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WEIERSTRASS REPRESENTATIONS FOR HARMONIC MORPHISMS ON EUCLIDEAN SPACES AND SPHERES

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Abstract.

We construct large families of harmonic morphisms which are holomorphic with respect to Hermitian structures by finding heierarchies of Weierstrass-type representations. This enables us to find new examples of complex-valued harmonic morphisms from Euclidean spaces and spheres.

1. Introduction.

Let $\phi : U \to N$ be a smooth mapping between Riemannian manifolds. Then ϕ is called a *harmonic morphism* if, for every real valued function harmonic on an open set $W \subset N$ with $\phi^{-1}(W)$ non-empty, the pull-back $f \circ \phi$ is harmonic on $\phi^{-1}(W)$ in M. Equivalently, ϕ is a harmonic morphism if and only if ϕ is both horizontally weakly conformal and harmonic [5, 9]. In the case when U is a domain in $\mathbf{R}^n$ with its standard Euclidean structure and N is the complex plane $\mathbf{C}$, the equations for horizontal weak conformality and harmonicity are, respectively,

$$\sum_{i=1}^{n} \left(\frac{\partial \phi}{\partial x^i} \right)^2 = 0 , \tag{1}$$

$$\triangle \phi \equiv \sum_{i=1}^{n} \frac{\partial^2 \phi}{(\partial x^i)^2} = 0 . \tag{2}$$

In fact, provided (1) is satisfied, (2) is equivalent to the minimality of the fibres at regular points [1].

An important example of a harmonic morphism is the following. Let U be

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a domain of $\mathbf{R}^{2m} \cong \mathbf{C}^m$ and suppose that ϕ is holomorphic with respect to a Kähler structure on $\mathbf{R}^{2m}$, then ϕ is a harmonic morphism [5]. In a more general setting, but restricting the dimension of the domain to be 4, the second author showed that, if U is a domain of a 4-dimensional Einstein manifold, then any submersive harmonic morphism ϕ is holomorphic with respect to an integrable Hermitian structure J on U. Furthermore, the fibres of ϕ are superminimal with repect to J [11]. Thus there is a strong relationship between holomorphic maps and harmonic morphisms.

In [4], we studied those harmonic morphisms defined on domains U of $\mathbf{R}^{2m}$, $(m \in \mathbf{N})$, which are holomorphic with respect to an integrable Hermitian structure on U, finding full global examples in the case $m \geq 3$, which are neither holomorphic with respect to a Kähler structure nor have superminimal fibres, as well as such examples which factor to a domain of $\mathbf{R}^{2m-1}$. In this paper we concentrate on the case of superminimal fibres and study reduction to odd-dimensional Euclidean spaces and to spheres, constructing large classes of examples in terms of holomorphic data.

More generally, we refer to a parametrization of solutions to Equations (1) and (2) in terms of holomorphic data as a "Weierstrass representation", after the local representation of Weierstrass for minimal surfaces in $\mathbf{R}^3$. We have observed an interesting duality between the theory of minimal surfaces and harmonic morphisms [2].

In [2], Weierstrass representations for harmonic morphisms defined on Euclidean spaces and spheres were obtained in the case when the fibres are totally geodesic. More precisely, if $(\xi_1, \ldots, \xi_n)$ is an n-tuple of meromorphic functions of z such that $\sum \xi_i^2 = 0$, then:

(i) the *inhomogeneous* equation

$$\xi_1 x^1 + \ldots + \xi_n x^n = 1 \tag{3}$$

locally determines all (submersive complex-valued) harmonic morphisms $z = z(x)$ on domains of $\mathbf{R}^n$ with totally geodesic fibres;

(ii) the *homogeneous* equation

$$\xi_1 x^1 + \ldots + \xi_n x^n = 0 \tag{4}$$

locally determines all (submersive complex-valued) harmonic morphisms $z = z(x)$ on domains of S^{n-1} with totally geodesic fibres. A similar representation for harmonic morphisms from real hyperbolic spaces H^{n-1} with totally geodesic fibres has been obtained by S. Gudmundsson [8].

Here we develop a far richer description of harmonic morphisms in terms of holomorphic data, obtaining a *heierarchy* of Weierstrass representations on domains of $\mathbf{R}^n$ such that:

(i) on $\mathbf{R}^{2m}$, the Weierstrass representation locally gives all (submersive

complex-valued) harmonic morphisms which are holomorphic with respect to a Hermitian structure with superminimal fibres;

(ii) reduction $\mathbf{R}^n \to \mathbf{R}^{n-1}$ commutes with the Weierstrass representation;

(iii) on $\mathbf{R}^3$ and $\mathbf{R}^4$, the Weierstrass representation describes *all* (submersive complex-valued) harmonic morphisms.

Finally we show how to construct harmonic morphisms from spheres using our Weierstrass representations by choosing the appropriate holomorphic function to be homogeneous. We refer the reader to the work of Gudmundsson for other constructions of harmonic morphisms on complex projective spaces and Kähler manifolds [6,7].

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2. Holomorphic harmonic morphisms and reduction.

By a Hermitian structure on an open subset U of $\mathbf{R}^{2m}$ we mean the smooth choice of an almost Hermitian structure on U which is integrable (cf. [4]). By a *holomorphic harmonic morphism* on U we mean a complex-valued harmonic morphism which is holomorphic with respect to some Hermitian structure on U. We briefly summarize the characterization of holomorphic harmonic morphisms given in [4] and describe how to reduce to other manifolds.

Let $(x^1, \ldots, x^{2m})$ be standard coordinates for $\mathbf{R}^{2m}$ and introduce complex coordinates $q^1 = x^1 + ix^2, q^2 = x^3 + ix^4$ etc. If J is a positive integrable Hermitian structure defined on an open subset $U \subset \mathbf{R}^{2m}$, then locally J is characterised by $m(m-1)/2$ functions $\mu_1, \ldots, \mu_{m(m-1)/2}$ holomorphic in m complex parameters $(z^1, \ldots, z^m)$ as follows:

Given $\mu = (\mu_1, \ldots, \mu_{m(m-1)/2})$, let $M = \left(M^i_{\bar{j}}(\mu)\right)$ be the skew symmetric matrix

$$(5) \qquad \left(M^i_{\bar{j}}(\mu)\right) = \begin{pmatrix} 0 & \mu_1 & \mu_2 & \cdots & \mu_{m-1} \\ -\mu_1 & 0 & \mu_m & \cdots & \mu_{2m-3} \\ -\mu_2 & -\mu_m & 0 & \cdots & \mu_{3m-6} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\mu_{m-1} & -\mu_{2m-3} & -\mu_{3m-6} & \cdots & 0 \end{pmatrix}.$$

