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SOME INEQUALITIES ARISING FROM VECTOR-VALUED DIRAC DELTAS

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Abstract.

A new vector-valued Dirac deltas technique is developed in order to obtain discrete inequalities. One of them is the following:

$$\left|\left\{x : \left|\sum_{k=1}^{N}\sum_{j\in Z}\frac{e^{ijx}}{a_k - x}b_k^j\right| > \lambda\right\}\right| \leq \frac{C}{\lambda}\sum_{k=1}^{N}\sum_{j\in Z}|b_k^j|,$$

with $a_k \in \mathsf{T} = [-\pi, \pi]$, $b_k^j \in \mathsf{C}$ and $N \in \mathsf{N}$.

1. Introduction.

Development of vector-valued Harmonic Analysis has had important applications to classical operators in the last few years, see [GC, R de F], [R de F, R, T], [R de F].

The vector-valued Harmonic Analysis philosophy is that, although the proofs in the vector-valued case are, in general, not essentially different from the classical ones, results that can be obtained as applications of this vector-valued analysis are relevant. Besides this, there are results which, up to now, may only be proved by this technique, see [R de F].

The purpose of this paper is to develop this philosophy in order to obtain discrete inequalities.

One of the first inequalities (in fact equality) of this type is due to Loomis, [L]; it states that for $x_1, x_2, \cdots, x_N \in \mathsf{R}$ and $\lambda > 0$,

$$\left|\left\{x \in \mathsf{R} : \left|\sum_{k=1}^{N}\frac{1}{x - x_j}\right| > \lambda\right\}\right| = \frac{2}{\lambda}N,$$

where $|E|$ means Lebesgue's measure of E.

This estimate being exact, rather than an inequality, suggests that the result has been obtained through an algebraic proof, as it is in fact the case.

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Later on, Guzmân, [G, Theorem 4.1.1 on p.75], using different techniques proved that the weak type (1, 1) property for maximal convolution operators is equivalent to the weak type (1, 1) property for these operators acting over finites sums of Dirac deltas.

In the last years, Guzman's Theorem was extended to more general situations, and it was used to obtain properties for maximal operators, as it can be seen in [Ca], [T,V] and [M,S].

Our method here is to develop the vector-valued version of Guzmân's Theorem (see Theorem 2.3 in the second section), and to apply it to different vector-valued maximal operators in order to obtain classical type discrete inequalities. In Section 3 the following results are proven:

(1.1) *There exists a constant $C > 0$ such that for all $N \in \mathbb{N}$*

$$\left|\left\{x : \left|\sum_{k=1}^{N}\sum_{j\in\mathbb{Z}}\frac{e^{ijx}}{a_k - x}b_k^j\right| > \lambda\right\}\right| \leq \frac{C}{\lambda}\sum_{k=1}^{N}\sum_{j\in\mathbb{Z}}|b_k^j|$$

holds for all sequences $\{a_k\}$ and $\{b_k^j\}$, with $a_k \in \mathbb{T} = [-\pi, \pi]$, $b_k^j \in \mathbb{C}$, $j \in \mathbb{Z}$ and $k = 1, \cdots, N$ (see Theorem (3.6)).

(1.2) *If $\Phi(x)$ is a bounded integrable function in $\mathbb{R}^n$ such that $\hat{\Phi}(0) = \int \Phi(x)dx = 0$, satisfying conditions* (3.13) *and* (3.14), *there are some constants C_1 and C_2 such that*

$$\left|\left\{x : \left(\sum_{j=-\infty}^{\infty}\left|\sum_{k=1}^{N}\Phi_{2^j}(x - a_k)\right|^2\right)^{1/2} > \lambda\right\}\right| \leq \frac{C_1}{\lambda}N$$

and

$$\left|\left\{x : \left(\int_0^{\infty}\left(\sum_{k=1}^{N}\Phi_t(x - a_k)\right)^2\frac{dt}{t}\right)^{1/2} > \lambda\right\}\right| \leq \frac{C_2}{\lambda}N$$

for all sequences $\{a_k\}_{k=1}^N$ with $a_k \in \mathbb{R}^n$ (see Theorem (3.22)).

(1.3) *There is a constant $C > 0$ such that for all sequences $\{a_k\}$ and $\{b_k^j\}$ with $a_k \in \mathbb{R}^n$, $b_k^j \in \mathbb{R}$, and $1 < q < \infty$, we have*

$$\left|\left\{x : \left(\sum_{j=1}^{\infty}\sup_{r>0}\left|\sum_{k=1}^{N}\frac{1}{r^n}\mathcal{X}_{B_r(a_k)}(x)b_k^j\right|^q\right)^{1/q} > \lambda\right\}\right| \leq \frac{C}{\lambda}\sum_{k=1}^{N}\left(\sum_{j=1}^{\infty}|b_k^j|^q\right)^{1/q}$$

(see Remark (3.28)). $B_r(a_k)$ represents as usual the ball centered in a_k with radius r.

2. Technical Results.

Let R^n be endowed with Lebesgue's measure. Given a Banach space B we shall call a *vector-valued Dirac delta* any expression of the type

$$f = b\delta_a \qquad b \in B, \qquad a \in \mathsf{R}^n,$$

where b is a vector of B, a is a point of R^n and δ_a is the standard Dirac delta associated with a.

Let B and F be Banach spaces and T a linear operator acting on strongly measurable functions $f : \mathsf{R}^n \to B$, with compact support , such that

$$Tf(x) = \int_{\mathsf{R}^n} K(x,y)(f(y))dy. \tag{2.1}$$

Here, K is a strongly measurable function, $K : \mathsf{R}^n \times \mathsf{R}^n \to \mathscr{L}(B,F)$, and $\mathscr{L}(B,F)$ is the Banach space of bounded linear operators from B to F. For the operators that we are going to consider, the integral in (2.1) is always well defined.

If $f = b\delta_a$, $\quad b \in B$, $\quad a \in \mathsf{R}^n$, is a vector-valued Dirac delta, we shall define the action of T over f as

$$Tf(x) = T(b\delta_a)(x) = K(x,a)(b). \tag{2.2}$$

For brevity, we shall write $Tf(x) = \int_{\mathsf{R}^n} K(x,y)f(y)dy$ and $T(b\delta_a)(x) = K(x,a)b$.

