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KOROVKIN THEORY IN LIMINAL JB-ALGEBRAS

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Abstract.

In this paper the author studies the Korovkin closures in liminal JB-algebras. The construction of a net $(P_i)_{i\in I}$ of positive linear contractions for a subalgebra B of a dual JB-algebra A such that P_i maps A into B and converges strongly on B to the identity on B makes it possible to compute the universal Korovkin closures in dual JB-algebras. As a consequence we obtain an equivalent condition for a dual JB-algebra to contain a finite universal Korovkin system.

1. Introduction.

A JB-algebra is a real Jordan Banach algebra for which the norm satisfies $\|a^2 - b^2\| \leq \max(\|a^2\|, \|b^2\|)$, $\|a^2\| = \|a\|^2$. Let A be a JB-algebra and $T \subseteq A$ be a non-empty test set. The Korovkin closure $\mathrm{Kor}_A(T)$ of the test set T with respect to A is by definition the set of all $x \in A$ that satisfy the following condition:

If $(P_i)_{i\in I}$ is a net of positive linear contractions $P_i : A \to A$ such that

$$\lim_{i\in I} \|P_i(y) - y\| = 0 \quad \forall y \in T,$$

then also

$$\lim_{i\in I} \|P_i(x) - x\| = 0.$$

One of the main problems in Korovkin theory is to characterize various kinds of Korovkin closures and with the aid of such characterizations to give necessary and sufficient conditions such that a space has a finite Korovkin system, i.e. there is a finite test set T with $\mathrm{Kor}_A(T) = A$. This has successfully been done for spaces of continuous functions using the so called uniqueness closures of a test set, cf. the recent monograph by F. Altomare and M. Campiti: [1].

The Korovkin theory of C^*−algebras has been studied for example in the papers [3,4,5], [11],[13], [15]. To compute the Korovkin closure $\mathrm{Kor}_A(T)$ for non-commutative C^*−algebras one needs new methods, [3,4,5]. The motiva-

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tion to study Korovkin theory for JB-algebras was the following result: Let A be a C^*-algebra and $T \subseteq A$ be a test set, then

$$J^*(T) \subseteq \mathrm{Kor}_A(T \cup \{t^* \circ t : t \in T\}).$$

Here $J^*(T)$ denotes the J^*-subalgebra of A generated by T. A J^*-subalgebra of A is a *-closed and norm-closed subspace of A which is also closed with respect to the special Jordan product $a \circ b := \frac{1}{2}(ab + ba)$. See [4, Theorem 1.2] for a more general result. The main tool to prove the above result is the Kadison-Schwarz inequality for positive linear contractions on C^*-algebras; this inequality is also true for JB-algebras, cf. section 2.

The main results of this paper are as follows: Let A be a dual JB-algebra and B be a JB-subalgebra. Then there exists a net of positive linear contractions $P_i : A \to B$, such that $\lim_{i \in I} \|P_i(x) - x\| = 0$ holds for every $x \in B$, in particular we have $\mathrm{Kor}_A(B) = B$.

For a general JB-algebra A let $\mathscr{F}$ be the system of finite intersections of the primitive ideals of A and define for a subset B of A

$$\mathscr{F}(B) := \bigcap_{I \in \mathscr{F}} (B + I).$$

Then for a liminal JB-algebra we prove:

$$J(T) \subseteq \mathrm{Kor}^u_A(T \cup \{t^2 : t \in T\}) \subseteq \mathscr{F}(J(T))$$

where T is a test set and $\mathrm{Kor}^u_A(T \cup \{t^2 : t \in T\})$ is the so called universal Korovkin closure (to be defined below). For C^*-algebras the above mentioned results have been proved by F. Beckhoff in [3].

2. Existence of positive projections.

In this paragraph we fix notation and recall some folk theorems about the existence of positive projections on JBW-algebras.

For the general theory of JB and JBW-algebras the reader is referred to the book [9] by Hanche-Olsen and Størmer. We use the same notation as there.

Let A be a JB-algebra and $T \subseteq A$ a test set. The universal Korovkin closure $\mathrm{Kor}^u_A(T)$ of T in A is the set of all $x \in A$ that satisfy the following condition:

If B is a JB-algebra, $S : A \to B$ is a Jordan homomorphism and $(P_i)_{i \in I}$ is a net of positive linear contractions $P_i : A \to B$ such that

$$\lim_{i \in I} \|P_i(y) - S(y)\| = 0 \quad \forall y \in T,$$

then also

$$\lim_{i \in I} \|P_i(x) - S(x)\| = 0.$$

Obviously one has the inclusions:

$$T \subseteq \mathrm{Kor}_A^u(T) \subseteq \mathrm{Kor}_A(T).$$

If A is a non-unital JB-algebra then let $\tilde{A}$ be the JB-algebra with the adjoint unit. Recall that a positive linear map $P : A \to B$ between JB-algebras A,B is automatically continuous (with the same proof as in C^*-algebras) and that in the unital case $\|P\| = \|P1\|$ holds. Further, the extension $\tilde{P} : \tilde{A} \to \tilde{B}$, $\tilde{P}(x + \lambda 1) := P(x) + \lambda\mu 1$ for a fixed $\mu \geq \|P\|$, $x \in A$, $\lambda \in R$ is a positive linear extension of P with $\|\tilde{P}\| = \mu \geq \|P\|$. A.G. Robertson and M.G. Youngson proved that the Kadison-Schwarz inequality is still valid in unital JB-algebras.

