}{2^j} \frac{2^{-k}}{1-r} \chi_{|x+\tau|\le 2^k(1-r)} \right) * f(\beta) \equiv \sup_{\substack{r\in B_j \\ |\tau|\le(1-r)2^{jp}}} K_{r,\tau} * f(\beta) \, .$$

We shall now show that we can use Lemma 2.3 on the operators T_j. We start with a rather crude estimate. Take an arbitrary term in the sum, say $\frac{1}{2^k(1-r)} \chi_{|x+\tau|\le 2^k(1-r)} * f(\beta)$. This term satisfies

$$\frac{1}{2^k(1-r)} \chi_{|x+\tau|\le 2^k(1-r)} * f(\beta) = \frac{1}{2^k(1-r)} \int_{\beta+\tau-2^k(1-r)}^{\beta+\tau+2^k(1-r)} f(\varphi) d\varphi$$

$$\le \frac{2^{jp}}{2^k} \frac{1}{2^{jp}(1-r)} \int_{\beta+\tau-2\,2^{jp}(1-r)}^{\beta+\tau+2\,2^{jp}(1-r)} f(\varphi) d\varphi$$

where the inequality holds because $2^k \le 2^{jp+1}$. This now gives that

$$T_{jk} f(\beta) \equiv \sup_{\substack{r\in B_j \\ |\tau|\le(1-r)2^{jp}}} \frac{1}{2^k(1-r)} \chi_{|x+\tau|\le 2^k(1-r)} * f(\beta) \le C \frac{2^{jp}}{2^k} Mf(\beta)$$

and thus we have that T_{jk} is of weak type $(1,1)$ with constant at most $C\frac{2^{jp}}{2^k}$ and of strong type (∞,∞) with constant ~ 1.

Marcinkiewicz's interpolation theorem now gives that

$$\|T_{jk}\|_\rho \le c(p,\rho) \frac{2^{jp/\rho}}{2^{k/\rho}} \, , \quad 1 < \rho < \infty \, .$$

This implies that

$$\|T_j\|_\rho \le \left\| \sum_{k=0}^{[jp+1]} \frac{1}{2^j} T_{jk} \right\|_\rho \le \sum_{k=0}^{[jp+1]} \frac{1}{2^j} \|T_{jk}\|_\rho$$

$$\le c(p,\rho) \sum_{k=0}^{[jp+1]} \frac{1}{2^j} \frac{2^{jp/\rho}}{2^{k/\rho}} = c(p,\rho) \sum_{k=0}^{[jp+1]} \frac{2^{j(p/\rho-1)}}{(2^{1/\rho})^k}$$

$$\sim c(p,\rho) 2^{j(p/\rho-1)} \le c(p,\rho) \ \text{ if } \rho \ge p \, .$$

This implies that if $\rho \ge p$ then T_j is of strong type (ρ,ρ) uniformly in j, which gives one of the conditions in Lemma 2.3 .

Thus it remains to prove the rest of the conditions of the lemma. The condition of measurability is obviously satisfied, as are the conditions of integrable kernels and non-negative operators.

We have that

$$T_jf(\beta) \sim \sup_{\substack{r\in B_j \\ |\tau|\le(1-r)2^{jp}}} \sum_{k=0}^{[jp+1]} \frac{1}{2^j}\frac{1}{2^k(1-r)}\chi_{|x+\tau|\le 2^k(1-r)} * f(\beta)$$

$$\le \sup_{r\in B_j} \sum_{k=0}^{[jp+1]} \frac{1}{2^j}\frac{1}{2^k(1-r)}\chi_{|x|\le(2^k+2^{jp})(1-r)} * f(\beta) .$$

Letting

$$\gamma_j' = 2^{jp+1}2^{-2^j} \quad \text{and}$$

$$I_j = \{(r,\tau) : r \in B_j ,\ |\tau| \le (1-r)2^{jp+1} \}$$

$$\subset \{(r,\tau) : r \in B_j ,\ |\tau| \le 2^{-2^{j-1}+jp+1} \} ,$$

we have, for $s = (r,\tau) \in I_j$,

$$\operatorname{supp} K_s \subset \{x : |x| \le 2\, 2^{[jp+1]}2^{-2^{j-1}} \} .$$

Since $\{\gamma_j'\}$ is decreasing for j sufficiently large, it can be modified to get a decreasing sequence $\{\gamma_j\}$with the desired properties, satisfying $\gamma_j \le C\cdot\gamma_j'$ if C is large enough.