As in [4] this matrix determines a positive almost Hermitian structure at any point of $\mathbf{R}^{2m}$, namely that with (1,0)-cotangent space given by $\operatorname{span}\{e^i = dq^i - M^i_{\bar{j}} dq^{\bar{j}} : i = 1, \ldots, m\}$.[1] Now suppose that the $\mu_i = \mu_i(z)$ are holomorphic functions on a domain V of $\mathbf{C}^m$ and let $h^1(z), \ldots, h^m(z)$ be fur-

[1] We use the double summation convention throughout.

ther holomorphic functions on V. Consider the following system of equations:

$$F^i(q,z) \equiv q^i - M^i_{\bar{j}}(z)q^{\bar{j}} - h^i(z) = 0, \quad (i = 1,\dots,m)\,. \tag{6}$$

where we write $M(z)$ for $M(\mu(z))$. The system (6) has the form

$$F(q,z) = 0 \tag{7}$$

where $F : \mathbf{R}^{2m} \times V \to \mathbf{C}^m$. Provided the determinant of the Jacobian matrix

$$K = (\partial_j F^i) \text{ where } \partial_j = \frac{\partial}{\partial z^j}$$

is non-zero, we can locally solve (7) for $z = z(q)$. Then, on suitable open sets, $z(q) = (z^1(q),\dots,z^m(q))$ form complex coordinates for the complex manifold (U,J) where $J = J(M)$ is given at $q \in U$ by $J_q = J(M(z(q)))$. Note that all complex coordinates with respect to any Hermitian structure are given locally this way. Indeed, if $q \mapsto M^i_{\bar{j}}(q)$ defines a Hermitian structure, then

$$w^i = q^i - M^i_{\bar{j}}(q)q^{\bar{j}} \quad (i = 1,\dots,m) \tag{8}$$

give complex coordinates (see, for example [4]). Any other complex coordinates $z = (z^1,\dots,z^m)$ are related by equations $w^i = h^i(z)$ for some holomorphic functions h^i, whence, writing $M^i_{\bar{j}}(q) = M^i_{\bar{j}}(z(q))$, we see that $z = z(q)$ satisfies (6).

More precisely, only those Hermitian structures with values at each point in a large cell of the space $SO(2m)/U(m)$ of all positive almost Hermitian structures are parametrized as above. However, by acting on $\mathbf{R}^{2m}$ by an isometry, we may always assume, at least locally, that J satisfies this condition. In the general case, we must allow the μ_i's to become infinite.

Let J be a Hermitian structure on $U \subset \mathbf{R}^{2m}$ characterized by Equation (6) above. Then any holomorphic map $\phi : (U,J) \to \mathbf{C}$ is a holomorphic function of the complex coordinates $(z^1,\dots,z^m)$. Now, in a neighbourhood of a regular point of ϕ, it is no loss of generality to suppose that $\phi = z^1$. Then the Laplacian of the function $z^1 = z^1(q)$ is calculated to be [4]:

$$\Delta z^1 = \frac{4}{\det K} \sum_{1 \le i < j \le m} (-1)^{i+j} \begin{vmatrix} \partial_2 M^i_{\bar{j}} & \partial_3 M^i_{\bar{j}} & \dots & \partial_m M^i_{\bar{j}} \\ \partial_2 F^{k_1} & \partial_3 F^{k_1} & \dots & \partial_m F^{k_1} \\ \vdots & \vdots & \ddots & \vdots \\ \partial_2 F^{k_{m-2}} & \partial_3 F^{k_{m-2}} & \dots & \partial_m F^{k_{m-2}} \end{vmatrix} \tag{9}$$

where $(k_1,\dots,k_{m-2}) = (1,\dots,\hat{i},\dots,\hat{j},\dots,m)$. Since z^1 is holomorphic it is horizontally weakly conformal, and so, if $\Delta z^1 = 0$ then $z^1 : U \to \mathbf{C}$ is a harmonic morphism.

A particular class of solutions is given by the case when each $\mu_i(z)$ depends on z^1 only, for then all the partial derivatives $\partial_2 M_j^i, \ldots, \partial_m M_j^i$ vanish. In this case the Hermitian structure J is constant along the fibres of z^1, equivalently the fibres of z^1 are *superminimal*. Conversely, by the above remarks, any (submersive complex-valued) harmonic morphism defined on an open subset of $\mathbf{R}^{2m}$, holomorphic with respect to a Hermitian structure and with superminimal fibres, may be described this way locally.

Our method of reducing holomorphic harmonic morphisms defined on domains in $\mathbf{R}^{2m}$ to other manifolds follows from the fundamental property that the composition of two harmonic morphisms is a harmonic morphism [5] and a converse to this [6]. In particular, we shall be concerned with the following projections which are all harmonic morphisms:

(i) *Orthogonal projection* $\pi_1^n : \mathbf{R}^n \to \mathbf{R}^{n-1}$ given by the formula $\pi_1^n(x^1, \ldots, x^n) = (x^2, \ldots, x^n)$.

(ii) *Radial projection* $\pi_r^n : \mathbf{R}^n \backslash \{0\} \to S^{n-1}$ given by the formula $\pi_r^n(x^1, \ldots, x^n) = (x^1, \ldots, x^n)/|x|$.

(iii) *Radial projection followed by the Hopf map* $\pi_H^{2m} : \mathbf{R}^{2m} \backslash \{0\} = \mathbf{C}^m \backslash \{0\} \to \mathbf{C}P^{m-1}$ given by the formula $\pi_H^{2m}(q^1, \ldots, q^m) = [q^1, \ldots, q^m]$.