2.1. THEOREM. *Let A, B be* JB-*algebras and $P : A \to B$ a positive linear map such that $\|P\| \leq 1$. Then for every $a \in A$: $P(a^2) \geq (P(a))^2$.*

PROOF. Adjoin identities to A and B and extend P to a unital map $\tilde{P} : \tilde{A} \to \tilde{B}$ as above. Then apply [14], Theorem 1.2.

Now an application of Lemma 1, Satz 1 and Bemerkung 3 in Kap. 3 of [3] yields the following

2.2. PROPOSITION. *Let T be a test set in a* JB-*algebra A. Then*

$$J(T) \subseteq \mathrm{Kor}_A^u(T \cup \{t^2 : t \in T\}) \subseteq \mathrm{Kor}_A(T \cup \{t^2 : t \in T\})$$

where $J(T)$ denotes the JB-*algebra generated by T.*

Now let A be a unital JB-algebra and φ a state, i.e. a positive linear functional on A such that $\|\varphi\| = 1$. If in addition for every symmetry $s \in A$ (i.e. $s^2 = 1$)

$$\varphi(U_s(a)) = \varphi(a) \quad \forall a \in A$$

holds, then φ is called a tracial state, cf [12] for a number of equivalent conditions. Here $U_x(y) = \{xyx\} = 2x \circ (x \circ y) - x^2 \circ y$ for $x, y \in A$.

2.3. PROPOSITION. *Let M be a* JBW-*algebra with a faithful normal tracial state τ. Let N be a* JBW-*subalgebra of M containing the identity of M. Then there exists a normal positive linear projection $Q : M \to N$ with the following properties:*

(i) *For $x \geq 0$ we have: $Q(x) = 0 \Leftrightarrow x = 0$.*

(ii) $\|Q(x)\| \leq \|x\| \quad (x \in M)$.

(iii) $Q(x) \circ a = Q(x \circ a)$ *for* $x \in M$, $a \in N$.
(iv) $Q(x)^2 \leq Q(x^2) \quad (x \in M)$.

In von Neumann algebras this result was proved by H. Umegaki, [18], compare [16], Proposition 4.4.12, too. The proof in Sakai's book even works in JBW-algebras. The main tool is Sakai's linear Radon-Nikodym theorem for normal positive linear functionals on von Neumann algebras. But also the latter result holds true in JBW-algebras with the same proof, cf. [16], Prop. 1.24.4.

We need a sufficient condition for the existence of a faithful normal tracial state τ on a JBW-algebra. In finite von Neumann algebras there always exists a faithful family of normal tracial states, compare [19] for a beautiful proof. In modular JBW-algebras, [9, 5.1.2], we have the

2.4. THEOREM. *Let M be a modular* JBW-*algebra and $\varphi \in M_*^+$, the positive part of the pre-dual of M, a linear positive normal functional on M. Then there exists a positive normal linear tracial state τ_φ on M such that $\tau_{\varphi|Z} = \varphi$ where Z denotes the center of M.*

Again, Yeadon's proof works in the case of modular JBW-algebras because modular JBW-algebras behave in the same way as finite von Neumann algebras. Recall that by definition two projections p, q in M are equivalent, $p \sim q$, if there exists a finite family of symmetries $s_1, \ldots, s_n \in M$ such that $U_{s_1} \ldots U_{s_n} p = q$. Therefore the general dimension theory in JBW-algebras is different from the dimension theory in von Neumann algebras. For example, in general JBW-algebras the following property does not hold:

(*) If $(p_i)_{i \in I}$ and $(q_i)_{i \in I}$ are families of projections in M with $p_i \perp p_j$, $q_i \perp q_j$ for $i \neq j$ and $p_i \sim q_i$ for $i \in I$ then $p := \sup_{i \in I} p_i \sim \sup_{i \in I} q_i =: q$.

i.e. equivalence is in general not completely additive. One needs the additional assumption $p \perp q$, cf. [9], Lemma 5.2.9 and [17], Theorem 9, p. 19. But in von Neumann algebras equivalence is completely additive, and this is used in the proof of Lemma 1 in [19]. That (*) is true in modular JBW-algebras seems to be a folk theorem, for I could not find a reference. In the special case of JW-algebras it can be deduced from results of Topping's memoir [17].

