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SEPARABLE MAXIMAL PLURISUBHARMONIC FUNCTIONS IN TWO COMPLEX VARIABLES

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Abstract.

We use separation of variables applied to the complex Monge-Ampère equation in $\mathbf{C}^2$ to construct explicit formulas for some maximal plurisubharmonic functions.

1. Introduction and results.

Plurisubharmonic functions play a role in pluripotential theory analogous to that of subharmonic functions in classical potential theory. The plurisubharmonic functions are precisely those subharmonic functions which are invariant under biholomorphic mappings and they are of importance in multidimensional complex analysis.

A natural counterpart of the class of harmonic functions in classical potential theory is the class of maximal plurisubharmonic functions in pluripotential theory. For instance, the real part of any holomorphic function is pluriharmonic and hence a maximal plurisubharmonic function. We mention that, for example, Lundin's formula [7] and results of Siciak [9] for the relative extremal function give further explicit examples but, despite the fact that maximal plurisubharmonic functions have been studied for quite some time, very few explicit formulas are known. In this paper we use separation of variables to construct explicit formulas for some *separable* maximal plurisubharmonic functions, i.e. functions of the form

$$u(z, w) = f(z)g(w)$$

In contrast to the classical case we do not assume any smoothness. We use the characterization of maximal plurisubharmonic functions as generalized solutions to the homogeneous complex Monge-Ampère equation.

In this section we state our results and in the next section we briefly state

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some general properties of maximal plurisubharmonic functions. The third section contains the proofs of our theorems.
The first result is a system of equations for f and g.

THEOREM 1.1. *Let Ω be a domain in $\mathbf{C}^2$. Assume that $u(z,w)$ is a bounded maximal plurisubharmonic function of the form*

$$u(z,w) = f(z)g(w)$$

in Ω. Then one of the following two cases holds:

(i) *There is a real constant $\alpha \neq 0$ such that*

$$(1) \qquad df \wedge d^c f = \alpha f dd^c f, \quad dg \wedge d^c g = (1/\alpha) g dd^c g, \quad \text{and} \quad \alpha fg < 0 \quad \text{in } \Omega$$

and each of the functions f, g are either sub- or superharmonic.

(ii) *One of the functions is constant and the other one is a bounded sub- or superharmonic function on Ω.*

Conversely, every pair of bounded sub- or superharmonic functions f, g satisfying equation (1) gives a maximal plurisubharmonic function $u(z,w) = f(z)g(w)$.

REMARK 1. Observe that if one of the functions f and g is harmonic, then we have case (ii).

REMARK 2. Note that if f is smooth, then the equation for f can be written

$$|\nabla f|^2 = \alpha f \Delta f$$

In the next theorem we determine all separable solutions of the system (1), i.e., solutions of the form $f(x+iy) = p(x)q(y)$.

THEOREM 1.2. *Assume that f is a bounded sub- or superharmonic function of the form*

$$f(x+iy) = p(x)q(y)$$

in a domain U in $\mathbf{C}$ such that

$$f dd^c f = (1/\lambda - 1) df \wedge d^c f$$

where $\lambda \neq 0$ is a real number.

Then there is a real constant ν and a point $z^0 = x^0 + iy^0 \in \mathbf{C}$ such that

(i) *if $\lambda\nu \geq 0$, then U is contained in the strip $\{x + iy \in \mathbf{C} : |x - x^0| < \frac{\pi|\lambda|}{2\sqrt{\lambda\nu}}\}$ or*

(ii) *if* $\lambda\nu < 0$, *then* U *is contained in the strip* $\{x + iy \in \mathbf{C} : |y - y^0| < \frac{\pi|\lambda|}{2\sqrt{-\lambda\nu}}\}$ *and* f *can be extended to the strip where it is given by the formula*

$$f(z) = f(z^0)\Big(\cos(\sqrt{\lambda\nu}(x - x^0)/\lambda)\cos(\sqrt{-\lambda\nu}(y - y^0)/\lambda)\Big)^{\lambda}$$

COROLLARY 1.1. *If* $u(z, w)$ *is a bounded maximal plurisubharmonic function of the form*

$$u(x_1 + iy_1, x_2 + iy_2) = p_1(x_1)q_1(y_1)p_2(x_2)q_2(y_2)$$

in a domain Ω *in* $\mathbf{C}^2$, *then there are constants* ν_1, ν_2 *and* $\lambda_1 \notin \{0, 1\}$ *and a point* $(x_1^0 + iy_1^0, x_2^0 + iy_2^0) \in \mathbf{C}^2$ *such that*

$$u(x_1 + iy_1, x_2 + iy_2) =$$

$$= u(z^0)\Big(\cos(\sqrt{\lambda_1\nu_1}(x_1 - x_1^0)/\lambda_1)\cos(\sqrt{-\lambda_1\nu_1}(y_1 - y_1^0)/\lambda_1)\Big)^{\lambda_1}$$

$$\cdot\Big(\cos(\sqrt{\lambda_2\nu_2}(x_2 - x_2^0)/\lambda_2)\cos(\sqrt{-\lambda_2\nu_2}(y_2 - y_2^0)/\lambda_2)\Big)^{\lambda_2}$$

where $\lambda_1 + \lambda_2 = 1$.