We denote by $L^p_B(\mathsf{R}^n)$, $p < \infty$, the Bochner-Lebesgue space consisting of all B-valued (strongly) measurable functions f defined in R^n such that

$$\|f\|_{L^p_B(\mathsf{R}^n)} = \left(\int_{\mathsf{R}^n} \|f(x)\|^p_B dx\right)^{\frac{1}{p}} < \infty.$$

Similarly, the weak $-$ $L^p_B(\mathsf{R}^n)$ space is formed by all B-valued functions f such that

$$\sup_{t>0} t|\{x \in \mathsf{R}^n : \|f(x)\|_B > t\}|^{\frac{1}{p}} < \infty.$$

We state now our main technical result.

THEOREM 2.3. *Let* R^n *be given, with the Lebesgue measure. Let* B *and* F *be Banach spaces and let* $\{T_j\}_j$ *be a family of linear operators as above, with kernels* $K_j : \mathsf{R}^n \times \mathsf{R}^n \to \mathscr{L}(B,F)$, *such that for every* $x \in \mathsf{R}^n$ *the function* $\|K(x,\cdot)\|_{\mathscr{L}(B,F)}$ *is locally integrable, and such that the operator*

$$T_j f(x) = \int_{\mathsf{R}^n} K_j(x,y)f(y)dy,$$

is well defined for all B-valued strongly measurable bounded functions f with compact support. Let T^* be the maximal operator associated,

$$T^*f(x) = \sup_j \|T_jf(x)\|_F.$$

We assume that the kernels K_j satisfy the following condition: for each j, given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y_1, y_2 \in \mathbb{R}^n$ with $|y_1 - y_2| < \delta$

$$\int_{\mathbb{R}^n} \|K_j(x, y_1) - K_j(x, y_2)\|_{\mathcal{L}(B,F)} dx < \varepsilon. \tag{2.4}$$

Then the following statements are equivalent:

(a) *The operator T^* is bounded from $L^1_B(\mathbb{R}^n)$ into weak-$L^1_F(\mathbb{R}^n)$.*

(b) *There exists a constant $C > 0$ such that for all sequences $\{a_k\}_1^N$ and $\{b_k\}_1^N$ with $a_k \in \mathbb{R}^n$, $b_k \in B$, $N \in \mathbb{N}$ and $\lambda > 0$,*

$$\left|\left\{x : \sup_j \left\|\sum_{k=1}^N K_j(x, a_k) b_k\right\|_F > \lambda\right\}\right| \le \frac{C}{\lambda}\sum_{k=1}^N \|b_k\|_B.$$

PROOF. We shall first prove (b) $\Rightarrow$ (a): In order to prove that the maximal operator T^* is of weak type $(1, 1)$ it is enough to show that the truncated maximal operators

$$T^*_M f(x) = \sup_{1\le j\le M} \|T_jf(x)\|_F$$

are uniformly of weak type $(1, 1)$. Moreover, by density of B-valued simple functions in $L^1_B(\mathbb{R}^n)$, we only need to prove the inequality

$$\left|\left\{x : T^*_M\left(\sum_{k=1}^N b_k\chi_{I_k}\right)(x) > \lambda\right\}\right| \le \frac{C}{\lambda}\sum_{k=1}^N \|b_k\chi_{I_k}\|_{L^1_B} \tag{2.5}$$

with C independent of M, where $b_k \in B$ and I_k are disjoint intervals in $\mathbb{R}^n$.

Given $\varepsilon > 0$ and $f(x) = \sum_k b_k\chi_{I_k}(x)$, we may suppose that the diameter of each I_k is small enough so that for $y_1, y_2 \in I_k$ the condition (2.4) is satisfied for this ε and for all K_j, $1 \le j \le M$.

On the other hand, if we consider $g = \sum_k b_k|I_k|\delta_{a_k}$, where $a_k \in I_k$, by (b) we have, for $0 < \alpha < \lambda$,

$$\begin{aligned}|\{x : T^*_M f(x) > \lambda\}| &\le \{x : T^*_M g(x) > \lambda - \alpha\}| + |\{x : T^*_M(f - g)(x) > \alpha\}| \\ &\le \frac{C}{\lambda - \alpha}\sum_k \|b_k\|_B|I_k| + |\{x : T^*_M(f - g)(x) > \alpha\}|.\end{aligned}$$

Then, by using Chebychev's inequality, the definition of T_j and Minkowski's inequality, we have

$$|\{x : T_M^*(f-g)(x) > \alpha\}| \leq \frac{1}{\alpha}\int_{\{x:T_M^*(f-g)(x)>\alpha\}} T_M^*(f-g)(x)dx$$

$$\leq \frac{1}{\alpha}\int_{\{x:T_M^*(f-g)(x)>\alpha\}} \sum_{j=1}^{M} \|T_j(f-g)(x)\|_F dx$$

$$= \frac{1}{\alpha}\int_{\{x:T_M^*(f-g)(x)>\alpha\}} \sum_{j=1}^{M} \|\sum_{k=1}^{N} \int_{I_k} (K_j(x,y) - K_j(x,a_k))b_k dy\|_F dx$$

$$\leq \frac{1}{\alpha}\sum_{j=1}^{M}\sum_{k=1}^{N}\int_{I_k}\int_{\{x:T_M^*(f-g)(x)>\alpha\}} \|(K_j(x,y) - K_j(x,a_k))b_k\|_F dxdy$$

$$\leq \frac{1}{\alpha}\sum_{j=1}^{M}\sum_{k=1}^{N}\int_{I_k}\int_{\mathbb{R}^n} \|(K_j(x,y) - K_j(x,a_k))\|_{\mathscr{L}\mathscr{L}(B,F)}\|b_k\|_B dxdy$$

$$\leq \frac{1}{\alpha}\sum_{j=1}^{M}\sum_{k=1}^{N}\int_{I_k} \varepsilon\|b_k\|_B dy.$$

Note that we have used (2.4) in the last inequality.

Now, we observe that

$$\frac{1}{\alpha}\sum_{j=1}^{M}\sum_{k=1}^{N}\int_{I_k} \varepsilon\|b_k\|_B dy = \frac{\varepsilon}{\alpha}M\sum_{k=1}^{N}\|b_k\|_B|I_k| = \frac{\varepsilon}{\alpha}M\|f\|_{L_B^1}$$

goes to zero when $\varepsilon \to 0$. Therefore, in order to prove (a) it is enough to take the limit when $\alpha \to 0$. This finishes the proof of (b) $\Rightarrow$ (a).

The proof of (a) $\Rightarrow$ (b) is similar and we shall sketch it.