The proof of (*) in modular JBW-algebras depends heavily on lattice theory. Let L be the projection lattice of M. Recall that two projections $p, q \in M$ are perspective iff there exists a common complement $e \in M$, i.e. e is a projection in L such that

$$p \wedge e = 0 = q \wedge e \quad \text{and} \quad p \vee e = 1 = q \vee e.$$

In [20] it is shown that in general JBW-algebras equivalence and perspectivity coincide. But by [10], §13 Theorem 14, perspectivity is completely additive in a complete orthocomplemented modular lattice. This shows that (*) holds in modular JBW-algebras.

Finally notice that Akemann's criterion for relatively weak compactness of a set K in the pre-dual of a von Neumann algebra that is needed in the proof of Lemma 2 in [19] is also true in the pre-dual of a JBW-algebra. This completes the proof of Theorem 2.4

Summing up, we have the following result: In a modular countably decomposable JBW-algebra there always exists a faithful normal tracial state. This is (as usual) easily seen by considering a maximal family $(\tau_i)_{i\in I}$ of tracial states with pairwise orthogonal (central) support projections e_i. Such a family has by the countable decomposability at most countable non-zero elements. Then build a suitable weighted sum.

3. Dual JB-algebras.

3.1. For the theory of dual JB-algebras the reader is referred to the article [6]. Let us recall some facts about dual JB-algebras.

Let S be a subset of a JB-algebra A; then define

$$S^\circ := \{a \in A \,:\, a \circ S = \{0\}\}.$$

A JB-algebra is called dual, if for every norm-closed quadratic ideal I in A we have $(I^\circ)^\circ = I$. A subspace I of A is called a quadratic ideal in A if, for each element $a \in I$ and $b \in A$, the element $U_a(b)$ lies in I.

Now let (A_λ) be a family of JB-algebras. By $(\sum A_\lambda)_0$ we denote the JB-algebra of all functions $f : \Lambda \to \bigcup_\lambda A_\lambda$ such that $f(\lambda) \in A_\lambda$ and f vanishes at infinity, i.e. for every $\epsilon > 0$ there is a finite index set $\Lambda_\epsilon \subseteq \Lambda$ such that $\|f(\lambda)\| < \epsilon$ for all $\lambda \in \Lambda \setminus \Lambda_\epsilon$. With the norm $\|f\| := \sup_\lambda \|f(\lambda)\|$ the algebra $(\sum_\lambda A_\lambda)_0$ is a JB-algebra. If the algebras A_λ are dual for every $\lambda \in \Lambda$, then $(\sum A_\lambda)_0$ is dual by [6, Lemma 1.5]. Any JB-subalgebra B of a dual JB-algebra A is dual, [6, Cor.2.6].

Every dual JB-algebra A is Jordan isomorphic to a dual JB-algebra of the form $(\sum A_\lambda)_0$ where the algebras A_λ are simple dual JB-algebras in A. Therefore A_λ is (isomorphic to) $H_3(O)$, the Hermitean 3×3 matrices over the Cayley numbers O, A_λ is a spin factor or A_λ is a (reversible) JC-algebra contained in $C(H_\lambda)_{\text{s.a.}}$, the self-adjoint part of the compact operators on some complex Hilbert space H_λ, cf. [6, Theorem 3.3] and the proof of [6, Corollary 1.4].

Now we come to the main result of the paper.

3.2. THEOREM. *Let A be a dual* JB-*algebra and B be a* JB-*subalgebra of A.*

Then there exists a net $(P_i)_{i\in I}$ of positive linear contractions $P_i : A \to B$ such that

$$\lim_{i\in I} \|P_i(b) - b\| = 0$$

for every $b \in B$.

PROOF. By 3.1 we can assume that $A = (\sum A_\lambda)_0$, where the A_λ are simple Jordan ideals in A. A_λ is exactly one of the following types:

(i) A_λ is a (reversible) JC-algebra of compact self-adjoint operators.

(ii) A_λ is a spin factor.

(iii) A_λ is $H_3(O)$.

Consider B^{**} as a subalgebra of A^{**} and define $\mathcal{P}$ to be exactly the set of projections $p \in B^{**}$ such that p is a finite supremum of minimal projections in B^{**}. As a subalgebra of a dual algebra B is dual, therefore the minimal projections of B^{**} are contained in B, [6, Prop. 2.1,Th. 3.3]. The idea for the proof is to apply the results of §2 to the reduced algebras $U_p(A) \supseteq U_p(B)$ for $p \in \mathcal{P}$. We divide the proof into several steps.

Step 1. Definitions.

