MATH. SCAND. 82 (1998), 89–100

ENDOMORPHISMS OF THE ALGEBRA OF ABSOLUTELY CONTINUOUS FUNCTIONS AND OF ALGEBRAS OF ANALYTIC FUNCTIONS

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Abstract.

Let $\mathscr{AC}$ be the algebra of absolutely continuous functions on the unit circle T. The main result of this paper is that a function $\tau \in \mathscr{AC}$ with $\tau(\mathsf{T}) \subseteq \mathsf{T}$ induces an endomorphism of $\mathscr{AC}$ by $f \mapsto f \circ \tau$ $(f \in \mathscr{AC})$ if and only if

$$\sup_{t \in \mathsf{T}} \#\big(\partial\big(\tau^{-1}(t)\big)\big) < \infty$$

(where ∂X denotes the topological boundary of X and $\#X$ the number of elements in X). We also discuss endomorphisms of the algebra $\mathscr{AC}^+ = \mathscr{AC} \cap \mathscr{A}(\overline{\Delta})$ (where $\mathscr{A}(\overline{\Delta})$ is the disc algebra) and of Lipschitz algebras on the closed unit disc $\overline{\Delta}$.

1. Introduction.

Let $\mathscr{B}$ be a unital, semisimple, commutative Banach algebra considered as an algebra of functions on its character space X. Suppose that X is a connected subset of C and that the function $\alpha : z \mapsto z$ belongs to $\mathscr{B}$ (so that $\mathscr{B}$ contains the polynomials). Let Ψ be a non-zero (necessarily continuous) endomorphism of $\mathscr{B}$. For $x \in X$, the map

$$f \mapsto \Psi(f)(x), \qquad f \in \mathscr{B}$$

is a character on $\mathscr{B}$. (The map is non-zero, since $\Psi(1) = 1$.) It follows that there exists $\tau(x) \in X$ such that $\Psi(f)(x) = f(\tau(x))$ for $f \in \mathscr{B}$. Hence

$$\Psi(f) = f \circ \tau \qquad \text{for } f \in \mathscr{B},$$

so the endomorphism is given by a change of variable. Also, $\tau = \Psi(\alpha) \in \mathscr{B}$. Conversely, if $\tau(X) \subseteq X$ and $f \circ \tau \in \mathscr{B}$ for every $f \in \mathscr{B}$ (in particular, $\tau \in \mathscr{B}$), then $\Psi_\tau(f) = f \circ \tau$ $(f \in \mathscr{B})$ defines an endomorphism of $\mathscr{B}$. In this case we say that τ *defines an endomorphism of* $\mathscr{B}$. It is interesting to note that

Received October 24, 1995.

the algebraic problem of determining the endomorphisms of $\mathscr{B}$ has thus been turned into the analytic problem of characterizing those functions $\tau \in \mathscr{B}$ with $\tau(X) \subseteq X$ for which all the composite functions $f \circ \tau$ $(f \in \mathscr{B})$ belong to $\mathscr{B}$. It also follows that the automorphisms of $\mathscr{B}$ correspond to those homeomorphisms τ of X for which τ and τ^{-1} both define endomorphisms of $\mathscr{B}$.

For the algebra $C(X)$ of all continuous functions on X, we see that every $\tau \in C(X)$ with $\tau(X) \subseteq X$ defines an endomorphism of $C(X)$. For other algebras this need not be the case. A notable example of this is the Wiener algebra $\mathscr{A}$ of absolutely convergent Fourier series on the unit circle T. Beurling and Helson ([1] or [4, p.86]) proved that every endomorphism of $\mathscr{A}$ corresponds to a linear change of variable, that is, if Ψ is an endomorphism of $\mathscr{A}$, then

$$\tau(\theta) = k\theta + \theta_0, \qquad \theta \in \mathsf{T}$$

for some $k \in \mathsf{Z}$ and $\theta_0 \in \mathsf{T}$ (identifying T with $[0, 2\pi]$). Conversely, every such function τ defines an endomorphism of $\mathscr{A}$. Note, in particular, that the automorphisms of $\mathscr{A}$ correspond to $k = \pm 1$. These results were generalized to the Beurling algebras by Leblanc ([6]).

The endomorphisms of Lipschitz algebras on T were characterized by Sherbert. Let $0 < \gamma \leq 1$ and let Λ_γ be the *Lipschitz algebra* of functions f on T satisfying

$$p_\gamma(f) = \sup\left\{\frac{|f(t) - f(s)|}{|t - s|^\gamma} : t, s \in \mathsf{T},\ t \neq s\right\} < \infty.$$

(We use the symbol $|\cdot|$ to denote the Euclidean distance on C as well as the usual metric on T.) Equipped with the norm

$$\|f\|_{\Lambda_\gamma} = \|f\|_\infty + p_\gamma(f), \qquad f \in \Lambda_\gamma,$$

Λ_γ becomes a Banach algebra with character space T. Sherbert ([9, Theorem 5.1]) proved that a function $\tau : \mathsf{T} \to \mathsf{T}$ defines an endomorphism of Λ_γ if and only if $\tau \in \Lambda_1$. Furthermore, for $0 < \gamma < 1$, we let λ_γ be the closed subalgebra of Λ_γ of functions f on T satisfying $|f(t) - f(s)| = o(|t - s|^\gamma)$ uniformly as $|t - s| \to 0$. Sherbert's proof can easily be modified to show that the condition $\tau \in \Lambda_1$ also is necessary and sufficient for τ to define an endomorphism of λ_γ.