Given the linear combination $\sum_k b_k\delta_{a_k}$, we consider a family of disjoint cubes Q_k, such that $a_k \in Q_k$ and if $y_1, y_2 \in Q_k$, then the property (2.4) is satisfied for $1 \leq j \leq M$. Now, we define the simple function

$$h(x) = \sum_{k=1}^{N} \frac{b_k}{|Q_k|}\chi_{Q_k}(x),$$

and then

$$\left|\left\{x : \sup_{1\leq j\leq M}\left\|\sum_{k=1}^{N} K_j(x,a_k)b_k\right\|_F > \lambda\right\}\right| = \left|\left\{x : T_M^*\left(\sum_{k=1}^{N} b_k\delta_{a_k}\right)(x) > \lambda\right\}\right|$$

$$\leq |\{x : T_M^* h(x) > \lambda - \alpha\}| + \left|\left\{x : T_M^*\left(\sum_{k=1}^{N} b_k\delta_{a_k} - h\right)(x) > \alpha\right\}\right|.$$

Now, observing that

$$\|h\|_{L^1_B} = \sum_{k=1}^{N} \|b_k\|_B,$$

the rest of the proof follows the same lines as $(b) \Rightarrow (a)$.

3. Applications.

A. *Carleson's maximal operator*

Let S^* be Carleson's maximal operator of the Fourier partial sums

$$S^*f(x) = \sup_{j\in \mathsf{Z}} \left| \text{p.v.} \int_{\mathsf{T}} \frac{e^{ijy}}{x-y} f(y)dy \right|, \qquad x \in \mathsf{T} = [-\pi, \pi].$$

It is well known, see [C],[H], that this operator is bounded from $L^p(\mathsf{T})$ into $L^p(\mathsf{T})$, $1 < p < \infty$, and it is not of weak type $(1,1)$.

It is clear that S^* maps $L^p(\mathsf{T})$ into $L^p(\mathsf{T})$, $1 < p < \infty$, if and only if for all finite subsets J of Z the operators

$$S_J^*f(x) = \sup_{j\in J} \left| \text{p.v.} \int_{\mathsf{T}} \frac{e^{ijy}}{x-y} f(y)dy \right|, \qquad x \in \mathsf{T} = [-\pi, \pi]$$

are uniformly bounded from $L^p(\mathsf{T})$ into $L^p(\mathsf{T})$.

On the other hand, given a finite subset J of Z if we consider the $\ell^\infty(J)$-valued operator

$$T_Jf(x) = \left\{ \text{p.v.} \int_{\mathsf{T}} \frac{e^{ijy}}{x-y} f(y)dy \right\}_{j\in J}, \tag{3.1}$$

we have $\|T_Jf(x)\|_{\ell^\infty(J)} = S_J^*f(x)$; then, as a direct consequence of Carleson-Hunt Theorem, T_J is bounded from $L^p(\mathsf{T})$ into $L^p_{\ell^\infty(J)}(\mathsf{T})$, $1 < p < \infty$, with operator norm independent of J.

The transpose operators defined by

$$U_J(\{g_j\}_{j\in J})(x) = \sum_{j\in J} \text{p.v.} \int_{\mathsf{T}} \frac{e^{ijx}}{y-x} g_j(y)dy \tag{3.2}$$

are uniformly bounded from $L^p_{\ell^1(J)}(\mathsf{T})$ into $L^p(\mathsf{T})$, $1 < p < \infty$, see [R de F,R,T, III.2].

Given a finite subset J of Z, the operators T_J and U_J can be handled as special cases of vector valued Calderón-Zygmund operators with variable kernels; the kernel of T_J is

$$(3.3)\qquad K_J(x,y)=\left\{\frac{e^{ijy}}{x-y}\right\}_{j\in J}\in\mathcal{L}(\mathsf{C},\ell^\infty(J))\cong\ell^\infty(J),$$

while the kernel of U_J is

$$(3.4)\qquad K_J^*(x,y)=\left\{\frac{e^{ijx}}{y-x}\right\}_{j\in J}\in\mathcal{L}(\ell^1(J),\mathsf{C})\cong\ell^\infty(J).$$

It is clear that, for $|x-y|>2|x-z|$

$$(3.5)\qquad \|K_J(x,y)-K_J(z,y)\|_{\ell^\infty(J)}=\|K_J^*(y,x)-K_J^*(y,z)\|_{\ell^\infty(J)}$$

$$=\left|\frac{1}{x-y}-\frac{1}{z-y}\right|\le C\frac{|x-z|}{|x-y|^2},$$

Then, the kernels K_J satisfy the standard estimates of Calderón-Zygmund kernels only in the first variable, while the K_J^* satisfy these estimates in the second variable. By using these estimates and the Carleson-Hunt Theorem, the general theory of vector valued Calderón-Zygmund operators can be applied to these operators; in particular it can be proved that the operators U_J are uniformly bounded from $L^1_{\ell^1(J)}(\mathsf{T})$ into weak-$L^1(\mathsf{T})$ and from $L^1_{\ell^1(J)}(\mathsf{T},\omega)$ into weak-$L^1(\mathsf{T},\omega)$ for every weight $\omega\in A_1$, see [R de F,R,T, III.2]. In fact, using the Rubio de Francia's Extrapolation Theorem for A_p-weights, see [GC,R de F, p.448] it can be proved that Carleson's Theorem is equivalent to the inequality

$$\omega\left(\left\{x\in\mathsf{T}:\left|\sum_{j\in\mathsf{Z}}\text{p.v.}\int_{\mathsf{T}}\frac{e^{ijx}}{y-x}g_j(y)dy\right|>\lambda\right\}\right)$$

$$\le\frac{C}{\lambda}\sum_{j\in\mathsf{Z}}\int_{\mathsf{T}}|g_j(x)|\omega(x)dx,\ \omega\in A_1,$$

i.e., the operators U_J are uniformly bounded from $L^1_{\ell^1(J)}(\mathsf{T},\omega)$ into weak-$L^1(\mathsf{T},\omega)$ for every weight $\omega\in A_1$.

We have the following

THEOREM 3.6. *There exists a constant $C>0$ such that for all sequences $\{a_k\}$ and $\{b_k^j\}$, with $a_k\in\mathsf{T}$, $b_k^j\in\mathsf{C}$, $j\in\mathsf{Z}$ and $k=1\cdots N$, we have*

$$\left|\left\{x:\left|\sum_{k=1}^N\sum_{j\in\mathsf{Z}}\frac{e^{ijx}}{a_k-x}b_k^j\right|>\lambda\right\}\right|\le\frac{C}{\lambda}\sum_{k=1}^N\sum_{j\in\mathsf{Z}}|b_k^j|.$$

PROOF. First of all it is clear that in order to prove the theorem it is enough to prove that for any finite subset J of Z we have

$$\left|\left\{x : \left|\sum_{k=1}^{N}\sum_{j\in J}\frac{e^{ijx}}{a_k - x}b_k^j\right| > \lambda\right\}\right| \leq \frac{C}{\lambda}\sum_{k=1}^{N}\sum_{j\in J}|b_k^j|,$$

with the constant C independent of J.