Let $p \in \mathcal{P}$ be a minimal projection, $p = (p_\lambda)$. By definition of $A = (\sum A_\lambda)_0$ for only finitely many λ we have $p_\lambda \neq 0$. For $p \in B^{**}$ define

$$\Lambda_p := \{\lambda \in \Lambda : p_\lambda \neq 0\} \quad , \quad p = (p_\lambda), \quad p_\lambda \in A_\lambda^{**}.$$

Λ_p is called the support of p.

Claim: Let $p \in \mathcal{P}$. Then Λ_p is a finite subset of Λ.

This is clear, since $p = \vee_{i=1}^n p_i$ for minimal projections p_i in B^{**}, and Λ_{p_i} is finite. By definition of p as the range projection of $\sum_{i=1}^n p_i$ it is easy to see that $\Lambda_p = \bigcup_{i=1}^n \Lambda_{p_i}$.

Step 2. For $p \in \mathcal{P}$ we prove that $U_p(A)$ and $U_p(B)$ are JBW-algebras. Moreover $p \in B \Leftrightarrow p \in A$.

To see this, let $p \in \mathcal{P}$ have the support $\Lambda_p = \{\lambda_1, \dots, \lambda_k\}$, $p = (p_\lambda)$. The projections $p_{\lambda_i} \in A_{\lambda_i}^{**} \subseteq A^{**}$, $i = 1, \dots, k$ are pairwise orthogonal and we have $p = \sum_{i=1}^k p_{\lambda_i}$. Therefore

$$(1) \qquad U_p(A) = U_{\sum_{i=1}^k p_{\lambda_i}}(A) = \sum_{i=1}^k U_{p_{\lambda_i}}(A) = \sum_{i=1}^k U_{p_{\lambda_i}}(A_{\lambda_i})$$

and this is a direct sum since $\{p_{\lambda_i} A p_{\lambda_j}\} = \{0\}$ for $i \neq j$ ($A_{\lambda_i}^{**}$ is an ideal in A^{**}). As p is a finite supremum of minimal projections of B, the λ-components of p are finite suprema of projections of A_λ. Now consider the three cases mentioned above:

(i) A_{λ_i} is a JC-algebra of compact self-adjoint operators. p_{λ_i} is a finite supremum of compact projections, i.e. it has a finite dimensional range in the complex Hilbert space H_{λ_i}, $A_{\lambda_i} \subseteq C(H_{\lambda_i})_{s.a.}$. We conclude that $U_{p_{\lambda_i}}(B) \subseteq U_{p_{\lambda_i}}(A)$ are finite-dimensional JB-algebras, in fact they are JBW-algebras.

(ii) A_{λ_i} is a spin factor, whence $A_{\lambda_i} = A_{\lambda_i}^{**}$. Any projection q in a spin factor different from $0, 1$ is minimal and maximal in A_{λ_i}. Summing up, we have

$$U_{p_{\lambda_i}}(A_{\lambda_i}) = \{0\} \quad \text{or} \quad A_{\lambda_i} \quad \text{or} \quad Rp_{\lambda_i},$$

in any case the JB-subalgebra $U_{p_{\lambda_i}}(B)$ of $U_{p_{\lambda_i}}(A)$ is strongly closed (in a spin factor the norm-topology and the strong topology coincide, [9, Prop. 6.1.7]), i.e. $U_{p_{\lambda_i}}(B) \subseteq U_{p_{\lambda_i}}(A)$ are JBW-algebras.

(iii) $A_{\lambda_i} = H_3(\mathrm{O})$ is finite-dimensional. The proof is then completed as in (i).

Further, $U_p(B) = \bigoplus_{i=1}^k U_{p_{\lambda_i}}(B)$ so that $U_p(A)$ and $U_p(B)$ are JBW-algebras, since the direct summands are JBW-algebras.

Now let p be in $B \subseteq A$. Then $U_p(B) \subseteq U_p(A)$ and both $U_p(B)$ and $U_p(A)$ are unital with unit p. If $p \notin B$, then by Hahn-Banach:

$$A \cap B^{**} = A \cap \overline{B}^{\sigma(B^{**},B^*)} = A \cap \overline{B}^{\sigma(A^{**},A^*)} = \overline{B}^{\sigma(A,A^*)} = B.$$

Since p is in B^{**}, we have $p \notin A$.

Step 3. For $p \in \mathscr{P}$ the algebra $U_p(A)$ is modular and countably decomposable.