REMARK 3. The functions are analogues of the maximal subharmonic (i.e. harmonic) functions

$$u(z) = u(z^0)\ \cos(\sqrt{\lambda\nu}(x - x^0)/\lambda)\cos(\sqrt{-\lambda\nu}(y - y^0)/\lambda)\ \text{ in }\ \mathbf{C}.$$

2. Maximal plurisubharmonic functions.

In this section we state some properties of maximal plurisubharmonic functions. If Ω is an open subset of $\mathbf{C}^n$, let $\mathrm{PSH}(\Omega)$ denote the class of plurisubharmonic functions on Ω and let $M_{k,k}(\Omega)$ denote the space of (k, k)-forms on Ω with Borel measure coefficients. Furthermore let $d = \partial + \overline{\partial}$ and $d^c = i(\overline{\partial} - \partial)$. Then $dd^c = 2i\partial\overline{\partial}$ and the complex Monge-Ampère operator $(dd^c)^n : \mathrm{PSH}(\Omega) \cap L^{\infty}_{loc}(\Omega) \to M_{n,n}(\Omega)$ is defined as follows: $dd^c u$ is defined in the sense of distributions and, for $2 \le k \le n$, $(dd^c u)^k$ is inductively defined as a positive (k, k)-current by an integration by parts formula

$$\int_{\Omega} (dd^c u)^k \wedge \theta = \int_{\Omega} u(dd^c u)^{k-1} \wedge dd^c\theta$$

where θ is an arbitrary $(n - k, n - k)$-form with compactly supported smooth coefficients, see [2].

DEFINITION 2.1. A plurisubharmonic function $u : \Omega \to \mathbf{R}$ is said to be maximal if for every relatively compact open subset G of Ω, and for each upper semicontinuous function v on $\bar{G}$ such that $v \in \mathrm{PSH}(G)$ and $v \leq u$ on ∂G, we have $v \leq u$ in G.

By $\mathrm{MPSH}(\Omega)$ we denote the class of maximal plurisubharmonic functions on Ω. In the complex plane, as a direct consequence of the definition, we have that $\mathrm{MPSH}(\Omega)$ is equal to the class of harmonic functions on Ω. This is no longer true in higher dimension. The class $\mathrm{PSH}(\Omega)$ is then a proper subclass of the class of subharmonic functions on Ω and there is no inclusion between $\mathrm{MPSH}(\Omega)$ and the class of harmonic functions on Ω. Notice that in the one-dimensional case the maximal plurisubharmonic functions are C^{∞} functions (even real analytic). This is in contrast to the case when $n \geq 2$ since one can easily see that there exist discontinuous maximal plurisubharmonic functions. For example, take any discontinuous subharmonic function $w : \mathsf{C} \to \mathsf{R}$ and define the function $u : \mathsf{C}^n \to \mathsf{R}$ by $u(z_1, ..., z_n) = w(z_n)$. The function u is clearly plurisubharmonic and, for each $a \in \mathsf{C}$, the function $(z_1, ..., z_{n-1}) \to u(z_1, ..., z_{n-1}, a)$ is constant and hence harmonic. This implies that u is a maximal plurisubharmonic function on C^n.

We mention two approximation theorems.

THEOREM 2.1. (Lelong [6]). *Let Ω be a pseudoconvex domain in C^n, $n \geq 2$. Then every locally bounded plurisubharmonic function on Ω can be approximated by continuous maximal plurisubharmonic functions in the L^1_{loc}-topology.*

THEOREM 2.2. (Sadullaev [8]). *Let Ω be an open set in C^n and let u be a maximal plurisubharmonic function on Ω. If D is a strictly pseudoconvex domain such that $\bar{D} \subset \Omega$, then $u|D$ is the limit of a decreasing sequence of continuous maximal plurisubharmonic functions on D.*

The class $\mathrm{MPSH}(\Omega)$ is in the one-dimensional case characterized by a linear operator since the harmonic functions are precisely the solutions to the Laplace equation. In general we have the following characterization theorem for $\mathrm{MPSH}(\Omega)$:

THEOREM 2.3. *Let Ω be an open subset of C^n and let u be a locally bounded plurisubharmonic function on Ω. Then u is maximal if and only if it satisfies the homogeneous complex Monge-Ampère equation $(dd^c u)^n = 0$.*

Thus locally bounded maximal plurisubharmonic functions in higher dimension are characterized by a nonlinear operator. The proof of the 'only if'-part can be found in Bremermann [3] when u is a C^2 function. Sadullaev proved the theorem for continuous maximal plurisubharmonic functions in [8]. The 'if'-part follows from Sadullaev's theorem [8] above, from the fact that the complex Monge-Ampère operator is continuous on decreasing sequences and from the solution to the generalized Dirichlet problem for

$(dd^c)^n$ obtained by Bedford and Taylor in [1]. The comparison theorem of Bedford and Taylor [2] gives the 'only if'-part in the general case.

We conclude this section by giving some examples of maximal plurisubharmonic functions. The function $\log|z|$ belongs to $\mathrm{MPSH}(\mathbf{C}^n \setminus \{0\})$. This can be seen from the fact that for any point $w \in \mathbf{C}^n \setminus \{0\}$, the one variable function $t \to \log|tw|$ is harmonic in $\mathbf{C}^n \setminus \{0\}$. Furthermore, if Ω is an open subset of $\mathbf{C}^n$, $n \geq 2$, and if v is a pluriharmonic function on Ω, then the function $\max\{0, v\}$ belongs to $\mathrm{MPSH}(\Omega)$.

Let Ω be an open subset of $\mathbf{C}^n$. If E is a subset of Ω, then the relative extremal function for E in Ω is defined by

$$u_{E,\Omega} = \sup\{v(z) : v \in \mathrm{PSH}(\Omega), v|E \leq -1, v \leq 0\}, \quad z \in \Omega$$

and

$$u^*_{E,\Omega}(z) = \limsup_{\zeta \to z} u_{E,\Omega}(\zeta), \quad z \in \Omega$$

is the upper semicontinuous regularization. If Ω is hyperconvex and E is relatively compact in Ω, then $u^*_{E,\Omega}$ belongs to $\mathrm{MPSH}(\Omega)$. For proofs of the above-mentioned examples, see e.g. [5].

If Ω is a domain in $\mathbf{C}^n$ and if a is a point in Ω, then the pluricomplex Green function of Ω with pole at a is defined by

$$g_\Omega(z, a) = \sup\{v(z) : v \in \mathrm{PSH}(\Omega), v < 0, v(z) - \log|z - a| \leq O(1) \text{ as } z \to a\}$$

for $z \in \Omega$. In [4] it was proved that if Ω is bounded, then $g_\Omega(z, a)$ belongs to $\mathrm{MPSH}(\Omega \setminus \{a\})$.

3. Separation of variables in the complex Monge-Ampère equation.

This section contains the proofs of our theorems.

Let Ω be an open set in $\mathbf{C}^n$ and let $u, v \in \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$. The product uv can then locally be written as a difference between two positive functions in $\mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ and $dd^c uv$ and $(dd^c uv)^2$ are well-defined as closed currents of order zero by bilinearity. In fact, given a relatively compact open subset Ω' of Ω, we can find a constant $K_{\Omega'} > 0$ such that $u + K_{\Omega'}$ and $v + K_{\Omega'}$ are positive on Ω' and $uv = \varphi_{\Omega'} - \psi_{\Omega'}$ where

$$\varphi_{\Omega'} = \tfrac{1}{2}(u + v + 2K_{\Omega'})^2$$

and

$$\psi_{\Omega'} = \tfrac{1}{2}\Big((u + K_{\Omega'})^2 + (v + K_{\Omega'})^2\Big) + K_{\Omega'}u + K_{\Omega'}v + {K_{\Omega'}}^2$$

are plurisubharmonic and bounded on Ω'. Now we define $(dd^c uv)^2$ on Ω' by bilinearity:

$$(dd^c uv)^2 = (dd^c \varphi_{\Omega'})^2 + (dd^c \psi_{\Omega'})^2 - 2dd^c \varphi_{\Omega'} \wedge dd^c \psi_{\Omega'} \tag{2}$$

and we have to show that this gives a well-defined measure on Ω, independent of the choice of Ω'. Let Ω'' be another relatively compact open subset of Ω, and let

$$\eta = 2(K_{\Omega''} - K_{\Omega'})(u + v + K_{\Omega''} + K_{\Omega'})$$

Then η is locally bounded on Ω and either plurisub- or plurisuperharmonic. We can assume that η is plurisubharmonic. Then it is easy to see that