So, for example, a harmonic morphism on a domain in S^{n-1} is equivalent to a harmonic morphism on a domain in $\mathbf{R}^n \backslash \{0\}$ invariant under radial projection. Precisely, if $\phi : W \subset S^{n-1} \to \mathbf{C}$ is a harmonic morphism, then the composition $\Phi = \phi \circ \pi_r^n|_{(\pi_r^n)^{-1}(W)}$ is a harmonic morphism on $(\pi_r^n)^{-1}(W)$ such that $\partial\Phi/\partial r = 0$, where $r = |x|$ denotes the radial coordinate. Conversely, if $\Phi : U \subset \mathbf{R}^n \to \mathbf{C}$ is a harmonic morphism on an open set U with $U \cap S^{n-1}$ non-empty and such that $\partial\Phi/\partial r = 0$, then it follows that the restriction $\phi = \Phi|_{U \cap S^{n-1}} : U \cap S^{n-1} \to \mathbf{C}$ is a harmonic morphism on the domain $U \cap S^{n-1} \subset S^{n-1}$. We formulate the invariance in a more general setting:

Let J be a Hermitian structure defined on a domain $U \subset \mathbf{R}^{2m}$ with associated complex coordinates $(z^1, \ldots, z^m)$ given locally by Equation (6). Let $v = a^j(x)(\partial/\partial x^j)$ be a vector field on U. If $\phi : (U, J) \to \mathbf{C}$ is a holomorphic function, then ϕ is *invariant under* v if and only if the directional derivative $d\phi(v)$ vanishes. As remarked above, in a neighbourhood of a regular point of ϕ we may assume without loss of generality that ϕ is the holomorphic function z^1.

PROPOSITION 2.1. *Let $\phi = z^1 : (U, J) \to \mathbf{C}$ be a holomorphic function, where $z = (z^1(q), \ldots, z^m(q))$ is a solution to* (6). *Then ϕ is invariant under $v = a^j(x)\frac{\partial}{\partial x^j}$ if and only if*

$$\begin{vmatrix} w^1(\alpha, z(q)) & \partial_2 F^1 & \dots & \partial_m F^1 \\ w^2(\alpha, z(q)) & \partial_2 F^2 & \dots & \partial_m F^2 \\ \vdots & \vdots & \ddots & \vdots \\ w^m(\alpha, z(q)) & \partial_2 F^m & \dots & \partial_m F^m \end{vmatrix} = 0 \tag{10}$$

for all $q \in U$, where $\alpha = \alpha^j \frac{\partial}{\partial q^j}$ is the complex vector field associated to v obtained by setting $\alpha^j = a^{2j-1} + ia^{2j}$ $(j = 1, \dots, m)$, and $w^i(q, z) \equiv q^i - M^i_{\bar{j}}(z)q^{\bar{j}}$ (thus $w^i(q, z)$ is the part of F^i which is homogeneous of degree 1 in q).

PROOF. The function z^1 is invariant under v if and only if

$$\alpha^I \frac{\partial z^1}{\partial q^I} = 0$$

where we sum over $I = 1, \dots, m, \bar{1}, \dots, \bar{m}$. But

$$\alpha^I \frac{\partial z^1}{\partial q^I} = -\alpha^I (K^{-1})^1_b \frac{\partial F^b}{\partial q^I} = -\alpha^I \frac{1}{\det K} \hat{K}^b_1 \frac{\partial F^b}{\partial q^I}$$

where $\hat{K}^b_a$ is the (b, a)-th entry in the matrix of cofactors of K. But since the expression $w^i(\alpha, z) = \alpha^i - M^i_{\bar{j}}(z)\alpha^{\bar{j}}$ is homogeneous of degree one in α,

$$\alpha^I \frac{\partial F^b}{\partial q^I} = \alpha^I \frac{\partial w^b}{\partial q^I} = w^b(\alpha, z)$$

and the result follows.

Special cases of the above proposition are as follows:

(i) *Orthogonal projection* $\pi^{2m}_1 : \mathbf{R}^{2m} \to \mathbf{R}^{2m-1}$ given by the formula $\pi^{2m}_1(x^1, x^2, \dots, x^{2m}) = (x^2, \dots, x^{2m})$. Putting $a = (1, 0, \dots, 0)$, we see that z^1 reduces to $\mathbf{R}^{2m-1}$ if and only if

$$\begin{vmatrix} 1 & \partial_2 F^1 & \dots & \partial_m F^1 \\ \mu_1 & \partial_2 F^2 & \dots & \partial_m F^2 \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{m-1} & \partial_2 F^m & \dots & \partial_m F^m \end{vmatrix} = 0 . \tag{11}$$

(ii) *Radial projection* $\pi^{2m}_r : \mathbf{R}^{2m} \backslash \{0\} \to S^{2m-1}$ given by the formula $\pi^{2m}_r(x^1, \dots, x^n) = (x^1, \dots, x^n)/|x|$. Put $a = (x^1, \dots, x^{2m})$ so that $\alpha^j = q^j$. In particular $\alpha^I \frac{\partial F^b}{\partial q^I} = w^b(q, z) = h^b(z)$, so that z^1 reduces to S^{2m-1} if and only if

$$\begin{vmatrix} h^1 & \partial_2 F^1 & \dots & \partial_m F^1 \\ h^2 & \partial_2 F^2 & \dots & \partial_m F^2 \\ \vdots & \vdots & \ddots & \vdots \\ h^m & \partial_2 F^m & \dots & \partial_m F^m \end{vmatrix} = 0 . \tag{12}$$

(iii) *Projection* $\pi_H^{2m} : \mathbf{R}^{2m}\backslash\{0\} \to \mathbf{C}P^{m-1}$ given by $\pi^{2m}(q^1,\dots,q^m) \mapsto [q^1,\dots,q^m]$. The fibres of this map are spanned by the vector fields $\alpha_1 = q^I \frac{\partial}{\partial q^I}$ and $\alpha_2 = iq^i \frac{\partial}{\partial q^i} - iq^{\bar{i}} \frac{\partial}{\partial q^i}$ so that z^1 reduces to $\mathbf{C}P^{m-1}$ if and only if (12) above holds and

$$\begin{vmatrix} q^1 & \partial_2 F^1 & \dots & \partial_m F^1 \\ q^2 & \partial_2 F^2 & \dots & \partial_m F^2 \\ \vdots & \vdots & \ddots & \vdots \\ q^m & \partial_2 F^m & \dots & \partial_m F^m \end{vmatrix} = 0 . \tag{13}$$

Finally, we wish to know when examples are genuinely new examples and not simply obtained by the composition with an orthogonal projection. We therefore make the definition:

DEFINITION 2.2. [4] Call a map $\phi : U \to \mathbf{C}$ from an open subset of $\mathbf{R}^n$ *full* if we cannot write it as $\phi = \psi \circ \pi_A$ for some orthogonal projection π_A onto a subspace A of $\mathbf{R}^n$ and map $\psi : \pi_A(U) \to \mathbf{C}$.

A test for fullness is given by Proposition 4.2 of [4].