Since minimal projections are modular, p is modular as a finite supremum of modular projections in $B^{**} \subseteq A^{**}$, [9, Theorem 7.6.4]. Therefore the JBW-algebra $U_p(A) \subseteq U_p(A^{**})$ is modular. Now choose a family (q_α) of pairwise orthogonal projections $q_\alpha \neq 0$ in $U_p(A)$. For $(q_\alpha) = (q_\lambda^{(\alpha)})_\lambda$ we have:

$$q_\alpha \leq p \quad \text{and} \quad q_\lambda^{(\alpha)} = 0 \quad \text{for} \quad \lambda \notin \Lambda_p.$$

Let p_{λ_i} be a compact projection in a JC-algebra of compact operators (case (i)), then $p_{\lambda_i} \geq q_{\lambda_i}^{(\alpha)}$ is true only for finitely many projections $q_{\lambda_i}^{(\alpha)}$. This is also true in case (iii) where p_{λ_i} is in $H_3(\mathrm{O})$. Finally, let p_{λ_i} be a projection in a spin factor (case(ii)), then $p_{\lambda_i} = 0, 1$ or p_{λ_i} is minimal. In this case p_{λ_i} can dominate only two orthogonal projections. Summing up, (q_α) is a finite family of orthogonal projections, in particular $U_p(A)$ is countably decomposable.

Step 4. Let p be in $\mathscr{P}$ but $p \notin B$ (therefore $p \notin A$). Then consider the JBW-algebras $U_p(B) \oplus p \subseteq U_p(A) \oplus p$ with the unit p adjoined. $U_p(A) \oplus p$ is modular and countably decomposable.

The JB-algebras $U_p(B) \oplus p \subseteq U_p(A) \oplus p$ are as subalgebras of A^{**} obviously $\sigma(A^{**}, A^*)$-closed, i.e. they are JBW-algebras by step 3. Further,

$U_p(A) \oplus p \subseteq U_p(A^{**})$, therefore $U_p(A) \oplus p$ is modular as a JBW-subalgebra of a modular JBW-algebra. Finally, let $x + \lambda p$, $x \in U_p(A)$, $\lambda \in R$ be a projection in $U_p(A) \oplus p$, then $\lambda = 0$ and x is a projection in $U_p(A)$ or $\lambda = 1$ and $x^2 = -x$. Since two projections of the second kind can not be orthogonal, $U_p(A) \oplus p$ is countably decomposable by 3.

Step 5. Now we construct positive linear maps $P_p : A \to B$ such that $\|P_p\| \leq 1$ for $p \in \mathscr{P}$.

For $p \in \mathscr{P}$ define:

$$M_p := \begin{cases} U_p(A) & \text{for} \quad p \in B, \\ U_p(A) \oplus p & \text{for} \quad p \notin B, \end{cases} \quad N_p := \begin{cases} U_p(B) & \text{for} \quad p \in B, \\ U_p(B) \oplus p & \text{for} \quad p \notin B. \end{cases}$$

Then M_p is a modular countably decomposable JBW-algebra and N_p is a JBW-subalgebra on M_p with the unit element of M_p. By the results of §2 there exists a positive projection $\tilde{Q}_p : M_p \to N_p$ such that $\|\tilde{Q}_p\| \leq 1$. Define $Q_p := \tilde{Q}_p$ if $p \in B$. Now let $p \notin B$; denote by q the unit element of the JBW-algebra $U_p(B)$. Then define $Q_p := U_q \circ \tilde{Q}_p : U_p(A) \to U_p(B)$ ($U_p(B)$ is an ideal in $U_p(B) \oplus p$). Recall that B as a dual JB-algebra is an ideal in B^{**}, i.e. the multiplier algebra $M(B)$ of B is equal to B^{**}, [6, Th. 3.3]. Since $p \in B^{**} = M(B)$ it follows $U_p(B) \subseteq B$, [8]. Finally define

$$P_p : A \to B, \quad P_p := Q_p \circ U_p.$$

U_p and U_q are positive contractions, therefore P_p is a positive contraction. Since $\mathscr{P}$ is a directed set $(P_p)_{p \in \mathscr{P}}$ is a net of positive linear contractions.

Step 6. To complete the proof we show:

$$\lim_{p \in \mathscr{P}} \|P_p(b) - b\| = 0 \quad \forall b \in B.$$

Let $p \in \mathscr{P}$ and $b \in B$. Then

$$P_p(b) = Q_p(U_p(b)) = U_p(b),$$

since Q_p contains $U_p(B)$ in its range.

Since B is dual, we may assume $B = (\sum_{i \in I} B_i)_0$. Here B_i is a simple ideal in B and as above we have the following three cases:

(i) B_i is a JC-algebra, $B_i \subseteq C(H_i)_{s.a.}$.

(ii) B_i is a spin factor.

(iii) $B_i = H_3(O)$.