Now we observe that the last inequality is equivalent to the following inequality

$$\left|\left\{x : \left|\sum_{k=1}^{N} K_J^*(x, a_k)\bar{b}_k\right| > \lambda\right\}\right| \leq \frac{C}{\lambda}\sum_{k=1}^{N}\|\bar{b}_k\|_{\ell^1(J)}, \tag{3.7}$$

with $a_k \in \mathsf{T}$, $\bar{b}_k = \{b_k^j\}_{j\in J}$, $\bar{b}_k \in \ell^1(J)$, $k = 1, \cdots, N$ and K_J^* as in (3.4).

By using Fatou's lemma, in order to prove (3.7) it is enough to prove that for any $\gamma > 0$, there exists a constant C, independent of γ, such that the inequality

$$\left|\left\{x : \left|\sum_{k=1}^{N} K_J^*(x, a_k)\bar{b}_k \chi_{\{|x-a_k|>\gamma\}}(x)\right| > \lambda\right\}\right| \leq \frac{C}{\lambda}\sum_{k=1}^{N}\|\bar{b}_k\|_{\ell^1(J)}$$

holds for $\{a_k\}_1^N \subset \mathsf{T}$ and $\{\bar{b}_k\}_1^N \subset \ell^1(J)$. In other words, we need to prove a uniform estimate over finite sets $J \subset \mathsf{Z}$ and $\gamma > 0$, for the operators

$$U_{J,\gamma}(\{g_j\}_{j\in J})(x) = \sum_{j\in J}\int_{|x-y|>\gamma}\frac{e^{ijx}}{y - x}g_j(y)dy$$

acting over Dirac's deltas.

In order to do this, we need to show that the operators $U_{J,\gamma}$ satisfy the conditions in Theorem 2.3, that is, $U_{J,\gamma}$ are uniformly bounded from $L^1_{\ell^1(J)}(\mathsf{T})$ into weak-$L^1(\mathsf{T})$ and their kernels

$$K_{J,\gamma}^*(x, y) = \left\{\frac{e^{ijx}}{y - x}\chi_{\{|x-y|>\gamma\}}\right\}_{j\in J,\gamma}$$

verify condition (2.4). The following lemma shows condition (2.4) for $K_{J,\gamma}^*$:

LEMMA 3.8. *For each pair (J, γ), given $\varepsilon > 0$, there exists $\delta > 0$ such that for all y_1, y_2 with $|y_1 - y_2| < \delta$,*

$$\int_{\mathsf{T}}\|K_{J,\gamma}^*(x, y_1) - K_{J,\gamma}^*(x, y_2)\|_{\mathscr{L}(\ell^1(J),\mathsf{C})}dx < \varepsilon.$$

PROOF. Let $\varepsilon > 0$ be fixed, and let y_1, y_2 be such that $y_1 \leq y_2$ and $y_2 - y_1 < \gamma$. As $\mathscr{L}(\ell^1(J), \mathsf{C}) \cong \ell^\infty(J)$, we have

$$(3.9)\qquad \int_{\mathsf{T}} \|K^*_{J,\gamma}(x,y_1)-K^*_{J,\gamma}(x,y_2)\|_{L(\ell^1(J),\mathsf{C})}dx$$

$$= \int_{\mathsf{T}} \sup_{j\in J}\left|\frac{e^{ijx}}{y_1-x}\chi_{\{|x-y_1|>\gamma\}}(x,y_1)-\frac{e^{ijx}}{y_2-x}\chi_{\{|x-y_2|>\gamma\}}(x,y_2)\right|dx$$

$$= \int_{\mathsf{T}} \left|\frac{1}{y_1-x}\chi_{\{|x-y_1|>\gamma\}}(x,y_1)-\frac{1}{y_2-x}\chi_{\{|x-y_2|>\gamma\}}(x,y_2)\right|dx$$

$$= \int_{y_1-\gamma}^{y_2-\gamma}\frac{1}{|y_2-x|}dx+\int_{y_1+\gamma}^{y_2+\gamma}\frac{1}{|y_1-x|}dx+\int_A\left|\frac{1}{y_1-x}-\frac{1}{y_2-x}\right|dx,$$

where A represents the complementary set of $(y_1-\gamma, y_2+\gamma)$.

As $|y_1-x|$ and $|y_2-x|$ are bigger than γ in $(y_1+\gamma, y_2+\gamma)\cup A$ and $(y_1-\gamma, y_2-\gamma)\cup A$, respectively, we finally have

$$(3.9)\qquad \leq \frac{2}{\gamma}|y_2-y_1|+\int_A\frac{|y_2-y_1|}{|y_1-x||y_2-x|}dx\leq C_\gamma|y_2-y_1|.$$

Then taking $\delta=\min(\frac{\varepsilon}{C_\gamma},\gamma)$ the lemma is proven.

Now, we shall see that the operators $U_{J,\gamma}$ are uniformly bounded from $L^1_{\ell^1(J)}(\mathsf{T})$ into weak-$L^1(\mathsf{T})$. We consider the $\ell^\infty(J)$-valued operators

$$T_{J,\gamma}f(x)=\left\{\int_{|x-y|>\gamma}\frac{e^{ijy}}{x-y}f(y)dy\right\}_{j\in J,\gamma},$$

and let M be the Hardy-Littlewood maximal operator. It is clear that, by using (3.5) and Jensen inequality, we have, for $1<q<\infty$,

$$\|T_{J,\gamma}f(x)\|_{\ell^\infty(J)}-M(\|T_Jf\|_{\ell^\infty(J)})(x)$$

$$\leq\frac{1}{\gamma}\int_{|z-x|\leq\frac{\gamma}{2}}(\|T_{J,\gamma}f(x)\|_{\ell^\infty(J)}-\|T_Jf(z)\|_{\ell^\infty(J)})dz$$

$$\leq\frac{1}{\gamma}\int_{|z-x|\leq\frac{\gamma}{2}}\|T_{J,\gamma}f(x)-T_Jf(z)\|_{\ell^\infty(J)}dz$$