3. Heierarchies of Weierstrass representations.

Let J be a Hermitian structure defined on a domain U of $\mathbf{R}^{2m}$ and suppose that $\phi : (U, J) \to \mathbf{C}$ is holomorphic. As in Section 2, in a neighbourhood of a regular point we may assume that $\phi = z^1$ where $z = (z^1,\dots,z^m)$ is a solution $z(q)$ to Equation (6):

$$F(q,z) \equiv q - M(z)\bar{q} - h(z) = 0$$

for some holomorphic functions $\mu_i(z), (i = 1,\dots m(m-1)/2)$, $h^i(z)$, $(i = 1,\dots,m)$. Throughout this section we assume that z^1 has superminimal fibres (and so, in particular, is harmonic). Then $M(z)$ depends on z^1 only and the Jacobian matrix takes the form

$$K = \begin{pmatrix} \partial_1 F^1 & \partial_2 h^1 & \dots & \partial_m h^1 \\ \partial_1 F^2 & \partial_2 h^2 & \dots & \partial_m h^2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 F^m & \partial_2 h^m & \dots & \partial_m h^m \end{pmatrix} .$$

By assumption, the determinant of K is non-zero on U, so that, at least one of the cofactors $\hat{K}_1^1, \hat{K}_1^2, \dots, \hat{K}_1^m$ is non-zero, say the cofactor $\hat{K}_1^a$ obtained by omitting row a and column 1 from K. Then, by the Implicit Function Theorem, we can locally solve the equations $F^1 = 0,\dots,F^{a-1} = 0$, $F^{a+1} = 0,\dots,F^m = 0$ for $z^2,\dots,z^m$ as holomorphic functions of z^1, on sub-

stituting these functions into the remaining equation $F^a = 0$, this takes the form (14) in the following proposition (cf. [4, Proposition 3.12]):

PROPOSITION 3.1. *Let $\mu_1(z^1), \dots, \mu_{m(m-1)/2}(z^1)$ be given holomorphic functions of one variable and $\Psi_{2m}(w^1, \dots, w^{m+1})$ a given holomorphic function of $m+1$ variables. Consider the equation:*

$$\widetilde{\Psi}_{2m}(q, z(q^1 - M^1_{\bar{j}}(z^1)q^{\bar{j}}, q^2 - M^2_{\bar{j}}(z^1)q^{\bar{j}}, \dots \tag{14}$$

$$\dots, q^m - M^m_{\bar{j}}(z^1)q^{\bar{j}}, z^1) = 0 .$$

Suppose that, at a point (q, z^1) satisfying $\widetilde{\Psi}_{2m}(q, z^1) = 0$,

$$\frac{\partial \widetilde{\Psi}_{2m}}{\partial z^1} \neq 0 .$$

Then the local solution $z^1 = z^1(q)$ to Equation (14) through that point is a holomorphic harmonic morphism with superminimal fibres. All submersive holomorphic harmonic morphisms with superminimal fibres are given this way locally.

We refer to Equation (14) as the *Weierstrass representation for holomorphic harmonic morphisms with superminimal fibres on* $\mathbf{R}^{2m}$.

EXAMPLE 3.2. If $m = 2$, then Equation (14) has the form

$$\Psi_4(q^1 - \mu_1(z^1)q^{\bar{2}}, q^2 + \mu_1(z^1)q^{\bar{1}}, z^1) = 0 ,$$

which is precisely the equation obtained in [11] describing *all* (submersive complex-valued) harmonic morphisms $z^1 = z^1(q)$ on domains of $\mathbf{R}^4$.

EXAMPLE 3.3. Let $\alpha_1, \alpha_2, \dots, \alpha_m$ be m holomorphic functions of z^1 and consider the particular form of Ψ_{2m} given by

$$\Psi_{2m}(w^1, \dots, w^m, z^1) = \alpha_1 w^1 + \alpha_2 w^2 + \cdots + \alpha_m w^m - 1 .$$

Then Equation (14) can be written

$$(\alpha_1 - M^k_{\bar{1}}\alpha_k)x^1 + \mathrm{i}(\alpha_1 + M^k_{\bar{1}}\alpha_k)x^2 + \cdots + (\alpha_m - M^k_{\bar{m}}\alpha_k)x^{2m-1} +$$

$$\mathrm{i}(\alpha_m + M^k_{\bar{m}}\alpha_k)x^{2m} = 1 .$$

Note that the sum of the squares of the coefficients of $x^1, \dots, x^{2m}$ equals $-4\sum M^k_{\bar{j}}\alpha_j\alpha_k$, which vanishes since $M^k_{\bar{j}}$ is antisymmetric in k and j. We thus obtain the local characterization (3) of those (submersive complex-valued) harmonic morphisms with totally geodesic fibres, on even-dimensional Euclidean spaces.

EXAMPLE 3.4. (see also [4], Example 4.9) Let $m = 3$ and set $\Psi_6(w^1, w^2, w^3, z^1) = w^1 w^2 - w^3, \mu_1 = \mu_2 = z^1, \mu_3 = 0$. Then (14) becomes the quadratic equation

$$(z^1)^2 q^{\bar{1}}(q^{\bar{2}} + q^{\bar{3}}) + (z^1)[q^{\bar{1}} + q^2(q^{\bar{2}} + q^{\bar{3}}) - |q^1|^2] + (q^3 - q^1 q^2) = 0 .$$

Any local solution is a full harmonic morphisms with superminimal fibres. It is easy to see that z^1 is not holomorphic with respect to any Kähler structure.

More generally, for arbitrary m, we can set $\Psi_{2m}(w^1, \ldots, w^m, z^1) = w^1 w^2 \ldots w^{m-1} - w^m$ to obtain generalizations to arbitrary even dimensions.