Now let $\epsilon > 0$ be given. Choose a finite index set $I_0 \subseteq I$ such that

$$\sup_{i \in I \setminus I_0} \|b_i\| < \frac{\epsilon}{2} \quad \text{where} \quad b = (b_i)_{i \in I}. \tag{2}$$

Let $i \in I_0$. In case (ii) define $p_i = 1_i$ where 1_i is the unit element of the spin factor B_i. In case (iii) $B_i = H_3(\mathrm{O})$ define $p_i = 1_i$. Then in both cases p_i is a finite supremum of minimal projections in B (recall that by [6, Cor. 2.2, Cor. 2.6] every projection $q \in B^{**}$ is a supremum of orthogonal minimal projections, in the above case this supremum is finite). Finally assume $B_i \subseteq C(H_i)_{\text{s.a.}}$. Then b_i is a compact operator, in particular b_i is of the following form (by spectral theory):

$$b = \sum_{n=1}^{\infty} \lambda_n \langle \cdot , e_n \rangle e_n, \quad , \quad \|e_n\| = 1, \tag{3}$$

where the series converges in the operator norm and e_n are pairwise orthogonal vectors in H_i. $(\lambda_n)_{n \in N}$ is a sequence converging to zero. Let $e_1, \ldots, e_N$ span the range spaces of the compact spectral projections $p_1^{(i)}, \ldots, p_n^{(i)}$ that correspond to pairwise different eigenvalues $\neq 0$ of b_i. We may assume

$$\sup_{n \geq N+1} |\lambda_n| < \frac{1}{4} \epsilon. \tag{4}$$

Now define $p_i = \sum_{k=1}^{N} p_k^{(i)}$.

Summing up, we defined for $i \in I_0$ projections $p_i \in B_i^{**}$, such that every p_i is a finite supremum of minimal projections of B. Put $p := (p_i)_{i \in I}$, $p_i := 0$ for $i \in I \setminus I_0$; then $p \in \mathscr{P}$.

Now let $q \in \mathscr{P}$, $q = (q_i)$, $p \leq q$. It follows

$$\begin{aligned} \sup_{i \in I \setminus I_0} \|(b - P_q(b))_i\| &= \sup_{i \in I \setminus I_0} \|b_i - U_{q_i}(b_i)\|) \\ &\leq \sup_{i \in I \setminus I_0} 2\|b_i\| \overset{(2)}{<} \epsilon. \end{aligned} \tag{5}$$

Let $i \in I_0$; then in the cases (ii) and (iii) we have $1_i \geq q_i \geq p_i \geq 1_i$, i.e. $q_i = 1_i$ and it follows

$$\|b_i - U_{q_i}(b_i)\| = \|b_i - U_{1_i}(b_i)\| = 0. \tag{6}$$

Finally assume $i \in I_0$ and $B_i \subseteq C(H_i)_{\text{s.a.}}$ (i.e. case (i)). Then a computation in $C(H_i)_{\text{s.a.}}$ yields:

$$\begin{aligned} \|b_i - U_{q_i}(b_i)\| &= \|b_i - q_i b_i q_i\| \\ &\leq \|b_i - b_i q_i\| + \|b_i q_i - q_i b_i q_i\| \\ &\leq \|b_i - b_i q_i\| + \|b_i - q_i b_i\| \, \|q_i\|) \\ &\leq 2\|b_i - b_i q_i\| \end{aligned} \tag{7}$$

since b_i, q_i are self-adjoint and $\|b_i - q_i b_i\| = \|(b_i - q_i b_i)^*\|$. For $\xi \in H_i$ we have:

$$\begin{aligned}
&\|b_i\xi - b_iq_i\xi\|^2\\
&= \left\| \sum_{n=1}^{\infty} \lambda_n \langle \xi, e_n\rangle e_n - \sum_{n=1}^{\infty} \lambda_n \langle q_i\xi, e_n\rangle e_n \right\|^2\\
&= \left\| \sum_{n\geq N+1} \lambda_n \langle \xi - q_i\xi, e_n\rangle e_n \right\|^2 \qquad \text{(since } q_i \geq p_i\text{)}\\
&= \sum_{n\geq N+1} |\lambda_n \langle \xi - q_i\xi, e_n\rangle|^2\\
&\leq \left(\sup_{n\geq N+1} |\lambda_n| \, \|\xi - q_i\xi\| \right)^2\\
&\leq \left(\sup_{n\geq N+1} |\lambda_n| \right)^2 2^2 \, \|\xi\|^2.
\end{aligned}$$

This inequality implies:

$$\|b_i - b_iq_i\| \leq 2 \sup_{n\geq N+1} |\lambda_n|. \tag{8}$$

so that

$$\|b_i - U_{q_i}(b_i)\| \overset{(7),(8)}{\leq} 4 \sup_{n\geq N+1} |\lambda_n| \overset{(4)}{\leq} \epsilon. \tag{9}$$

The equations (5),(6) and (9) yield for every projection $q \in \mathscr{P}$ such that $q \geq p$ the inequality:

$$\|b - P_q(b)\| = \|b - U_q(b)\| = \sup_{i\in I} \|b_i - U_{q_i}(b_i)\| < \epsilon.$$

This completes the proof.