$$=\frac{1}{\gamma}\int_{|z-x|\leq\frac{\gamma}{2}}\left\|\int_{|x-y|>\gamma}\left(\frac{e^{ijy}}{x-y}-\frac{e^{ijy}}{z-y}\right)f(y)dy\right\|_{\ell^\infty(J)}dz$$

$$+\frac{1}{\gamma}\int_{|z-x|\leq\frac{\gamma}{2}}\|T_J(f\chi_{B_\gamma(x)})(z)\|_{\ell^\infty(J)}dz$$

$$\leq \frac{1}{\gamma}\int_{|z-x|\leq\frac{\gamma}{2}}\int_{|x-y|>\gamma}\left|\frac{1}{x-y}-\frac{1}{z-y}\right||f(y)|dydz$$
$$+\left(\frac{1}{\gamma}\int_{|z-x|\leq\frac{\gamma}{2}}\|T_J(f\chi_{B_\gamma(x)})(z)\|^q_{\ell^\infty(J)}dz\right)^{\frac{1}{q}}.$$

Now, by Carleson's Theorem and (3.5) again we finally have that (see [R.de F, G-C, on p. 204] for a similar argument)

(3.10) $$\|T_{J,\gamma}f(x)\|_{\ell^\infty(J)} \leq C_1M(\|T_Jf\|_{\ell^\infty(J)})(x)+C_2M_qf(x),$$

where $M_qf(x)=(M|f|^q)(x))^{\frac{1}{q}}$.

The last inequality combined with Carleson's Theorem and the fact that Hardy-Littlewood maximal operator is bounded from $L^p(\mathrm{T})$ into $L^p(\mathrm{T})$, with $1<p<\infty$, gives the uniform boundedness of the operators $T_{J,\gamma}$ from $L^p(\mathrm{T})$ into $L^p_{\ell^\infty(J)}(\mathrm{T})$, with $1<p<\infty$.

By duality we obtain that the transpose operators

$$U_{J,\gamma}(\{g_j\}_{j\in J})(x)=\sum_{j\in J}\int_{|x-y|>\gamma}\frac{e^{ijx}}{y-x}g_j(y)dy$$

are uniformly bounded from $L^p_{\ell^1(J)}(\mathrm{T})$ into $L^p(\mathrm{T})$, with $1<p<\infty$. The kernels $K^*_{J,\gamma}$ of $U_{J,\gamma}$ satisfy, for any $y,z\in\mathrm{T}$

$$\int_{|x-y|>2|y-z|}\|K^*_{J,\gamma}(x,y)-K^*_{J,\gamma}(x,z)\|_{\ell^\infty(J)}dx$$
$$=\int_{|x-y|>2|y-z|}\left|\frac{1}{x-y}\chi_{\{|x-y|>\gamma\}}(x,y)-\frac{1}{x-z}\chi_{\{|x-z|>\gamma\}}(x,z)\right|dx$$
$$\leq\int_{|x-y|>2|y-z|}\left|\frac{1}{x-y}-\frac{1}{x-z}\right|dx$$
$$+\int_{\substack{|x-y|>2|y-z|\\|x-y|>\gamma>|x-z|}}\frac{1}{|x-y|}dx+\int_{\substack{|x-y|>2|y-z|\\|x-z|>\gamma>|x-y|}}\frac{1}{|x-z|}dx$$
$$\leq C+\frac{1}{\gamma}\int_{|x-z|<\gamma}dx+\frac{1}{\gamma}\int_{|x-y|<\gamma}dx\leq C.$$

Then the kernels $K^*_{J,\gamma}$ of operators $U_{J,\gamma}$ satisfy a uniform Hörmander type condition in the second variable. This fact tells us that operators $U_{J,\gamma}$ are uniformly bounded from $L^1_{\ell^1(J)}(\mathsf{T})$ into weak-$L^1(\mathsf{T})$ (see [R.de F,R,T, III.2]). Therefore we can apply Theorem 2.3 and the proof of Theorem 3.6 is finished.

REMARK 3.11. We believe that the inequality (3.10) is interesting by itself when working with Carleson operator; in fact, taking the supremum on γ and J we can observe that, for $1 < q < \infty$ the following inequality is proven

$$\sup_{n\in\mathsf{Z},\gamma}\left|\int_{|x-y|>\gamma}\frac{e^{inx}}{x-y}f(y)dy\right| \leq C_1 M(S^*f)(x) + C_2 M_q f(x).$$

We may finally observe that Theorem 3.6 can be written in the form of the following corollary; this indicates in which sense Theorem 3.6 generalizes the Loomis (in)-equality.

COROLLARY 3.12. *Let $A(\mathsf{T})$ be the space of functions $f, f : \mathsf{T} \to \mathsf{C}$, such that the Fourier-series of f converges absolutely,* i.e., $A(\mathsf{T}) = \{f : \mathsf{T} \to \mathsf{C} : \|f\|_{A(\mathsf{T})} = \sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty\}$. There exists a constant C such that for all sequences $\{a_k\}$ with $a_k \in \mathsf{T}$, for all $B_k \in A(\mathsf{T})$, $k = 1, \cdots, N$, and for $\lambda > 0$ we have

$$\left|\left\{x : \left|\sum_{k=1}^{N}\frac{B_k(x)}{a_k - x}\right| > \lambda\right\}\right| \leq \frac{C}{\lambda}\sum_{k=1}^{N}\|B_k\|_{A(\mathsf{T})}.$$

B. *$\mathcal{G}$-functions.*

Let $\Phi(x)$ be a bounded integrable function on R^n such that $\hat{\Phi}(0) = \int \Phi(x)dx = 0$, and let us assume that, for some $0 < \beta < \alpha$, it satisfies

(3.13) $$|\Phi(x)| \leq C\min(|x|^{-n-\alpha}, |x|^{-n-\beta}), \qquad x \in \mathsf{R}^n$$

and

(3.14) $$\int |\Phi(x+h) - \Phi(x)|dx \leq C|h|^{\alpha}, \qquad h \in \mathsf{R}^n.$$

Then, the operators

$$\mathcal{G}f(x) = \left(\sum_{j=-\infty}^{\infty} |\Phi_{2^j} * f(x)|^2\right)^{\frac{1}{2}}$$

and

$$\Delta f(x) = \left(\int_0^\infty |\Phi_t * f(x)|^2 \frac{dt}{t}\right)^{\frac{1}{2}}$$

are bounded in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, and of weak type $(1,1)$, see [GC,R de F], V 5. As usual Φ_t denotes the dilation of Φ: $\Phi_t(x) = t^{-n}\Phi(\frac{x}{t})$.