$$\varphi_{\Omega''} - \varphi_{\Omega'} = \eta = \psi_{\Omega''} - \psi_{\Omega'}$$

on $\Omega' \cap \Omega''$ and if we replace $\varphi_{\Omega'}$ and $\psi_{\Omega'}$ by $\varphi_{\Omega''} = \varphi_{\Omega'} + \eta$ and $\psi_{\Omega''} = \psi_{\Omega'} + \eta$ respectively in the right-hand side of (2), we get the same measure on $\Omega' \cap \Omega''$. Thus the definitions of $(dd^c uv)^2$ coincide on $\Omega' \cap \Omega''$. This shows that $(dd^c uv)^2$ is well-defined as a measure on Ω. Furthermore, $du \wedge d^c u$ is defined by

$$du \wedge d^c u = \tfrac{1}{2} dd^c u^2 - u dd^c u$$

Note that $dd^c uv$ and $(dd^c uv)^2$ are not positive in general. However $du \wedge d^c u$ is always positive.

REMARK 4. Let Ω be an open set in $\mathbf{C}^n$ and $1 \leq k \leq n$. If u_j and v_j, $1 \leq j \leq k$, are locally bounded plurisubharmonic functions on Ω, then we can with the same method define $dd^c u_1 v_1 \wedge \ldots \wedge dd^c u_k v_k$ as a closed (k,k)-current of order zero on Ω.

PROOF OF THEOREM 1.1.

The currents $dd^c f$ and $dd^c g$ are defined, since f and g are bounded plurisub- or plurisuperharmonic as functions of (z, w).

Let $\omega = \omega(f, g) = d(f + g) \wedge d^c(f + g) - df \wedge d^c f - dg \wedge d^c g$. If f and g are smooth functions, then

$$\omega = df \wedge d^c g + dg \wedge d^c f \tag{3}$$

and

$$\omega \wedge \omega = -2(df \wedge d^c f) \wedge (dg \wedge d^c g) \tag{4}$$

By definition, ω and $\omega \wedge \omega$ can be written in terms of dd^c which means that they are continuous on decreasing sequences. Therefore, by regularization in (3) and (4), we see that

$$\omega(u, v) = \mu_1 dz \wedge d\overline{w} + \mu_2 d\overline{z} \wedge dw$$

for some signed measures μ_1 and μ_2, and that equation (4) is valid for all f and g.

We have

$$dd^c fg = \tfrac{1}{2} dd^c (f+g)^2 - \tfrac{1}{2} dd^c f^2 - \tfrac{1}{2} dd^c g^2 = d(f+g) \wedge d^c(f+g)$$

$$+ (f+g) dd^c (f+g) - df \wedge d^c f - f dd^c f - dg \wedge d^c g - g dd^c g$$

$$= \omega + f dd^c g + g dd^c f$$

and since

$$f dd^c g = \mu_3 dw \wedge d\overline{w}$$

$$g dd^c f = \mu_4 dz \wedge d\overline{z}$$

for some measures μ_3 and μ_4, we get

$$0 = (dd^c fg)^2 = \omega \wedge \omega + 2fg dd^c f \wedge dd^c g = 2fg dd^c f \wedge dd^c g - 2df \wedge d^c f \wedge dg \wedge d^c g$$

By the Lebesgue-Radon-Nikodym theorem there are unique decompositions in absolute continuous and singular parts:

(5) $$f dd^c f = h df \wedge d^c f + \sigma$$

and

(6) $$g dd^c g = k dg \wedge d^c g + \mu$$

so

$$df \wedge d^c f \wedge dg \wedge d^c g = h df \wedge d^c f \wedge k dg \wedge d^c g + \text{ singular part}$$

By the uniqueness of the Lebesgue-Radon-Nikodym decomposition the singular part must be zero, and from this it follows that σ and μ are zero. Thus $h(z)k(w) \equiv 1$ on (supp $df \wedge d^c f$) $\times$ (supp $dg \wedge d^c g$) so $h \equiv$ constant $= \alpha$ and $k \equiv 1/\alpha$.

Assume now that supp $df \wedge d^c f \wedge dg \wedge d^c g \neq \emptyset$. Take a connected component $A_z \times B_w$ of supp $df \wedge d^c f \wedge dg \wedge d^c g$. The equations (5) and (6) are then fulfilled, for some constant $\alpha \neq 0$, on $A_z \times B_w$. The separability of u gives that f fulfills (5) on $\Omega \cap (A_z \times \mathbf{C})$ and that g fulfills (6) on $\Omega \cap (\mathbf{C} \times B_w)$. By exhausting Ω in the same manner we see that f and g fulfill (5) and (6) respectively on Ω. Hence we get the same constant α in Ω. On the other hand, if supp $df \wedge d^c f \wedge dg \wedge d^c g = \emptyset$, then either supp $df \wedge d^c f = \emptyset$ or supp $dg \wedge d^c g = \emptyset$. This means that one of the functions f and g is constant.