What remains is to prove the integrability of $K_s^*(x)$. Notice that

$$K_{r,\tau}(x) = \sum_{k=0}^{N(p,r)} \frac{1}{\log\dfrac{1}{1-r}}\frac{1}{2^k(1-r)}\chi_{|x+\tau|\le 2^k(1-r)} \le$$

$$\le \sum_{k=0}^{N(p,r)} \frac{1}{\log\dfrac{1}{1-r}}\frac{1}{2^k(1-r)}\chi_{|x|\le 2^k(1-r)+\tau} .$$

We now take $N = 1$ in the definition of $K_{r,\tau}^*(x)$ and let $r \in B_j$. Thus we get

$$K_{r,\tau}^*(x) = \sup_{|y|\le\gamma_{j+1}} K_{r,\tau}(x+y) .$$

Then

$$\int K^*_{r,\tau}(x)dx = \frac{1}{\log\frac{1}{1-r}} \int \sup\left\{ \sum_{k=0}^{N(p,r)} \frac{1}{2^k(1-r)} \chi_{|x+\tau+y|\le 2^k(1-r)} \; ; \; |y| \le \gamma_{j+1} \right\} dx$$

$$\le \frac{1}{\log\frac{1}{1-r}} \int \sum_{k=0}^{N(p,r)} \frac{1}{2^k(1-r)} \chi_{|x+\tau|\le 2^k(1-r)+\gamma_{j+1}} dx =$$

$$= \frac{1}{\log\frac{1}{1-r}} \sum_{k=0}^{N(p,r)} \frac{1}{2^k(1-r)} \int \chi_{|x|\le 2^k(1-r)+\gamma_{j+1}} dx = C(p,r,j) \le C(p)\,.$$

This implies that the last condition is satisfied, and Lemma 2.3 gives that T^2 is at least of weak type (p,p). This altogether gives that M_{A^p} is of weak type (p,p), and the theorem is proved.

Finally in this section, I am going to show that the L^p weakly tangential regions are the largest possible convergence regions in the disk for our operators if we want the regions to be regular in the sense that the boundary of the regions is described by a monotone function for $1-r$ sufficient small.

Assume that Ω is a convergence region for $\mathscr{P}_0 f(z)$ corresponding to the point $1 = e^0 \in \mathrm{T}$ with boundary defined by $|\arg(z)| = g(1-r)$ where $r = |z|$ and g is a monotone increasing function. Let Ω^x be the corresponding convergence region for $x = e^{i\beta}$, $\beta \in (0, 2\pi]$. Then we have that the maximal function $M_\Omega \mathscr{P}_0 f(z)$, where $M_\Omega u(x) = \sup_{z\in\Omega^x} |u(z)|$ is of weak type $(p,p), 1 \le p < \infty$ (see [Ga, result 1.1.2 and section 3]). We will now use this to prove the statement.

Let $Q = \{re^{i\theta} \in \mathrm{U} \;:\; |\theta| \le t\,,\; 1-r \le t\}$, and let $\Omega(t) = \Omega \cap \{1-r = t\}$ for an arbitrary $t \in (0,1)$. It is easy to see that

$$|\Omega(t)| \le |\{x \in \mathrm{T};\; M_\Omega \chi_Q(x) \ge 1/2\}|,$$

because $M_\Omega \chi_Q(x) = 1$ if $Q \cap \Omega^x \ne \emptyset$. In fact, we can replace $1/2$ with any number γ such that $0 < \gamma \le 1$ without violating the inequality.