Now, we consider the families of Hilbert-valued (ℓ^2 and $L^2((0,\infty),\frac{dt}{t})$) operators

$$T_M f(x) = \{\Phi_{2^j} * f(x)\}_{j=-M}^{M}$$

and

$$S_M f(x) = \{\Phi_t * f(x)\}_{t\in[\frac{1}{M},M]}.$$

The operators T_M (resp. S_M) are bounded from $L^p(\mathbb{R}^n)$ into $L^p_{\ell^2}(\mathbb{R}^n)$, $1 < p < \infty$, and from $L^1(\mathbb{R}^n)$ into weak-$L^1_{\ell^2}(\mathbb{R}^n)$, (resp. from $L^p(\mathbb{R}^n)$ into $L^p_{L^2(\frac{dt}{t})}(\mathbb{R}^n)$, $1 < p < \infty$, and from $L^1(\mathbb{R}^n)$ into weak-$L^1_{L^2(\frac{dt}{t})}(\mathbb{R}^n)$). Moreover

$$\mathcal{G}f(x) = \sup_{M\in\mathbb{N}} \|T_M f(x)\|_{\ell^2} \tag{3.15}$$

and

$$\Delta f(x) = \sup_{M\in\mathbb{N}} \|S_M f(x)\|_{L^2(\frac{dt}{t})}.$$

The kernels of the operators T_M are functions $K_M(x,y) : \mathbb{C} \to \ell^2$ such that

$$K_M(x,y)\lambda = \{\Phi_{2^j}(x-y)\lambda\}_{j=-M}^{M}, \qquad \lambda \in \mathbb{C}. \tag{3.16}$$

On the other hand, the kernels of S_M are the functions $L_M(x,y) : \mathbb{C} \to L^2(\frac{dt}{t})$ such that

$$L_M(x,y)\lambda = \{\Phi_t(x-y)\lambda\}_{t\in[\frac{1}{M},M]}.$$

These kernels satisfy the lemma:

LEMMA 3.17. *For each M, given $\varepsilon > 0$, there exists $\delta > 0$ such that if* $|y_1 - y_2| < \delta$

$$\int_{\mathbb{R}^n} \|K_M(x,y_1) - K_M(x,y_2)\|_{\mathcal{L}(\mathbb{C},\ell^2)} dx < \varepsilon \tag{3.18}$$

and

$$\int_{\mathbb{R}^n} \|L_M(x,y_1) - L_M(x,y_2)\|_{\mathcal{L}(\mathbb{C},L^2(\frac{dt}{t}))} dx < \varepsilon. \tag{3.19}$$

PROOF. As $\mathscr{L}(\mathsf{C},\ell^2) \cong \ell^2$, in order to prove (3.18) we must estimate the integral

$$\int \left(\sum_{j=-M}^{M} |\Phi_{2^j}(x-y_1) - \Phi_{2^j}(x-y_2)|^2 \right)^{\frac{1}{2}} dx.$$

But changing variables and using (3.14) we have

$$\int |\Phi_{2^j}(x-y_1) - \Phi_{2^j}(x-y_2)|dx = \int |\Phi_{2^j}(x-(y_1-y_2)) - \Phi_{2^j}(x)|dx$$

$$= \int \left| 2^{-jn} \left(\Phi(\frac{x}{2^j} - \frac{y_1-y_2}{2^j}) - \Phi(\frac{x}{2^j}) \right) \right| dx$$

$$= \int \left| \Phi(x - \frac{y_1-y_2}{2^j}) - \Phi(x) \right| dx \le C \left| \frac{y_1-y_2}{2^j} \right|^{\alpha};$$

then, for δ sufficiently small we obtain (3.18).

On the other hand, as $\mathscr{L}(\mathsf{C}, L^2(\frac{dt}{t})) \cong L^2(\frac{dt}{t})$, we have

$$\int_{\mathsf{R}^n} \|L_M(x,y_1) - L_M(x,y_2)\|_{\mathscr{L}(\mathsf{C},L^2(\frac{dt}{t}))} dx$$

$$= \int_{\mathsf{R}^n} \left(\int_{1/M}^{M} |\Phi_t(x-y_1) - \Phi_t(x-y_2)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} dx$$

$$= \int_{\mathsf{R}^n} \left(\int_{1/M}^{M} |\Phi_t(x-(y_1-y_2)) - \Phi_t(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} dx$$

$$\le C \left(\int_{|x|\le 2|y|} \int_{1/M}^{M} |\Phi_t(x-y) - \Phi_t(x)|^2 \frac{dt}{t} dx \right)^{\frac{1}{2}} (2|y|)^{\frac{n}{2}}$$

$$+ \left(\int_{|x|>2|y|} |x|^{-n-\frac{\beta}{2}} dx \right)^{1/2} \left(\int_{|x|>2|y|} \int_{1/M}^{M} |\Phi_t(x-y) - \Phi_t(x)|^2 |x|^{n+\frac{\beta}{2}} \frac{dt}{t} dx \right)^{\frac{1}{2}}$$

$$= I + II.$$

In the last expresion, we have written $y = y_1 - y_2$ and we have used Holder's inequality for $\beta > 0$.

If we use the fact that Φ is bounded, and (3.14), we have:

$$I \leq C\left(\int_{\mathbb{R}^n}\int_{1/M}^{M} |\Phi_t(x-y) - \Phi_t(x)| \frac{dt}{t} dx\right)^{\frac{1}{2}} |y|^{\frac{n}{2}} \tag{3.20}$$

$$= C\left(\int_{\mathbb{R}^n}\int_{1/M}^{M} \left|\Phi\left(x - \frac{y}{t}\right) - \Phi(x)\right| \frac{dt}{t} dx\right)^{\frac{1}{2}} |y|^{\frac{n}{2}}$$

$$\leq C\left(\int_{1/M}^{M} \frac{|y|^{\alpha} dt}{t^{1+\alpha}}\right)^{\frac{1}{2}} |y|^{\frac{n}{2}} \leq C_M |y|^{(\alpha+n)/2}.$$