Now suppose that z^1 factors through the projection $\pi_1^{2m} : \mathbf{R}^{2m} \to \mathbf{R}^{2m-1}$. That is, from Equation (11),

$$\begin{vmatrix} 1 & \partial_2 h^1 & \ldots & \partial_m h^1 \\ \mu_1 & \partial_2 h^2 & \ldots & \partial_m h^2 \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{m-1} & \partial_2 h^m & \ldots & \partial_m h^m \end{vmatrix} = 0 .$$

After elementary row operations this becomes the $(m-1) \times (m-1)$ determinant:

$$\begin{vmatrix} \partial_2(h^2 - \mu_1 h^1) & \ldots & \partial_m(h^2 - \mu_1 h^1) \\ \partial_2(h^3 - \mu_2 h^1) & \ldots & \partial_m(h^3 - \mu_2 h^1) \\ \vdots & \ddots & \vdots \\ \partial_2(h^m - \mu_{m-1} h^1) & \ldots & \partial_m(h^m - \mu_{m-1} h^1) \end{vmatrix} = 0 . \tag{15}$$

We consider the special solution of (15) given by

$$\partial_m(h^2 - \mu_1 h^1) = \partial_m(h^3 - \mu_2 h^1) = \ldots = \partial_m(h^m - \mu_{m-1} h^1) = 0 , \tag{16}$$

that is

$$\begin{cases} h^2 - \mu_1 h^1 &= \alpha^2(z^1, \ldots, z^{m-1}) \\ h^3 - \mu_2 h^1 &= \alpha^3(z^1, \ldots, z^{m-1}) \\ \vdots & \vdots \\ h^m - \mu_{m-1} h^1 &= \alpha^m(z^1, \ldots, z^{m-1}) \end{cases} \tag{17}$$

are all holomorphic functions independent of z^m. Consider the matrix

$$A = (\partial_i \alpha^j)_{i=2,\ldots,m-1}^{j=2,\ldots,m} .$$

Then amongst the minors $A^a = \det(\partial_i \alpha^j)_{i=2,\ldots,m-1}^{j=2,\ldots,\hat{a},\ldots,m}$ at least one is not identically zero otherwise $\det K$ would vanish. So amongst the $m-1$ equations (17), there are $m-2$ for which we can eliminate $z^2, \ldots, z^{m-1}$. Substituting $h^i = q^i - M^i_{\bar{j}}(z^1) q^{\bar{j}}$ into the remaining equation gives the functional relation:

$$\Phi_{2m-1}\Big(q^2 - q^{\bar{j}}M^2_{\bar{j}}(z^1) - \mu_1(z^1)\big(q^1 - q^{\bar{j}}M^1_{\bar{j}}(z^1)\big), \dots$$

$$\dots, q^m - q^{\bar{j}}M^m_{\bar{j}}(z^1) - \mu_{m-1}(z^1)\big(q^1 - q^{\bar{j}}M^1_{\bar{j}}(z^1)\big), z^1\Big) = 0\,,$$

where $\Phi_{2m-1} = \Phi_{2m-1}(u_1, \dots, u_m)$ is a holomorphic function of m complex variables.

If now this equation is satisfied, the restriction of z^1 to any level hypersurface $x^1 =$ constant is a harmonic morphism by Section 2. For convenience we take $x^1 = 0$, and then, setting $q^1 = \mathrm{i}x^2$, we obtain the following characterization:

PROPOSITION 3.5. *Let $\mu_1(z^1), \dots, \mu_{m(m-1)/2}(z^1)$ be given holomorphic functions of one variable and $\Phi_{2m-1} = \Phi_{2m-1}(u^1, \dots, u^m)$ a holomorphic function of m variables. Consider the equation*

$$\tilde{\Phi}_{2m-1}(q, z^1) \equiv \Phi_{2m-1}\Bigg(q^2 - 2\mathrm{i}\mu_1(z^1)x^2 - \sum_{\bar{j}\geq 2}\Big(M^2_{\bar{j}}(z^1) - \mu_1(z^1)M^1_{\bar{j}}(z^1)\Big)q^{\bar{j}}, \cdots$$

$$\dots, q^m - 2\mathrm{i}\mu_{m-1}(z^1)x^2 - \sum_{\bar{j}\geq 2}\Big(M^m_{\bar{j}}(z^1) - \mu_{m-1}(z^1)M^1_{\bar{j}}(z^1)\Big)q^{\bar{j}}, z^1\Bigg) = 0\,.$$

Suppose that, at a point $(q, z^1) = (x^2, q^2, \dots q^m, z^1)$ satisfying $\tilde{\Phi}_{2m-1}(q, z^1) = 0$,

$$\frac{\partial\tilde{\Phi}_{2m-1}}{\partial z^1} \neq 0\,.$$

Then the local solution $z^1 = z^1(q)$ to Equation (18) through that point is a harmonic morphism $\phi : U \subset \mathbf{R}^{2m-1} \to \mathbf{C}$ whose lift $\Phi = \phi \circ \pi_1^{2m}$ to $\mathbf{R}^{2m}$ is a holomorphic harmonic morphism which has superminimal fibres and satisfies the simplifying assumption (16). All such submersive harmonic morphisms on domains of $\mathbf{R}^{2m-1}$ are given this way locally.

EXAMPLE 3.6. If $m = 2$, then the assumption (16) is automatic and (18) takes the form:

$$\Phi_3(q^2 - 2\mathrm{i}\mu_1(z^1)x^2 - \big(\mu_1(z^1)\big)^2 q^{\bar{2}}, z^1) = 0$$

which is the local representation of *all* (submersive complex-valued) harmonic morphisms $z^1 = z^1(x^2, q^2)$ on domains of $\mathbf{R}^3$ [2].

REMARK. By [2, Lemma 4.3] and [3, Theorem 2.19], any harmonic morphism from a domain of $\mathbf{R}^3$ is locally a submersive complex-valued harmonic morphism followed by a weakly conformal map. In higher dimen-

sions, little is known about the behaviour of a harmonic morphism near a critical point: for some partial results in 4 dimensions, see [11].

EXAMPLE 3.7. Let $\alpha_1, \alpha_2, \ldots, \alpha_{m-1}$ be $m-1$ holomorphic functions of z^1 and consider the particular form of Φ_{2m-1} given by

$$\Phi_{2m-1}(w^1, \ldots, w^{m-1}, z^1) = \alpha_1 w^1 + \alpha_2 w^2 + \cdots + \alpha_{m-1} w^{m-1} - 1 .$$

Then Equation (18) becomes

$$\left(-\sum_{j=2}^{m-1} 2\mathrm{i}\alpha_j \mu_j\right) x^2 + \left(\alpha_1 - \sum_k \alpha_k \left(M_{\bar{2}}^k - \mu_k M_{\bar{2}}^1\right)\right) x^3$$

$$+ \mathrm{i}\left(\alpha_1 + \sum_k \alpha_k \left(M_{\bar{2}}^k - \mu_k M_{\bar{2}}^1\right)\right) x^4 + \ldots$$

$$\cdots + \left(\alpha_{m-1} - \sum_k \alpha_k \left(M_{\bar{m}}^k - \mu_k M_{\bar{m}}^1\right)\right) x^{2m-1}$$

$$+ \mathrm{i}\left(\alpha_1 + \sum_k \alpha_k \left(M_{\bar{m}}^k - \mu_k M_{\bar{m}}^1\right)\right) x^{2m} = 1 .$$

As in Example 14, the sum of the squares of the coefficients vanishes and we retrieve the local characterization (3) of those (submersive complex-valued) harmonic morphisms with totally geodesic fibres, now defined on odd-dimensional Euclidean spaces.