Let $f(\varphi) = \log\frac{1}{t} \cdot \chi_{|\varphi|\le t}$. We want to prove that

$$|\{x \in \mathrm{T};\; M_\Omega \chi_Q(x) \ge \gamma\}| \le |\{x \in \mathrm{T};\; M_\Omega \mathscr{P}_0 f(x) \ge \gamma\}|.$$

From this and the weak type (p,p) estimates for $M_\Omega \mathscr{P}_0$ mentioned above, we would get that

$$|\Omega(t)| \le C\|f\|_p^p = C(1-r)\left(\log\frac{1}{1-r}\right)^p,$$

which would prove our claim.

First observe that $|\{x \in \mathsf{T};\ M_{\Omega}\chi_Q(x) \geq 1/2\}| = 2(g(t)+t)$.Next, we know that

$$\mathscr{P}_0 f(z) \sim \frac{1}{\log\frac{1}{1-r}} \int_{\mathsf{T}} \frac{f(\varphi)}{1-r+|\theta-\varphi|} d\varphi.$$

If we consider this estimation in the point $z = (1-t)e^{it}$ we can easily conclude that $P_0 f((1-t)e^{it}) > \gamma$ for some fix $\gamma > 0$. But it is easy to see that $(1-t)e^{it} \in (\Omega^x)$ if $|x| \leq g(t)+t$. From this it follows that

$$|\{x \in \mathsf{T};\ M_{\Omega}\chi_Q(x) \geq \gamma\}| \leq |\{x \in \mathsf{T};\ M_{\Omega}\mathscr{P}_0 f(x) \geq \gamma\}|,$$

so we have proved that the L^p weakly tangential regions are the best possible convergence regions in the above meaning.

3. The two dimensional results for L^1 functions.

In this section we will prove the convergence for $\mathscr{P}_{\lambda_1,0} f(z_1,z_2)$ in products of weakly tangential regions and nontangential regions in the bidisk, if $f \in L^1(\mathsf{T}^2)$.

THEOREM 3.1. *Let $f \in L^1(\mathsf{T}^2)$, and let $(z_1,z_2) = (r_1e^{i\theta_1}, r_2e^{i\theta_2})$. Define*

$$A_{\beta_1,\beta_2} = \{(z_1,z_2) \in \mathsf{U}^2;$$

$$1-r_1 \sim 1-r_2\ , |\beta_1-\theta_1| \leq (1-r_1), |\beta_2-\theta_2| \leq (1-r_2)\log\frac{1}{1-r_2}\}.$$

Then $\mathscr{P}_{\lambda_1,0} f(z_1,z_2)$ converges to $f(e^{i\beta_1},e^{i\beta_2})$ as (z_1,z_2) tends to $(e^{i\beta_1},e^{i\beta_2})$ in A_{β_1,β_2}, for a.a. $(e^{i\beta_1},e^{i\beta_2}) \in \mathsf{T}^2$.

As usual, the theorem follows from a maximal function estimate:

THEOREM 3.2. *Let*

$$M_A f(\beta_1,\beta_2) = \sup_{\substack{(z_1,z_2)\in A_{\beta_1,\beta_2} \\ r_1>1/2,r_2>1/2}} \mathscr{P}_{\lambda_1,0}|f|(z_1,z_2).$$

Then M_A is of weak type $(1,1)$

In the proof of this estimate we will use the following proposition:

PROPOSITION 3.3. *Let $g(r) : (0,1) \to \mathsf{R}_+$ be such that*
i) *$g(r)\cdot r$ is increasing on $(0,\alpha)$ for some $\alpha > 0$ and $g(r) \geq 1$ if $0 < r < 1$.*
ii) $\lim_{r\to 0} g(r)\cdot r = 0$.

Let $\mathscr{A}_{\beta_1,\beta_2}$ *be the collection of all intervals on* T^2 *of the type*

$$\{(\varphi_1,\varphi_2):\ |\beta_1-\varphi_1|\le r,\ |\beta_2-\varphi_2|\le g(r)\cdot r\},$$

and let

$$M_{\mathscr{A}}f(\beta_1,\beta_2)=\sup_{A\in\mathscr{A}_{\beta_1,\beta_2}}\frac{1}{|A|}\int_A|f(\varphi_1,\varphi_2)|d\varphi_1 d\varphi_2.$$

Then $M_{\mathscr{A}}$ *is of weak type* $(1,1)$.