3.3. Proposition. *Let A be a dual* JB-*algebra, B be a* JB-*subalgebra of A and $T \subseteq A$ be a test set. Then*

$$B = \mathrm{Kor}_A^u(B) = \mathrm{Kor}_A(B)$$

and

$$J(T) = \mathrm{Kor}_A^u(T \cup \{t^2 : t \in T\}) = \mathrm{Kor}_A(T \cup \{t^2 : t \in T\}).$$

Proof. The inclusions $B \subseteq \mathrm{Kor}_A^u(B) \subseteq \mathrm{Kor}_A(B)$ are always true. Assume $x \in \mathrm{Kor}_A(B)$. With Theorem 3.2 choose a net $(P_i)_{i\in I}$ of positive linear contractions $P_i : A \to B$ such that

$$\lim_{i \in I} \|P_i(b) - b\| = 0 \quad \forall b \in B.$$

Since $x \in \mathrm{Kor}_A(B)$ it follows

$$\lim_{i \in I} \|P_i(x) - x\| = 0$$

and $P_i(x) \in B$ implies $x \in B$. This proves the first equality. Finally

$$\mathrm{Kor}_A(T \cup \{t^2 : t \in T\}) \subseteq \mathrm{Kor}_A(J(T)) = J(T)$$

by the first part of the proof. Since

$$J(T) \subseteq \mathrm{Kor}^u_A(T \cup \{t^2 : t \in T\}) \subseteq \mathrm{Kor}_A(T \cup \{t^2 : t \in T\})$$

the proof is complete.

The above proposition shows that a dual JB-algebra A has a finite universal Korovkin system T iff $J(T)$ is equal to A, i.e. T generates A as a JB-algebra.

4. Korovkin closures in liminal JB-algebras.

In this paragraph we give an estimation of the universal Korovkin closure from above for liminal JB-algebras. Here the development follows closely that of [3] where the corresponding C^*-algebra versions of the following results were established. In some cases the proofs in [3] have to be modified to obtain the corresponding JB-versions. Only in these cases a proof will be given.

4.1. Let A be a JB-algebra. Recall that a Type I factor representation of A is a Jordan homomorphism $\pi : A \to M$ such that M is a Type I JBW-factor and $\pi(A)$ is weak* dense in M. A primitive ideal I of A is by definition the kernel of a Type I factor representation. A JB-algebra has a faithful family of Type I factor representations: Let φ be a pure state of A, then $\pi_\varphi : A \to c(\varphi) \circ A^{**}$, $\pi_\varphi(x) := c(\varphi) \circ x$ where $c(\varphi)$ is the central support of φ, is a Type I factor representation, [2, Prop. 5.6, Prop. 8.7]. A liminal JB-algebra is a JB-algebra such that all quotients A/I with respect to primitive ideals I are dual JB-algebras.

Now denote by $\mathscr{F}$ the system of all finite intersections of primitive ideals; for a subset B of A define

$$\mathscr{F}(B) := \bigcap_{I \in \mathscr{F}} (B + I).$$

In [3, Kap. X, Satz 1] F. Beckhoff proved that $\mathscr{F}(\cdot)$ is a closure operation, i.e. the map $B \mapsto \mathscr{F}(B)$ defines the closed sets of a topology on A. His proof carries over to JB-algebras, but we will not use this fact.

4.2. LEMMA. *Let $I \subseteq A$ be a norm-closed Jordan ideal in the* JB-*algebra A. Further, let $T \subseteq A$ be a test set and $\pi_I : A \to A/I$ be the canonical homomorphism. Then*

$$J(\pi_I(T)) = \pi_I(J(T)). \tag{1}$$

Moreover

$$\pi_I(\mathrm{Kor}^u_A(T)) \subseteq \mathrm{Kor}^u_{A/I}(\pi_I(T)). \tag{2}$$

The proof is easy: To prove (1) use the fact that the image of the JB-algebra $J(T)$ under a Jordan homomorphism is a JB-algebra (i.e. $\pi_I(J(T))$ is norm-closed). The inclusion in (2) can be checked by using only the definition of the respective Korovkin closures.

