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ON THE CONVOLUTION BANACH ALGEBRA $l^1(0,1)$.

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0. In part 1 of this paper it is shown that the convolution Banach algebra $l^1(\mathsf{Q}_1) = l^1((0,1) \cap \mathsf{Q})$ contains an element g such that the linear span of the convolution powers of g is dense in $l^1(\mathsf{Q}_1)$. This is then used to find a Hilbert space operator with a particularly interesting invariant subspace structure. The function g_o, defined as g on Q_1, and as 0 on $(0,1) \setminus \mathsf{Q}_1$, generates a dense ideal in the convolution Banach algebra $l^1(0,1)$. In part 2, a family of elements of $l^1(0,1)$ with this property is obtained by a different and more general method.

1. For any set E and any p, $1 \leq p < \infty$, $l^p(E)$ denotes the Banach space of complex-valued functions f on E with

$$\|f\|^p = \sum_{x \in E} |f(x)|^p < \infty.$$

In [1, Problem 2'], K. R. Davidson raised the question, whether there is a bounded linear operator on $l^2(\mathsf{Q})$, for which the family of non-trivial invariant subspaces consists of all subspaces which are either of the form

$$\{f \in l^2(\mathsf{Q}),\ \operatorname{supp} f \subseteq (-\infty,\ t)\},\ t \in \mathsf{R},$$

or of the form

$$\{f \in l^2(\mathsf{Q}),\ \operatorname{supp} f \subseteq (-\infty,\ t]\}\ t \in \mathsf{Q}.$$

Due to the existence of an order-reversing bijection F of Q onto $\mathsf{Q}_1 = \mathsf{Q} \cap (0,1)$, for instance

$$x \mapsto \frac{-x}{2(1+|x|)} + \frac{1}{2},\ x \in \mathsf{Q},$$

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his question is answered in the affirmative by the following theorem.

THEOREM 1. *There is an element* $g \in l^1(\mathsf{Q}_1)$ *such that convolution with g is an operator on* $l^2(\mathsf{Q}_1)$, *for which the family of non-trivial invariant subspaces consists of all subspaces which are either of the form*

$$\{f \in l^2(\mathsf{Q}_1), \operatorname{supp} f \subseteq (t,1)\},\ t \in (0,1), \tag{1}$$

or of the form

$$\{f \in l^2(\mathsf{Q}_1), \operatorname{supp} f \subseteq [t,1)\},\ t \in \mathsf{Q}_1. \tag{2}$$

Here convolution is defined by

$$f \star g\,(x) = \sum_{0<y<x,\, y\in\mathsf{Q}_1} f(x-y)\,g(y),\ x \in \mathsf{Q}_1.$$

To prove the theorem we need the following lemma.

LEMMA 1. *The convolution Banach algebra* $l^1(\mathsf{Q}_1)$ *contains an element g, such that the linear span of the convolution powers* $g^{\star m}, m \geq 1$, *is dense in* $l^1(\mathsf{Q}_1)$.

PROOF OF LEMMA 1. Let e_p denote the element with value 1 at the point $(p!)^{-1}$, and value 0 elsewhere. Our g will be of the form

$$\sum_{p\geq 2} a_p\, e_p,$$

where the positive coefficients a_p are determined by the following iterative procedure, starting with $a_2 = \frac{1}{2}$. Suppose that

$$g_n = \sum_2^n a_p\, e_p$$

has been defined for a certain $n \geq 2$. Since $a_n \neq 0$, the powers $g_n^{\star m}$, $1 \leq m \leq n! - 1$, form a base in the subspace $l_n^1(\mathsf{Q}_1)$, formed by the elements in $l^1(\mathsf{Q}_1)$, vanishing outside $((n!)^{-1}\mathsf{Z}) \cap \mathsf{Q}_1$. Since this space is finite-dimensional and due to the submultiplicativity of the norm in $l^1(\mathsf{Q}_1)$, we can fix a constant d_n, such that every element in $l_n^1(\mathsf{Q}_1)$ of norm ≤ 1 has distance $\leq n^{-1}$ from the linear span of $\{h^{*m}\}, 1 \leq m \leq n! - 1$, if $h \in l^1(\mathsf{Q}_1)$ satisfies

$$\|h - g_n\| \leq d_n. \tag{3}$$

Then we choose $a_{n+1} \in (0, 2^{-n-1})$ using all previously chosen d_m, so that

$$a_{n+1} < d_m\, 2^{m-n-1}, 1 \leq m \leq n. \tag{4}$$

Let us now prove that the linear span of the convolution powers of g is dense in $l^1(\mathsf{Q}_1)$. By (4), we have for every $n \geq 2$

$$\|g - g_n\| = \sum_{p>n} a_p < \sum_{p>n} d_n\, 2^{n-p} = d_n.$$

Hence (3) is satisfied for $h = g$, and since $\|g\| \leq 1$, we find that all points in the closed unit ball in $l^1_n(\mathsf{Q}_1)$, have distance $\leq n^{-1}$ from the linear span of the powers of g. Since n can be chosen arbitrarily, the lemma is proved.