On the other hand, by (3.13) we have

$$II \leq C|y|^{-\frac{\beta}{4}}\left(\int_{|x|>2|y|}\int_{1/M}^{M} t^{-2n}\left|\Phi\left(\frac{x}{t} - \frac{y}{t}\right) - \Phi\left(\frac{x}{t}\right)\right|^2 |x|^{n+\frac{\beta}{2}} \frac{dt}{t} dx\right)^{\frac{1}{2}}$$

$$\leq C|y|^{-\frac{\beta}{4}}\left(\int_{1/M}^{M} t^{\frac{\beta}{2}}\left(\int_{|x|>\frac{2|y|}{t}}\left|\Phi\left(x - \frac{y}{t}\right) - \Phi(x)\right|^2 |x|^{n+\frac{\beta}{2}} dx\right)\frac{dt}{t}\right)^{\frac{1}{2}}$$

$$= C|y|^{-\frac{\beta}{4}}\left(\int_{1/M}^{M} t^{\frac{\beta}{2}} J(t)\frac{dt}{t}\right)^{\frac{1}{2}}.$$

But $|\Phi(x - \frac{y}{t}) - \Phi(x)| \leq C|x|^{-n-\beta} \leq c|x|^{-n-\beta/2}(\frac{|y|}{t})^{-\beta/2}$ when $|x| > \frac{2|y|}{t}$ (by (3.13)), and therefore by (3.14) we have

$$J(t) \leq C\int_{|x|>\frac{2|y|}{t}}\left|\Phi\left(x - \frac{y}{t}\right) - \Phi(x)\right|\left(\frac{|y|}{t}\right)^{-\beta/2} dx = C\left(\frac{|y|}{t}\right)^{\alpha-\beta/2},$$

and then

$$II \leq C_M |y|^{\frac{\alpha}{2}-\frac{\beta}{2}}. \tag{3.21}$$

Combining the inequalities (3.20) and (3.21) we complete the proof of (3.19) and Lemma (3.17).

This lemma allows us to obtain the following theorem:

THEOREM 3.22. *Let $\Phi(x)$ be a bounded integrable function in* R^n *such that* $\hat{\Phi}(0) = \int \Phi(x)dx = 0$, *satisfying conditions* (3.13) *and* (3.14). *Then, there are some constants* C_1 *and* C_2 *such that*

$$\left|\left\{x : \left(\sum_{j=-\infty}^{\infty}\left|\sum_{k=1}^{N}\Phi_{2^j}(x-a_k)\right|^2\right)^{1/2} > \lambda\right\}\right| \leq \frac{C_1}{\lambda}N$$

and

$$\left|\left\{x : \left(\int_0^{\infty}\left(\sum_{k=1}^{N}\Phi_t(x-a_k)\right)^2\frac{dt}{t}\right)^{1/2} > \lambda\right\}\right| \leq \frac{C_2}{\lambda}N$$

for all $\{a_k\}_{k=1}^N, a_k \in \mathsf{R}^n$.

PROOF. Since $\mathcal{G}$ maps $L^1(\mathsf{R}^n)$ into weak-$L^1(\mathsf{R})$ then, by (3.14), we have

$$|\{x : \sup_M \|T_M f(x)\|_{\ell^2} > \lambda\}| \leq \frac{C_1}{\lambda}\|f\|_{L^1}, \qquad \lambda > 0.$$

Therefore, by using Lemma (3.17), we can apply Theorem (2.3) and we get

$$\left|\left\{x : \sup_M\left\|\sum_{k=1}^{N}K_M(x,a_k)\lambda_k\right\|_{\ell^2} > \lambda\right\}\right| \leq \frac{C_1}{\lambda}\sum_{k=1}^{N}|\lambda_k|;$$

if we choose $\lambda_k = 1$, $k = 1,\cdots,N$, through the action of K_M (see (3.16)) and the monotone convergence theorem, we obtain the first part of the theorem.

The proof of the second part is analogous, using the operator Δ and the kernels L_M.

C. U.M.D. *Banach lattices and Hardy-Littlewood maximal operator.*
In this section X will denote a Banach lattice of measurable functions in a σ-finite measure space $(\Omega, d\omega)$. As usual, $|\cdot|$ will denote the absolute value in X: $|x| = \sup\{x, -x\}$, which, as a function in Ω, has the obvious definition: $|x|(\omega) = |x(\omega)|$. We shall identify $L^p_X(\mathsf{R}^n)$, $1 \leq p < \infty$, with a lattice of functions $f(t)(\omega) = f(t,\omega)$, $t \in \mathsf{R}^n, \omega \in \Omega$, see [R de F 2].

Given any operator T bounded in $L^p(\mathsf{R}^n)$, non-necessarily linear, we can define its extension in

$$X \otimes L^p(\mathsf{R}^n) = \{\sum_{\text{finite}} b_i\varphi_i : b_i \in X, \varphi_i \in L^p(\mathsf{R}^n)\}$$

in the following form

$$\tilde{T}f(t,\omega) = T(f(\cdot,\omega))(t), \quad f \in X \otimes L^p(\mathsf{R}^n), \quad t \in \mathsf{R}^n, \omega \in \Omega.$$

We shall say that T extends in $L^p_X(\mathsf{R}^n)$ if there exists a constant C such that

$$\|\tilde{T}f\|_{L^p_X(\mathsf{R}^n)} \leq C\|Tf\|_{L^p_X(\mathsf{R}^n)}, \quad f \in X \otimes L^p(\mathsf{R}^n);$$

then $\tilde{T}$ can be extended to general $f \in L^p_X(\mathsf{R}^n)$ by a limit process.

If R_k, $k = 1, \cdots, n$, is a Riesz transform, i.e.,

$$R_k f(t) = \text{p.v.} \int_{\mathsf{R}^n} c_n y_k |y|^{-n-1} f(t-y)dy,$$

then it is well known that R_k extends in $L^p_X(\mathsf{R}^n)$ if and only if $X \in$ U.M.D. (the class of spaces which have the unconditionality property for martingale differences), see [Bk], [B] and [R de F 2].

Any U.M.D. space X is reflexive, see [A]; therefore, if X is a U.M.D. lattice of functions, then X satisfies the Fatou's property, see [L-T], i.e.:

Everytime we have a sequence of functions $\{x_n\} \subset X$, *such that* $x_n(\omega) \geq 0$ *for* a.e. ω, $x_n(\omega) \nearrow x(\omega)$ *for* a.e. ω *and also* $\sup_n \|x_n\|_X < \infty$, *then we have* $x \in X$ *and* $\|x\|_X = \lim_n \|x_n\|_X$.

It is a simple consequence of Lebesgue's monotone convergence theorem for scalar functions that the space $L^p_X(\mathsf{R}^n)$ has the Fatou's property provided X has this property.