EXAMPLE 3.8. Let $m = 3$ and set $\Phi_5(w^1, w^2, z^1) = w^1 w^2 - 1, \mu_1 = \mu_2 = z^1, \mu_3 = 0$. Then (18) becomes the quartic equation:

$$(z^1)^4 \left(q^{\bar{2}} + q^{\bar{3}}\right)^2 - 4i(z^1)^3 x^2 \left(q^{\bar{2}} + q^{\bar{3}}\right)$$
$$+ (z^1)^2 \left(q^2 + q^3\right)\left(q^{\bar{2}} + q^{\bar{3}}\right) - 2\mathrm{i}(z^1) x^2 \left(q^2 + q^3\right) + \left(q^2 q^3 - 1\right) = 0 .$$

Any local solution $z^1 = z^1(x^2, q^2, q^3)$ is a full harmonic morphism defined on a domain of $\mathbf{R}^5$.

Suppose that we now reduce once more by the vector field $\partial/\partial x^2$. Then Condition (10) becomes:

$$\begin{vmatrix} -1 & \partial_2 h^1 & \ldots & \partial_m h^1 \\ \mu_1 & \partial_2 h^2 & \ldots & \partial_m h^2 \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{m-1} & \partial_2 h^m & \ldots & \partial_m h^m \end{vmatrix} = 0 , \tag{19}$$

and (11) and (19) are satisfied if and only if (11) holds and

$$\begin{vmatrix} \partial_2 h^2 & \dots & \partial_m h^2 \\ \partial_2 h^3 & \dots & \partial_m h^3 \\ \vdots & \ddots & \vdots \\ \partial_2 h^m & \dots & \partial_m h^m \end{vmatrix} = 0. \tag{20}$$

We consider the special case of (20) given by

$$\partial_m h^2 = \partial_m h^3 = \dots \partial_m h^m = 0. \tag{21}$$

Note, in particular, that (21) implies (16) and so such maps are a subset of those satisfying (18). Then

$$\begin{cases} h^2 &= \beta^2(z^1, \dots, z^{m-1}) \\ h^3 &= \beta^3(z^1, \dots, z^{m-1}) \\ \vdots & \ddots \quad \vdots \\ h^m &= \beta^m(z^1, \dots, z^{m-1}) \end{cases}$$

are all independent of z^m. By eliminating $z^2, \dots, z^{m-1}$ we now obtain a representation of z^1 in the form

$$\Psi_{2m-2}\left(q^2 - M^2_{\bar{j}}(z^1)q^{\bar{j}}, q^3 - M^3_{\bar{j}}(z^1)q^{\bar{j}}, \dots, q^m - M^m_{\bar{j}}(z^1)q^{\bar{j}}, z^1\right) = 0$$

where $\Psi_{2m-2}(w^2, \dots, w^{m+2})$ is a holomorphic function of m complex variables. But this is precisely (14) with m replaced by $m-1$.

To sum up, we define data for the Weierstrass representation:

Data For each $m = 2, 3, \dots$, let $\mu_1(z^1), \dots, \mu_{m(m-1)/2}(z^1)$ be given holomorphic functions of one variable and let $M = M(z^1)$ be the matrix given by (5). Suppose that either:

(i) Ψ_{2m} is a holomorphic function of $m+1$ complex variables, or

(ii) Φ_{2m-1} is a holomorphic function of m complex variables.

THEOREM 3.9. *Suppose that we are given the data above.*

(a) *The equation*

$$\Psi_{2m}\left(q^1 - M^1_{\bar{j}}(z^1)q^{\bar{j}}, q^2 - M^2_{\bar{j}}(z^1)q^{\bar{j}}, \dots, q^m - M^m_{\bar{j}}(z^1)q^{\bar{j}}, z^1\right) = 0$$

locally determines all submersive holomorphic harmonic morphisms with superminimal fibres defined on domains of $\mathbf{R}^{2m}$.

(b) *The equation*

$$\Phi_{2m-1}\left(q^2 - 2\mathrm{i}\mu_1(z^1)x^2 - \sum_{\bar{j}\geq 2}\left(M^2_{\bar{j}}(z^1) - \mu_1(z^1)M^1_{\bar{j}}(z^1)\right)q^{\bar{j}}, \ldots\right.$$

$$\left.\ldots, q^m - 2\mathrm{i}\mu_{m-1}(z^1)x^2 - \sum_{\bar{j}\geq 2}\left(M^m_{\bar{j}}(z^1) - \mu_{m-1}(z^1)M^1_{\bar{j}}(z^1)\right)q^{\bar{j}}, z^1\right) = 0$$

locally determines all submersive harmonic morphisms on domains of R^{2m-1} *which are the reduction of holomorphic harmonic morphisms on domains of* R^{2m} *with superminimal fibres satisfying the simplifying assumption* (16). *Amongst the solutions are the lifts of all submersive holomorphic harmonic morphisms with superminimal fibres defined on domains of* R^{2m-2}.

(c) *The equations* $\Phi_3 = 0$ *and* $\Psi_4 = 0$ *locally describe all submersive harmonic morphisms on domains of* R^3 *and* R^4, *respectively.*

Schematically we represent the above heierarchy of representations by the inclusions:

$$\cdots \supset \{\Psi_{2m} = 0\} \supset \{\Phi_{2m-1} = 0\} \supset \{\Psi_{2m-2} = 0\} \supset \cdots \supset \{\Psi_4 = 0\} \supset \{\Phi_3 = 0\}.$$

4, Reduction to Euclidean spheres.

Let U be an open subset of R^{2m} on which is defined a Hermitian structure J and let $\phi = z^1 : U \to \mathsf{C}$ be a harmonic morphism holomorphic with respect to J and with superminimal fibres. In particular by Proposition 3.1 we can suppose that z^1 is determined by (14):

$$\Psi(w^1, w^2, \ldots, w^m, z^1) = 0$$

where $w^i = q^i - M^i_{\bar{j}}q^{\bar{j}}$ and $M^i_{\bar{j}} = M^i_{\bar{j}}(z^1)$. Then z^1 is invariant under radial projection if and only if

$$q^I \frac{\partial z^1}{\partial q^I} = 0$$

where, as usual, we sum over $I = 1, \ldots, m, \bar{1}, \ldots, \bar{m}$.