PROOF OF THEOREM 1. Take g as in Lemma 1. Obviously the subspaces (1) and (2) of $l^2(\mathsf{Q}_1)$ are invariant under convolution with g. Conversely let $L \subseteq l^2(\mathsf{Q}_1)$ be a (closed) subspace, invariant under convolution with g. Since the operator norm for convolution is $\leq$ the corresponding l^1 norm, L is invariant under convolution with any element in the (closed) subspace of $l^1(\mathsf{Q}_1)$, generated by g, hence invariant under convolution with any element $h \in l^1(\mathsf{Q}_1)$. In particular, choosing h so that it vanishes except at one point, we find that L is invariant under all right translations τ_y, $y \in \mathsf{Q}_1$, where $\tau_y f(x) = 0$, $0 < x \leq y$, $\tau_y f(x) = f(x - y)$, $x \geq y$. But the (non-trivial) right translation invariant subspaces of $l^2(\mathsf{Q}_1)$ are exactly the subspaces of the theorem. This follows directly from the corresponding result for $l^2(0,1)$, which is known. It was announced in Helson [4] and can be derived from his theory of cocycles as given in Helson [5]. It is also a direct consequence of Theorem 1 in Domar [3].

2. The function g, constructed in the lemma, is of interest, too, for the discussion of the ideal structure of the convolution Banach algebra $l^1(0,1)$, with the obvious analogous definition of convolution. Let g_o denote the function on $(0,1)$, coinciding with g on Q_1 and taking the value 0 elsewhere. It follows from the lemma that the ideal generated by g_o is dense in $l^1(0,1)$. Equivalently, the linear span of the right translates of g_o is dense in $l^1(0,1)$. We will now construct a more general class of functions with this property.

LEMMA 2. *Let* $f \in l^1(0,1)$, *with* $\inf \operatorname{supp} f = 0$. *Suppose that there is a positive sequence* $\{t_n\}$, *tending to* ∞ *as* $n \to \infty$, *and a sequence* $\{a_n\}$ *of points in* $(0,1)$, *with* $a_n \to 0$, *as* $n \to \infty$, *such that, for* f_n, *defined by*

$$f_n(x) = f(x)\, \exp(-t_n x),\ x \in (0,1),$$

we have

$$2|f_n(a_n)| > \|f_n\|, \tag{5}$$

for every n. Then the linear span of the right translates of f is dense in $l^1(0,1)$.

PROOF OF LEMMA 2. If the right translates of f do not span a dense subspace of $l^1(0,1)$, then there is an element h, not $\equiv 0$, in the dual space $l^\infty(0,1)$, such that

$$\int_y^1 h(x)f(x-y)\,dx = 0,\; y \in (0,1). \tag{6}$$

Let us define h_n by

$$h_n(x) = h(x)\,\exp(t_n x),\; x \in (0,1).$$

There is a sequence $\{b_n\}$ in (0,1), satisfying $\liminf b_n > 0$, as $n \to \infty$, and

$$|h_n(b_n)|\,\|h_n\|^{-1} \to 1, n \to \infty.$$

Then (5) gives, if n is sufficiently large,

$$2|h_n(b_n)|\,|f_n(a_n)| > \sup|h_n|\,\|f_n\|. \tag{7}$$

and by (6),

$$\int_y^1 h_n(x)f_n(x-y)\,dx = 0,\; y \in (0,1).$$

Choosing n so large that $b_n - a_n > 0$, we obtain, for $y = b_n - a_n$ in the equality above,

$$2|h_n(b_n)|\,|f_n(a_n)| \le sup|h_n|\,\|f_n\|,$$

which contradicts (7), and we have proved that the linear span of the right translates of f is dense in $l^1(0,1)$.

THEOREM 2. *Let $\{a_n\}$, $n \ge 1$, be a sequence in $(0,1)$, converging to 0. Then there is a function $f \in l^1(0,1)$, with $f(a_n) > 0$, $n \ge 1$, $f(x) = 0$ elsewhere, and such that. for every g, with $\inf \operatorname{supp} g = 0$, $\operatorname{supp} g \subseteq \operatorname{supp} f$, $g(a_n) = o(f(a_n))$, $n \to \infty$, the linear span of the right translates of g is dense in $l^1(0,1)$.*

PROOF OF THEOREM 2. By a straightforward inductive procedure we can define a function f and a sequence $\{t_n\}$, satisfying the assumptions of Lemma 2 with respect to our given sequence. For every integer $n > 0$, we can then find a positive number u_n, such that

$$m \mapsto |g(a_m)||f(a_m)|^{-1}\exp\{-u_n a_m\}$$

takes its maximum c_n for $m = p(n) = p > n$. Defining

$$g_n(x) = g(x)\exp\{-(t_p + u_n)x\}, x \in (0,1),$$

we obtain from (5)

$$2|g_n(a_p)| = 2c_n|f_p(a_p)| > c_n\|f_p\| \geq \|g_n\|.$$

Hence g fulfils the conditions of Lemma 2 with respect to the sequences $\{t_{p(n)} + u_n\}$ and $\{a_{p(n)}\}, n \geq 1$, and Theorem 2 is proved.

In particular, if $a_n = 2^{-n}$, $n \geq 1$, easy estimates show that it is possible to take $f(a_n) = a_n^4$, $n \geq 1$, in Theorem 2.

It does not seem to be known, whether there exists an element $h \in l^1(0,1)$, with $\inf \operatorname{supp} h = 0$, such that the span of its right translates is not dense in $l^1(0,1)$. It should be observed that the function f in Lemma 2 in fact satisfies a much stronger property: if $h_y(x)$, with $x, y \in (0,1)$, are complex numbers of modulus 1, then the linear span of $\{h_y\,\tau_y f,\ y \in (0,1)\}$ is dense in $l^1(0,1)$. Hence we can make a corresponding extension of Theorem 2. Our discussion is related to the proof of Theorem 5 of [2], and that theorem can be extended in a similar way.

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