We consider now the Hardy-Littlewood maximal operator in R^n:

$$Mf(t) = \sup_{r>0} \frac{1}{r^n} \left| \int_{B_r(t)} f(y)dy \right|,$$

where $t, y \in \mathsf{R}^n$ and $B_r(t) = \{y : |y - t| < r\}$.

For a U.M.D. Banach lattice X, we shall consider also its X-valued extension,

$$\tilde{M}f(t) = \sup_{r>0} \frac{1}{r^n} \left| \int_{B_r(t)} f(y)dy \right|, \quad t, y \in \mathsf{R}^n$$

with $|\int_{B_r(t)} f(y)dy|$ denoting the absolute value in X; equivalently

$$\tilde{M}f(t,\omega) = \sup_{r \in \mathsf{R}} \left| \frac{1}{r^n} \int_{B_r(t)} f(y,\omega)dy \right|, \quad t, y \in \mathsf{R}^n, \quad \omega \in \Omega.$$

It is well known, see [R de F 2], that if $X \in U.M.D.$ then $\tilde{M}$ maps $L^p_X(\mathsf{R}^n)$ into $L^p_X(\mathsf{R}^n)$, $1 < p < \infty$. By Fatou's property this is equivalent to the uniform boundedness of the operators

$$\tilde{M}_J f(t,\omega) = \sup_{r \in J} \left| \frac{1}{r^n} \int_{B_r(t)} f(y,\omega)dy \right|$$

from $L^p_X(\mathsf{R}^n)$ into $L^p_X(\mathsf{R}^n)$, $1 < p < \infty$, for any finite subset J of R. Applying a vector-valued Calderón-Zygmund technique this is finally equivalent to the uniform boundedness of the operators $\tilde{M}_J$ from $L^1_X(\mathsf{R}^n)$ into weak-$L^1_X(\mathsf{R}^n)$, see [GC,M,T]. This says that the $X(\ell^\infty(J))$-valued operators

$$T_J(f)(t) = \left\{ \frac{1}{r^n} \int_{B_r(t)} f(y)dy \right\}_{r \in J} \tag{3.23}$$

are uniformly bounded from $L^1_X(\mathsf{R}^n)$ into weak-$L^1_{X(\ell^\infty(J))}(\mathsf{R}^n)$, where by $X(\ell^\infty(J))$ we denote the space of sequences $\{\bar{b} = \{b_r\} : b_r \in X, r \in J\}$ with norm

$$\|\bar{b}\|_{X(\ell^\infty(J))} = \|\sup_{r\in J} |b_r|\|_X.$$

The operators T_J have kernels $K_J(t,y) : X \to X(\ell^\infty(J))$ with the action defined by

$$K_J(t,y)(b) = \left\{ \frac{1}{r^n} \chi_{B_r(y)}(t) b \right\}_{r\in J}, \quad b \in X. \tag{3.24}$$

LEMMA 3.25. *For each finite subset J of* R, *given $\varepsilon > 0$, there is a $\delta > 0$ such that if $|y_1 - y_2| < \delta$ then*

$$\int_{\mathsf{R}^n} \|K_J(t,y_1) - K_J(t,y_2)\|_{\mathcal{L}(X,X(\ell^\infty(J)))} dt < \varepsilon.$$

PROOF. Since the action of $K_J(t,y)$ over X is diagonal we have

$$\int_{\mathsf{R}^n} \|K_J(t,y_1) - K_J(t,y_2)\|_{\mathcal{L}(X,X(\ell^\infty(J)))} dt$$
$$\leq \int_{\mathsf{R}^n} \sup_{r\in J} \left| \frac{1}{r^n} \chi_{B_r(y_1)}(t) - \frac{1}{r^n} \chi_{B_r(y_2)}(t) \right| dt.$$

Now, a standard computation shows that we can choose δ such that for $|y_1 - y_2| < \delta$ the last expresion is less than ε.

THEOREM 3.26. *Let X be a* U.M.D. *Banach lattice. Then, there is a constant $C > 0$ such that for all sequences $\{a_k\}$ and $\{b_k\}$ with $a_k \in \mathrm{R}^n$, $b_k \in X$, we have*

$$\left|\left\{t : \left\|\sup_{r>0}\left|\sum_{k=1}^{N} \frac{1}{r^n}\chi_{B_r(a_k)}(t)b_k\right|\right\|_X > \lambda\right\}\right| \leq \frac{C}{\lambda}\sum_{k=1}^{N} \|b_k\|_X. \tag{3.27}$$

PROOF. Since the operators T_J are uniformly bounded from $L^1_X(\mathrm{R}^n)$ into weak-$L^1_{X(\ell^\infty)}(\mathrm{R}^n)$, we have

$$|\{t : \|T_J(t)\|_{X(\ell^\infty(J))} > \lambda\}| \leq \frac{C}{\lambda}\|f\|_{L^1_X};$$

then by Lemma (3.25) we can apply Theorem (2.3) and we have that there is a constant $C > 0$ such that

$$\left|\left\{t : \left\|\sum_{k=1}^{N} K_J(t, a_k)b_k\right\|_{X(\ell^\infty(J))} > \lambda\right\}\right| \leq \frac{C}{\lambda}\sum_{k=1}^{N} \|b_k\|_X;$$

this is to say

$$\left|\left\{t : \left\|\sup_{r\in J}\left|\sum_{k=1}^{N} \frac{1}{r^n}\chi_{B_r(a_k)}(t)b_k\right|\right\|_X > \lambda\right\}\right| \leq \frac{C}{\lambda}\sum_{k=1}^{N} \|b_k\|_X.$$

Now, as X satisfies Fatou's Property, we have that

$$\left|\left\{t : \left\|\sup_{r\in \mathrm{Q}^+}\left|\sum_{k=1}^{N} \frac{1}{r^n}\chi_{B_r(a_k)}(t)b_k\right|\right\|_X > \lambda\right\}\right| \leq \frac{C}{\lambda}\sum_{k=1}^{N} \|b_k\|_X.$$

The proof is finished by observing that

$$\sup_{r\in \mathrm{Q}^+}\left|\sum_{k=1}^{N} \frac{1}{r^n}\chi_{B_r(a_k)}(t)b_k(\omega)\right| = \sup_{r>0}\left|\sum_{k=1}^{N} \frac{1}{r^n}\chi_{B_r(a_k)}(t)b_k(\omega)\right|.$$

REMARK 3.28. For $X = \ell^q$, $1 < q < \infty$, we get (1.3).

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