PROOF. The assumptions on g and $\mathscr{A}_{\beta_1,\beta_2}$ give that the sets in $\mathscr{A}_{\beta_1,\beta_2}$ are convex and nested. Theorem 3.2.10 in [dGu] gives that $M_{\mathscr{A}}$ is of weak type $(1,1)$, with constant less than 25 and not depending on g.

PROOF OF THEOREM 3.2. Assume that $f\ge 0$. First we conclude that $M_A f\sim M_{A'}f$, where

$$A'_{\beta_1,\beta_2}=\Bigg\{(re^{i\theta_1},re^{i\theta_2}):1/2<r<1,$$

$$|\beta_1-\theta_1|\le(1-r),|\beta_2-\theta_2|\le(1-r)\left(\log\frac{1}{1-r}\right)\Bigg\},$$

because of Harnacks inequality, and the fact that $1-r_1\sim 1-r_2$.

Next we have that

$$M_{A'}f(\beta_1,\beta_2)$$

$$\sim\sup_{r,\theta_1,\theta_2\in A'_{\beta_1,\beta_2}}\frac{(1-r)^{2\lambda_1}}{\log\frac{1}{1-r}}\int_{\mathsf{T}^2}\frac{f(\varphi_1,\varphi_2)d\varphi_1 d\varphi_2}{(1-r+|\theta_1-\varphi_1|)^{2\lambda_1+1}(1-r+|\theta_2-\varphi_2|)}$$

$$=\sup_{r,\theta_1,\theta_2\in A'_{\beta_1,\beta_2}}\frac{(1-r)^{2\lambda_1}}{\log\frac{1}{1-r}}\left[\int_{B_1}+\int_{B_2}+\int_{B_3}+\int_{B_4}\right]$$

$$\times\frac{f(\varphi_1,\varphi_2)d\varphi_1 d\varphi_2}{(1-r+|\theta_1-\varphi_1|)^{2\lambda_1+1}(1-r+|\theta_2-\varphi_2|)},$$

where

$$B_1 = \left\{ (\varphi_1, \varphi_2) : |\beta_1 - \varphi_1| \leq 2(1-r), |\beta_2 - \varphi_2| \leq 2(1-r)\left(\log \frac{1}{1-r}\right) \right\},$$

$$B_2 = \left\{ (\varphi_1, \varphi_2) : |\beta_1 - \varphi_1| \leq 2(1-r), |\beta_2 - \varphi_2| > 2(1-r)\left(\log \frac{1}{1-r}\right) \right\},$$

$$B_3 = \left\{ (\varphi_1, \varphi_2) : |\beta_1 - \varphi_1| > 2(1-r), |\beta_2 - \varphi_2| \leq 2(1-r)\left(\log \frac{1}{1-r}\right) \right\},$$

$$B_4 = \left\{ (\varphi_1, \varphi_2) : |\beta_1 - \varphi_1| > 2(1-r), |\beta_2 - \varphi_2| > 2(1-r)\left(\log \frac{1}{1-r}\right) \right\}.$$

Here, of course,

$$r, \theta_1, \theta_2 \in A'_{\beta_1,\beta_2}$$

means that

$$(re^{i\theta_1}, re^{i\theta_2}) \in A'_{\beta_1,\beta_2}.$$

Let

$$I_i(f) = \frac{(1-r)^{2\lambda_1}}{\log \frac{1}{1-r}} \int_{B_i} \frac{f(\varphi_1,\varphi_2)\,d\varphi_1 d\varphi_2}{(1-r+|\theta_1-\varphi_1|)^{2\lambda_1+1}(1-r+|\theta_2-\varphi_2|)}$$

and let $T^i f$ be the corresponding maximal function, $i = 1, \ldots, 4$.