LEMMA 4.1. *With the data above, the equation*

$$q^I \frac{\partial z^1}{\partial q^I} = 0,$$

is satisfied if and only if the condition

$$w^i \frac{\partial \Psi}{\partial w^i} = 0 \text{ whenever } \Psi = 0 \tag{22}$$

is satisfied.

PROOF. Differentiating (14) with respect to q^k yields

$$\frac{\partial \Psi}{\partial w^i}\frac{\partial w^i}{\partial q^k} + \frac{\partial \Psi}{\partial z^1}\frac{\partial z^1}{\partial q^k} = 0,$$

and, differentiating w^i,

$$\frac{\partial w^i}{\partial q^k} = \delta^i_k - \frac{\partial M^i_{\bar{j}}}{\partial z^1}\frac{\partial z^1}{\partial q^k} q^{\bar{j}}.$$

Also differentiating (14) and w^i with respect to $q^{\bar{k}}$ yields

$$\frac{\partial \Psi}{\partial w^i}\frac{\partial w^i}{\partial q^{\bar{k}}} + \frac{\partial \Psi}{\partial z^1}\frac{\partial z^1}{\partial q^{\bar{k}}} = 0$$

and

$$\frac{\partial w^i}{\partial q^{\bar{k}}} = -M^i_{\bar{k}} - \frac{\partial M^i_{\bar{j}}}{\partial z^1}\frac{\partial z^1}{\partial q^{\bar{k}}} q^{\bar{j}}.$$

Combining the first and third equations gives

$$\frac{\partial \Psi}{\partial w^i} q^K \frac{\partial w^i}{\partial q^K} + \frac{\partial \Psi}{\partial z^1} q^K \frac{\partial z^1}{\partial q^K} = 0,$$

so that $q^K \dfrac{\partial z^1}{\partial q^K} = 0$ if and only if $q^K \dfrac{\partial w^i}{\partial q^K}\dfrac{\partial \Psi}{\partial w^i} = 0$. But substituting the expressions for $\dfrac{\partial w^i}{\partial q^K}$ above gives

$$q^K \frac{\partial w^i}{\partial q^K}\frac{\partial \Psi}{\partial w^i} = \left(q^i - M^i_{\bar{k}} q^{\bar{k}}\right)\frac{\partial \Psi}{\partial w^i} - \left(\frac{\partial M^i_{\bar{j}}}{\partial z^1} q^{\bar{j}}\frac{\partial \Psi}{\partial w^i}\right) q^K \frac{\partial z^1}{\partial q^K},$$

and the result follows.

We therefore have

THEOREM 4.3. *Let $\phi = z^1 : U \to \mathbf{C}$ be a harmonic morphism from a domain of $\mathbf{R}^{2m}$ determined by* (14). *Then ϕ reduces to S^{2m-1} if and only if the condition* (22):

$$w^i \frac{\partial \Psi}{\partial w^i} = 0 \text{ whenever } \Psi = 0$$

is satisfied.

Clearly Condition (22) is satisfied if Ψ is homogeneous in $(w^1, \ldots, w^m)$. In fact a partial converse holds:

PROPOSITION 4.3. *Let Ψ be an irreducible polynomial in m complex variables $w^1, \ldots, w^m$. Then Condition* (22) *implies that Ψ is homogeneous in $w^1, \ldots, w^m$.*

The proof follows by combining the following two lemmas:

LEMMA 4.4 *Let Ψ be an irreducible polynomial in m complex variables $w^1, \ldots, w^m$. Then Condition* (22) *implies that*

$$w^i \frac{\partial \Psi}{\partial w^i} = k\Psi \tag{23}$$

for some $k \in \mathbf{C}$.

PROOF. Since Ψ is irreducible, by the Nullstellensatz (see, e.g. [10]), Condition (22) implies that

$$w^i \frac{\partial \Psi}{\partial w^i} = \alpha \Psi$$

for some polynomial α. But if Ψ has degree k_j in w^j, $(j = 1, \ldots, m)$, then so does $w^i \dfrac{\partial \Psi}{\partial w^i}$ and therefore α is constant.

LEMMA 4.5. *Let Ψ be an irreducible analytic function in m complex variables $w^1, \ldots, w^m$. Then*

$$w^i \frac{\partial \Psi}{\partial w^i} = k\Psi \text{ for some } k \in \mathbf{C}$$

if and only if Ψ is homogeneous.

PROOF. For fixed $(w_0^1, \ldots, w_0^m)$, set $\Psi_t = \Psi(tw_0^1, \ldots, tw_0^m) = \Psi(w^1, \ldots, w^m)$ where $w^j = tw_0^j$. Then by the chain rule,

$$\frac{d\Psi_t}{dt} = \frac{\partial \Psi}{\partial w^i} \frac{dw^i}{dt} = w_0^i \frac{\partial \Psi}{\partial w^i}$$

so that

$$t \frac{d\Psi_t}{dt} = w^i \frac{\partial \Psi}{\partial w^i} = k\Psi_t .$$

Hence

$$\frac{d\Psi_t}{\Psi_t} = \frac{k dt}{t}\,.$$

Integrating from $t = 1$ to $t = T$ gives:

$$\ln \Psi_t - \ln \Psi_1 = k \ln T$$

so that $\Psi_t = \Psi_1 T^k$, i.e.

$$\Psi(tw_0^1, \ldots, tw_0^m) = t^k \Psi(tw^1, \ldots, tw^m)\,,$$

showing that Ψ is homogeneous of degree k.

REMARK. Lemma 4.4 and the Proposition are false for Ψ an irreducible *analytic* function. For example $\Psi = e^{w^1} w^1$ is irreducible (since e^{w^1} is a unit) and

$$w^1 \frac{\partial \Psi}{\partial w^1} = (1 + w^1)\Psi\,,$$

so that Condition (22) is satisfied but not (23), and Ψ is not homogeneous.

The above results give a method for constructing examples on odd-dimensional spheres:

EXAMPLE 4.6. Consider the homogeneous analogue of Example 3.3, thus, as in that example, choose m holomorphic functions $\alpha_1, \ldots, \alpha_m$ and set $\Psi(w^1, \ldots, w^m, z^1) = \alpha_1 w^1 + \cdots + \alpha_m w^m$. Then $\Psi = 0$ determines locally all (submersive complex-valued) harmonic morphisms on S^{2m-1} with totally geodesic fibres.