PROPOSITION. *Let A be a liminal* JB-*algebra and $T \subseteq A$ be a test set. Then*

$$J(T) \subseteq \mathrm{Kor}^u_A(T \cup \{t^2 : t \in T\}) \subseteq \mathscr{F}(J(T)).$$

PROOF. Because of Proposition 2.2 only the second inclusion has to be proved. Let $I = \bigcap_{j=1}^n I_j$ be a finite intersection of primitive Jordan ideals $I_j := \ker\pi_j$ where $\pi_j : A \to c(\varphi_j) \circ A^{**}$ are Type I factor representations corresponding to pure states φ_j. Since A is liminal, the algebras A/I_j are dual. Define

$$\pi : A \to \bigoplus_{j=1}^n \pi_j(A) \quad , \quad x \mapsto (\pi_1(x), \ldots, \pi_n(x)).$$

Then π is a Jordan homomorphism and obviously $I = \ker\pi$. By [6, Lemma 1.5] the algebra $\bigoplus_{j=1}^n \pi_j(A) \simeq \bigoplus_{j=1}^n A/I_j$ is dual as a finite direct sum of the dual JB-algebras A/I_j. Moreover $A/\ker\pi \simeq \pi(A)$ is a JB-subalgebra of the dual JB-algebra $\bigoplus_{j=1}^n \pi_j(A)$, i.e. $A/\ker\pi$ is dual, [6, Cor. 2.6]. With Lemma 4.2 (2), Proposition 3.3 and Lemma 4.2 (1) we obtain the following chain of inclusions:

$$\begin{aligned}\pi_I(J(T)) &\subseteq \pi_I(\mathrm{Kor}^u_A(T\cup\{t^2 : t\in T\}))\\ &\overset{4.2(2)}{\subseteq} \mathrm{Kor}^u_{A/I}(\pi_I(T\cup\{t^2 : t\in T\}))\\ &= \mathrm{Kor}^u_{A/I}(\pi_I(T)\cup\{\pi_I(t)^2 : t\in T\})\\ &\overset{3.3}{=} J(\pi_I(T))\\ &= \pi_I(J(T)).\end{aligned}$$

Since for a subset B in A we have

$$\mathscr{F}(B) = \bigcap_{J\in\mathscr{F}} \pi_J^{-1}\pi_J(B),$$

it follows

$$\mathrm{Kor}^u_A(T\cup\{t^2 : t\in T\}) \subseteq \pi_I^{-1}\pi_I(J(T)),$$

but $I\in\mathscr{F}$ was arbitrary thus

$$\mathrm{Kor}^u_A(T\cup\{t^2 : t\in T\}) \subseteq \mathscr{F}(J(T)).$$

This completes the proof.

4.4. Remark. Consider the situation of Proposition 4.3; as a test set let $T := B$ be a JB-subalgebra of the liminal JB-algebra A. Since $\pi_I(B)$ for $I\in\mathscr{F}$ is a JB-algebra in A/I, $\pi_I^{-1}\pi_I(B) = B+I$ is a JB-algebra in A. Thus $\mathscr{F}(B) = \bigcap_{I\in\mathscr{F}}(B+I)$ is a JB-algebra that contains B as a subalgebra: $B\subseteq\mathscr{F}(B)$.

Claim: B separates $P(\mathscr{F}(B))\cup\{0\}$ where $P(\mathscr{F}(B))$ is the set of pure states of $\mathscr{F}(B)$.

To see this, consider pure states φ,ψ of $\mathscr{F}(B)$ and choose pure state extensions $\tilde\varphi,\tilde\psi$ of φ,ψ to A respectively. Define $I := \ker\pi_{\tilde\varphi}\cap\ker\pi_{\tilde\psi}$ where $\pi_{\tilde\varphi},\pi_{\tilde\psi}$ are the corresponding Type I factor representations. Using the definition of the central supports of $\tilde\varphi,\tilde\psi$ respectively, it is easy to see that $\ker\pi_{\tilde\varphi}\subseteq\ker\tilde\varphi$ and $\ker\pi_{\tilde\psi}\subseteq\ker\tilde\psi$. Assume $\varphi_{|B} = \psi_{|B}$. Then for $x\in\mathscr{F}(B)\subseteq B+I$ we have $x = y+i$, $y\in B$, $i\in I$ and

$$\varphi(x) = \tilde\varphi(y)+\tilde\varphi(i) = \psi(y) = \tilde\psi(y)+\tilde\psi(i) = \psi(x).$$

Thus $\varphi=\psi$. Similarly, B separates a pure state φ on $\mathscr{F}(B)$ and the zero linear functional. This completes the proof of the claim.

In the C^*-algebra case for a liminal C^*-algebra A one could now apply the Stone-Weierstrass theorem for Type I C^*-algebras to obtain the equality $B = \mathscr{F}(B)$ since then $\mathscr{F}(B)$ would be a liminal, hence Type I C^*-algebra, cf. [7, 11.1.8, 4.2.4].

PROBLEM. Are there versions of the Stone-Weierstrass theorem for liminal (or more general, Type I) JB-algebras ?

If the answer is positive, then for liminal JB-algebras one would have $J(T) = \mathscr{F}(J(T))$ and therefore

$$\mathrm{Kor}_A^u(T \cup \{t^2 : t \in T\}) = J(T).$$

In particular a liminal JB-algebra would have a finite universal Korovkin system iff A is finitely generated as a JB-algebra.

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