We now want to prove that each operator T^i is of weak type $(1,1)$. In order to do so we will need some estimates of the operators $I_i(f)$:

We have that

$$I_1(f) = \frac{(1-r)^{2\lambda_1}}{\log \frac{1}{1-r}} \int_{B_1} \frac{f(\varphi_1,\varphi_2)\,d\varphi_1 d\varphi_2}{(1-r+|\theta_1-\varphi_1|)^{2\lambda_1+1}(1-r+|\theta_2-\varphi_2|)}$$

$$\leq \frac{(1-r)^{2\lambda_1}}{\log \frac{1}{1-r}} \int_{B_1} \frac{f(\varphi_1,\varphi_2)\,d\varphi_1 d\varphi_2}{(1-r)^{2\lambda_1+1}(1-r)} = \frac{1}{(1-r)^2 \log \frac{1}{1-r}} \int_{B_1} f(\varphi_1,\varphi_2)\,d\varphi_1 d\varphi_2,$$

$$I_3(f) = \frac{(1-r)^{2\lambda_1}}{\log \frac{1}{1-r}} \int_{B_3} \frac{f(\varphi_1,\varphi_2)\,d\varphi_1 d\varphi_2}{(1-r+|\theta_1-\varphi_1|)^{2\lambda_1+1}(1-r+|\theta_2-\varphi_2|)}$$

$$\sim \frac{(1-r)^{2\lambda_1}}{(1-r)\log \frac{1}{1-r}} \int_{B_3} \frac{f(\varphi_1,\varphi_2)\,d\varphi_1 d\varphi_2}{|\beta_1-\varphi_1|^{2\lambda_1+1}}$$

$$\sim \sum_{k=1}^{[\log\frac{1}{1-r}]} \frac{(1-r)^{2\lambda_1}}{(1-r)\log\frac{1}{1-r}} \int_{\substack{|\beta_2-\varphi_2|\leq 2(1-r)\log\frac{1}{1-r} \\ 2^{k-1}(1-r)\leq|\beta_1-\varphi_1|\leq 2^k(1-r)}} \frac{f(\varphi_1,\varphi_2)d\varphi_1 d\varphi_2}{(2^k(1-r))^{2\lambda_1+1}} \leq$$

$$\leq \sum_{k=1}^{[\log\frac{1}{1-r}]} \frac{1}{(2^k)^{2\lambda_1}} \frac{1}{(1-r)\log\frac{1}{1-r}} \int_{\substack{|\beta_2-\varphi_2|\leq 2(1-r)\log\frac{1}{1-r} \\ |\beta_1-\varphi_1|\leq 2^k(1-r)\log\frac{1}{1-r}}} \frac{f(\varphi_1,\varphi_2)d\varphi_1 d\varphi_2}{(2^k(1-r))}$$

and

$$I_4(f) \sim \sum_{k=1}^{[\log\frac{1}{1-r}]} \frac{1}{(2^k)^{2\lambda_1}} \frac{1}{(1-r)\log\frac{1}{1-r}} \int_{|\beta_1-\varphi_1|\leq 2^k(1-r)\bigcap B_4} \frac{f(\varphi_1,\varphi_2)d\varphi_1 d\varphi_2}{1-r+|\beta_2-\varphi_2|}.$$

Finally, the estimate for $I_2(f)$ is

$$I_2(f) \sim \frac{1}{(1-r)\log\frac{1}{1-r}} \int_{B_2} \frac{f(\varphi_1,\varphi_2)d\varphi_1 d\varphi_2}{1-r+|\beta_2-\varphi_2|}.$$

We will now show that the operators T^i are of weak type $(1,1)$, which will give us the wanted weak type $(1,1)$ estimate for $M_{A'}$

T^1f The function $g(r) = (1-r)\log\frac{1}{1-r}$ satisfies the condition in Proposition 3.3 which together with the estimate above implies that T^1f is of weak type $(1,1)$.

T^3f Proposition 3.3 and the summation theorem for operators of weak type (p,p) (Theorem 3.8.2 in [dGu]) gives that T^3f is of weak type $(1,1)$.

T^4f This operator can be majorized by a weak type $(1,1)$ operator if we can prove the same for T^2f because T^4f is a sum of operators similar to T^2f with coefficients which allow the use of the summation theorem for weak type (p,p) operators.

T^2f To prove that T^2f is a weak type $(1,1)$ operator is the hard part of the proof. We will use the decomposition lemma given in section 2 to obtain this estimate.