It follows from the definition of maximality that the other function can be any bounded sub- or superharmonic function.

The converse is obvious by the formulas above. This proves the theorem.

PROOF OF THEOREM 1.2.

Put $\alpha = (1 - 1/\lambda)$. Then we get the equation

$$q(y)^2 T_\lambda(p)(x) + p(x)^2 T_\lambda(q)(y) = 0$$

where

$$T_\lambda(h) := hh'' + (1 - 1/\lambda)h'^2$$

in the sense of distributions. Thus $(T_\lambda(p), p^2)$ and $(T_\lambda(q), -q^2)$ are parallel vectors, but since they depend on different variables their direction must be constant, so

$$T_\lambda(p) = -\nu p^2 \quad \text{and} \quad T_\lambda(q) = \nu q^2$$

Now it is sufficient to solve the equation for p. Since the equation is invariant under translations, we change coordinates so that p is defined for $x = 0$ and $p(0) \neq 0$. The integral curves in the phase plane with coordinates (p, p') are generated by the vector field

$$p' \frac{\partial}{\partial p} - \left(\nu p + (1 - 1/\lambda)p'^2/p\right) \frac{\partial}{\partial p'}$$

which is smooth except at $p = 0$. Let $p_0 = p(0)$ and $p_0' = p'(0)$. Then there is a unique integral curve through the point (p_0, p_0'). We let

$$p(x) = p_0 \left(\frac{\cos(\sqrt{\lambda\nu}(x - x^0)/\lambda)}{\cos(\sqrt{\lambda\nu}x^0/\lambda)} \right)^\lambda$$

where x^0 is a point that fulfills

$$p_0' = p_0 \sqrt{\lambda\nu} \tan(\sqrt{\lambda\nu}x^0/\lambda)$$

and is chosen in the following way:

If $\lambda\nu < 0$, then x^0 is uniquely defined; If $\lambda\nu > 0$, then we choose x^0 so that the projection of U on the x-axis is contained in the interval $\{x \in R : |x - x^0| < \frac{\pi|\lambda|}{2\sqrt{\lambda\nu}}\}$; If $\nu = 0$, then the function f is constant. It is now easy to verify that $T_\lambda(p) = -\nu p^2$. Thus the maximal domain to which p can be extended is the interval above and U must be contained in the corresponding strip stated in the theorem. This concludes the proof.

REMARK 5. The vector field is singular at $p = 0$, but if we make a change of coordinates to (p, s) where $s = p'/p$ we get a resolution of this singularity; the vector field in these coordinates is

$$sp\frac{\partial}{\partial p} - (\nu - s^2/\lambda)\frac{\partial}{\partial s}$$

and the solutions can be found by integrating this vector field.

REFERENCES

1. E. Bedford and B. A. Taylor, *The Dirichlet problem for the complex Monge-Ampère operator,* Invent. Math. 37 (1976), 1–44.
2. E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions,* Acta Math. 149 (1982), 1–40.
3. H. J. Bremermann, *On a generalized Dirichlet problem for plurisubharmonic functions and pseudoconvex domains, characterization of Shilov boundaries,* Trans. Amer. Math. Soc. 91 (1959), 246–276.
4. M. Klimek, *Extremal plurisubharmonic functions and invariant pseudodistances,* Bull. Soc. Math. France 113 (1985), 231–240.
5. M. Klimek, *Pluripotential Theory,* Oxford University Press, 1991.
6. P. Lelong, *Discontinuité et annulation de l'operateur de Monge-Ampère complexe,* Séminaire d'analyse P. Lelong – P. Dolbeault – H. Skoda, années 1981/83 (P. Dolbeault P. Lelong and H. Skoda, eds.), Lectures Notes in Math. 1028, 1983.
7. M. Lundin, *The extremal plurisubharmonic function for convex symmetric subsets of R^n*, Michigan Math. J. 32 (1985), 107–201.
8. A. Sadullaev, *Plurisubharmonic measures and capacities on complex manifolds,* Russian Math. Surveys 36 (1981), 61–119.
9. J. Siciak, *On some extremal functions and their applications in the theory of analytic functions of several complex variables,* Trans. Amer. Math. Soc. 105 (1962), 322–57.

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