If we now choose Ψ to be an irreducible polynomial of degree ≥ 2, we obtain new examples of harmonic morphisms on odd-dimensional spheres:

EXAMPLE 4.7. Let $\Psi(w^1, w^2, w^3, z^1) = (w^1)^2 - (z^1)^2((w^2)^2 + (w^3)^2)$. Then the corresponding harmonic morphism $z^1 = z^1(q)$ defined on a suitable domain of S^5 is given implicitly by the equation

$$(q^1 - \mu_1 q^{\bar{2}} - \mu_2 q^{\bar{3}})^2 = (z^1)^2\Big((q^2 + \mu_1 q^{\bar{1}} - \mu_3 q^{\bar{3}})^2 + (q^3 + \mu_2 q^{\bar{1}} + \mu_3 q^{\bar{2}})^2\Big)$$

where μ_1, μ_2, μ_3 are arbitrary holomorphic functions of z^1. The generic regular fibre of z^1 extends to a compact minimal submanifold of S^5, which, after the change of coordinates $X = q^1 - \mu_1 q^{\bar{2}} - \mu_2 q^{\bar{3}}$, $Y = z^1(q^2 + \mu_1 q^{\bar{1}} - \mu_3 q^{\bar{3}})$, $Z = z^1(q^3 + \mu_2 q^{\bar{1}} + \mu_3 q^{\bar{2}})$, is the level set $F = 0$ of the polynomial function $F : \mathbf{C}^3 \to \mathbf{C}$, $F(X, Y, Z) = X^2 - Y^2 - Z^2$, holomorphic with respect to a Kähler structure on $\mathbf{R}^6$. However, this Kähler structure varies from fibre to

fibre and the minimal submanifolds are not the level sets of a function holomorphic with respect to a *fixed* Kahler structure. In particular we obtain a foliation of an open set of S^5 by minimal codimension 2 submanifolds.

In a similar vein, we can construct examples on even-dimensional spheres by choosing the function Φ in the Weierstrass representation (18) to be homogeneous in the first $m-1$ variables. Firstly, choosing Φ to be linear we have

EXAMPLE 4.8. Let $\alpha_1, \ldots, \alpha_{m-1}$ be $m-1$ holomorphic functions of z^1 and let Φ be given by $\Phi(w^1, \ldots, w^{m-1}, z^1) = \alpha_1 w^1 + \cdots + \alpha_{m-1} w^{m-1}$. Then $\Phi = 0$ determines locally all (submersive complex-valued) harmonic morphisms on S^{2m-2} with totally geodesic fibres.

Secondly, if we choose Φ to be an irreducible polynomial of degree ≥ 2, we obtain new examples of harmonic morphisms on even-dimensional spheres, for example:

EXAMPLE 4.9. Let $m = 4$ and set $\mu_1 = \mu_2 = \mu_3 = z^1$, $\mu_4 = \mu_5 = \mu_6 = (z^1)^2$. Let Φ be given by

$$\Phi(w^1, w^2, w^3, z^1) = (w^1)^2 + w^2 w^3 .$$

Then the corresponding harmonic morphism $z^1 = z^1(q)$ defined on a suitable domain of S^6 is given implicitly by the quartic equation

$$\begin{aligned}&(z^1)^4((q^{\bar 2})^2 + (2q^{\bar 2} + q^{\bar 3})(2q^{\bar 2} + 2q^{\bar 3} + q^{\bar 4})) + (z^1)^3(-4\mathrm{i}x^2 q^{\bar 2} - 6\mathrm{i}x^2 q^{\bar 3} - 2\mathrm{i}x^2 q^{\bar 4})\\ &+ (z^1)^2(-8(x^2)^2 + 2|q^2|^2 + 2|q^3|^2 + q^4(2q^{\bar 2} + q^{\bar 3}) + q^3(2q^{\bar 2} + q^{\bar 4}))\\ &+ 2\mathrm{i}z^1 x^2(2q^2 - q^3 - q^4) + (q^2)^2 + q^3 q^4 = 0 .\end{aligned}$$

Finally, to find examples with *non-superminimal* fibres which reduce to S^{2m-1} seems much harder; we have no general theory but give one family of examples:

EXAMPLE 4.10. Let $m = 4$ and let $\mu_1 = \mu_1(z^1, z^2)$ be an arbitrary holomorphic function of two variables. Set $\mu_2 = z^1, \mu_3 = \mu_4 = 0, \mu_5 = (z^1)^p$, $\mu_6 = (z^1)^q, h^3 = (z^1)^r z^2, h^4 = z^2$ with $\partial_4 h^2 = 0$ and $\partial_4 h^1 \neq 0$, for integers p, q, r not yet specified. Then $\triangle z^1 = 0$, $\det K$ is not identically zero and Condition (12) for reduction to S^7 is satisfied. Note that the fibres of z^1 are superminimal if and only if $\partial_2 \mu_1 = 0$.

The map z^1 is defined by the 3rd and 4th equations of the system (6):

$$\begin{cases} q^3 + \mu_2 q^{\bar 1} - \mu_4 q^{\bar 2} - \mu_6 q^{\bar 4} - h^3 = 0 \\ q^4 + \mu_3 q^{\bar 1} + \mu_5 q^{\bar 2} + \mu_6 q^{\bar 3} - h^4 = 0 \end{cases};$$

that is,

$$(z^1)^{r+q}q^{\bar{3}} + (z^1)^{r+p}q^{\bar{2}} + (z^1)^r q^4 + (z^1)^q q^{\bar{4}} - z^1 q^3 = 0. \tag{24}$$

The test for fullness (that z^1 does not factor through any orthogonal projection -- cf. [4, Proposition 4.2]) requires that the equation in the complex vector $\alpha \in \mathbf{C}^4$:

$$(z^1)^{r+q}\alpha^{\bar{3}} + (z^1)^{r+p}\alpha^{\bar{2}} + (z^1)^r\alpha^4 + (z^1)^q\alpha^{\bar{4}} - z^1\alpha^3 = 0$$

has only the trivial solution $\alpha = (\alpha^1, \alpha^2, \alpha^3, \alpha^4) = 0$. In general p, q, r can be chosen so that this is the case, e.g. $r = 1, p \geq 1, q \geq p + 2$ will suffice, giving a family of full harmonic morphisms on domains of the sphere S^7. The fibres are totally geodesic and (29) is of the form (4) of the Introduction. Similar constructions are possible for $m = 5, 6, \ldots$.

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