Let $x_1 = \beta_1 - \varphi_1$, $x_2 = \beta_2 - \varphi_2$. Then

$$I_2(f) \le C\frac{1}{(1-r)\log\frac{1}{1-r}}\int_{B_2}\frac{f(\varphi_1,\varphi_2)d\varphi_1 d\varphi_2}{1-r+|x_2|}$$

$$= \frac{1}{(1-r)\log\frac{1}{1-r}}\int_{|x_2|\ge 2(1-r)\log\frac{1}{1-r}}\frac{f(\varphi_1,\varphi_2)}{1-r+|x_2|}\chi_{|x_1|\le 2(1-r)}d\varphi_1 d\varphi_2.$$

We now introduce a partition which is specially designed to be used with Lemma 2.2. Define $e_k = e^{-2^k}$ for $k = 0, 1, 2, \ldots$ and e_k is arbitrary but greater than 1 if $k < 0$. If we assume that $e_k < 1 - r < e_{k-1}$, we have

$$\log\frac{1}{1-r} \sim 2^k \text{ and } e_k 2^{k+1} \le c|x_2|.$$

If we also make the partition

$$e_{k-\nu}2^{k+1} \le |x_2| \le e_{k-\nu-1}2^{k+1}, \quad \nu = 0, 1, \ldots, k,$$

we can consider the kernel

$$\frac{c}{1-r}\chi_{|x_1|\le 2(1-r)}\sum_{\nu=0}^{k}\frac{2^{-k}}{1-r+|x_2|}\chi_{e_{k-\nu}2^{k+1}\le|x_2|\le e_{k-\nu-1}2^{k+1}},$$

which can be estimated with

$$\frac{c}{1-r}\chi_{|x_1|\le 2(1-r)}\sum_{\nu=0}^{k}\frac{2^{-k}}{e_{k-\nu}2^{k+1}+|x_2|}\chi_{|x_2|\le e_{k-\nu-1}2^{k+1}},$$

because of the lower bounds of the partition.

Accordingly we will consider the majorizing operator

$$\sum_{\nu=0}^{\infty}2^{-\nu}\sup_{k>\nu}\sup_{e_k\le t\le e_{k-1}} f*\left[\frac{c}{t}\chi_{|\varphi_1|\le 2t}\frac{2^{-k+\nu}}{e_{k-\nu}2^{k+1}+|\varphi_2|}\chi_{|\varphi_2|\le e_{k-\nu-1}2^{k+1}}\right]$$

$$= \sum_{\nu=0}^{\infty}2^{-\nu}\sup_{k>\nu}\sup_{e_k\le t\le e_{k-1}} f*K_t^\nu = \sum_{\nu=0}^{\infty}2^{-\nu}\sup_{k>\nu}T_k^\nu f,$$

where $t = 1 - r$, and the last equalities define K_t^ν and T_k^ν. Lemma 2.3 will give that the operators $\sup_{k>\nu} T_k^\nu$ are of weak type $(1, 1)$, uniformly in ν, and thus show that T^2 is of weak type $(1, 1)$.

We have to show the following:
i) K_t^ν is nonnegative and integrable on T^2.
ii) $T_k^\nu f$ is measurable if f is measurable.
iii) T_k^ν is of weak type $(1,1)$ uniformly in ν and k.
iv) For $i=1,2$ there exist $\{\gamma_{ki}\}_{k=1}^{\infty}$, $\gamma_{ki}\geq\gamma_{k+1,i}>0$, such that
  (a) supp $K_t^\nu\subset\{(\varphi_1,\varphi_2)\in\mathsf{T}^2 : |\varphi_i|\leq\gamma_{ki}, i=1,2\}$
  if $t\in s_k=[e_k,e_{k-1})$.
  (b) $\int K_t^* d\varphi_1 d\varphi_2\leq C_0$
  if $t\in\cup_k s_k$, where for $t\in s_k$,
  $K_t^*=\sup_{y=(y_1,y_2)}\{K_t^\nu(\varphi+y);\ |y_i|\leq\gamma_{k+N,i}, i=1,2\}$
  for some fixed natural number N.

We see that i) and ii) follows directly from the definitions of K_t^ν and T_k^ν.
iii) Let $k>\nu$. We have that

$$T_k^\nu f(\beta)=\sup_{t\in s_k}\frac{c}{t}\int_{|x_1|\leq 2t,\ |x_2|\leq e_{k-\nu-1}2^{k+1}}\frac{2^{-k+\nu}}{e_{k-\nu}2^{k+1}+|x_2|}f(\varphi_1,\varphi_2)d\varphi_1 d\varphi_2$$
$$=\sup_{t\in s_k}\frac{c}{t}\int_{|x_1|\leq 2t}f*\Lambda_{k,\nu}(\varphi_1,\beta_2)d\varphi_1\,,$$

where the convolution is with respect to φ_2 and

$$\Lambda_{k,\nu}(\varphi_2)=\frac{2^{-k+\nu}}{e_{k-\nu}2^{k+1}+|\varphi_2|}\chi_{|\varphi_2|\leq e_{k-\nu-1}2^{k+1}}\,.$$

This gives that

$$\|\Lambda_{k,\nu}\|_{L^1}=\int_{|\varphi_2|\leq e_{k-\nu-1}2^{k+1}}\frac{2^{-k+\nu}}{e_{k-\nu}2^{k+1}+|\varphi_2|}d\varphi_2$$
$$=2\int_{0<\varphi_2\leq e_{k-\nu-1}2^{k+1}}\frac{2^{-k+\nu}}{e_{k-\nu}2^{k+1}+\varphi_2}d\varphi_2\leq C_0\,.$$

Now Young's inequality implies that $f*\Lambda_{k,\nu}(\beta_2)\in L^1(\mathsf{T})$ and this, together with weak type $(1,1)$ estimates for Hardy-Littlewoods maximal functions, gives that T_k^ν is of weak type $(1,1)$ uniformly in ν and k.

iv) Let $\{\gamma_{k1}\}_{k=1}^{\infty}=\{2e_{k-1}\}_{k=1}^{\infty}$, $\{\gamma_{k2}\}_{k=1}^{\infty}=\{2^{k+1}e_{k-\nu-1}\}_{k=1}^{\infty}$.

Then iv) a) is obvious if we choose γ_{k1} suitably for $k \leq 0$. Thus it remains to prove that iv) b) is satisfied. Let $N = 1$ and let $t \in s_k$. Then

$$K_t^* = \sup_y \left\{ \frac{c}{t} \chi_{|\varphi_1+y_1|\leq 2t} \frac{2^{-k+\nu}}{e_{k-\nu}2^{k+1} + |\varphi_2 + y_2|} \chi_{|\varphi_2+y_2|\leq e_{k-\nu}2^{k+1}} \ : \right.$$

$$\left. |y_i| \leq \gamma_{k+1,i}\, , i = 1, 2 \right\}$$

$$\leq \sup_{y_2} \left\{ \frac{c}{t} \chi_{|\varphi_1|\leq 2t+2e_k} \frac{2^{-k+\nu}}{e_{k-\nu}2^{k+1} + |\varphi_2 + y_2|} \chi_{|\varphi_2+y_2|\leq e_{k-\nu}2^{k+1}} \ : |y_2| \leq 2^{k+2}e_{k-\nu} \right\}$$

$$\leq \frac{c}{t} \chi_{|\varphi_1|\leq 3t} \frac{2^{-k+\nu}}{e_{k-\nu}2^{k+1} + |\varphi_2|} \chi_{|\varphi_2|\leq e_{k-\nu-1}2^{k+1}+e_{k-\nu}2^{k+2}}$$

where the first inequality is due to the definition of γ_{ki}. Thus

$$\int K_t^* d\varphi_1 d\varphi_2 \leq C_0 \text{ for all } t.$$

With all this done, we can use Lemma 3.4 to get

$$\sup_{k>\nu} T_k^\nu \text{ is of weak type } (1,1) \text{ uniformly in } \nu$$

Thus

$$\sum_{\nu=0}^{\infty} 2^{-\nu} \sup_{k>\nu} T_k^\nu \text{is of weak type } (1,1)$$

which gives that T^2 is majorized by a weak type $(1,1)$ operator, and thus

$$M_A \text{ is of weak type } (1,1).$$

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