In this paper we shall describe the endomorphisms of two other Banach algebras of functions on T. The algebras that we shall consider are the algebra of continuous functions of bounded variation and its closed subalgebra of absolutely continuous functions. For each of these algebras and for the Lipschitz algebras, we also consider the closed subalgebra of functions that extend continuously to analytic functions on the open unit disc, and discuss the endomorphisms of these subalgebras.

2. Endomorphisms of the algebra of absolutely continuous functions.

Let $\mathscr{BV}$ be the Banach algebra of continuous functions of bounded variation on T equipped with the norm

$$\|f\|_{\mathscr{BV}} = \|f\|_\infty + \operatorname{Var}(f),$$

where $\operatorname{Var}(f)$ is the total variation of $f \in \mathscr{BV}$. It is well known that the algebra $\mathscr{AC}$ of absolutely continuous functions on T is a closed subalgebra of $\mathscr{BV}$ with $\operatorname{Var}(f) = \int_{\mathsf{T}} |f'(t)|\, dt$ for $f \in \mathscr{AC}$. Also, a result of Banach (see [3, Theorem 18.25]) states that a real-valued continuous function on T is absolutely continuous if and only if it is of bounded variation and maps sets of measure zero to sets of measure zero. The character space of $\mathscr{BV}$ and $\mathscr{AC}$ is easily seen to be T. (For more information about $\mathscr{BV}$ and $\mathscr{AC}$, see, for example, [3].)

In general, it does not seem to be known exactly when the composite of two absolutely continuous functions is absolutely continuous. For a function $f : \mathsf{T} \to \mathsf{C}$, it is not hard to see ([3, Exercise 18.38]) that $f \circ \tau \in \mathscr{AC}$ for every $\tau \in \mathscr{AC}$ with $\tau(\mathsf{T}) \subseteq \mathsf{T}$ if and only if $f \in \Lambda_1$. To describe the endomorphisms of $\mathscr{AC}$, we need to solve the somehow converse problem of finding those functions $\tau : \mathsf{T} \to \mathsf{T}$ for which $f \circ \tau \in \mathscr{AC}$ for every $f \in \mathscr{AC}$. In the following characterization of such functions τ, we need the so-called Banach indicatrix (see [3, Exercise 17.34]). Let $g : \mathsf{T} \to \mathsf{R}$ be a continuous function and define the *Banach indicatrix* B_g by

$$B_g(y) = \#\big(g^{-1}(y)\big), \qquad y \in \mathsf{R},$$

where $\#X$ denotes the number of elements in X (with the convention that $\#X = \infty$ when X is infinite). Then B_g is measurable and

$$\operatorname{Var}(g) = \int_{\mathsf{R}} B_g(y)\, dy,$$

so g is of bounded variation if and only if B_g is integrable. Similarly when g maps into T.

THEOREM 2.1. *Let $\tau \in \mathscr{AC}$ with $\tau(\mathsf{T}) \subseteq \mathsf{T}$. Then τ defines an endomorphism of $\mathscr{AC}$ if and only if*

$$\sup_{t \in \mathsf{T}} \#\big(\partial\big(\tau^{-1}(t)\big)\big) < \infty$$

(*where ∂X is the topological boundary of X*).

Proof. Let $N(\tau) = \sup_{t\in\mathsf{T}} \#(\partial(\tau^{-1}(t)))$. First, suppose that $N(\tau) < \infty$. Since the interior of $\tau^{-1}(t)$ is non-empty for at most countably many values $t \in \mathsf{T}$, we deduce that $B_\tau(t) \leq N(\tau)$ for all but countably many values $t \in \mathsf{T}$. Let $f \in \mathscr{AC}$. By separating into real and imaginary parts, we may assume that f is real-valued. For $y \in \mathsf{R}$, we have

$$(f \circ \tau)^{-1}(y) = \bigcup_{t\in f^{-1}(y)} \tau^{-1}(t).$$

Hence

$$B_{f\circ\tau}(y) \leq N(\tau)\, B_f(y)$$

for all but countably many values y, so we deduce that $f \circ \tau \in \mathscr{BV}$. Since $f \circ \tau$ maps sets of measure zero to sets of measure zero, it follows that $f \circ \tau \in \mathscr{AC}$. Hence τ defines an endomorphism of $\mathscr{AC}$.

Conversely, suppose that τ defines an endomorphism of $\mathscr{AC}$ and let $t_0 \in \mathsf{T}$. Suppose that there are $2n - 1$ distinct points in $\partial(\tau^{-1}(t_0))$. Then there exists $\theta_1, \ldots, \theta_n \in \tau^{-1}(t_0)$ with $0 \leq \theta_1 < \theta_2 < \ldots < \theta_n < 2\pi$ such that, for $k = 1, \ldots, n$, there exists $\rho_k \in (\theta_k, \theta_{k+1})$ such that $\tau(\rho_k) \neq t_0$ (with $\theta_{n+1} = \theta_1$). For $\gamma > 0$, let $f_\gamma(t) = |t - t_0|^\gamma$ for $|t - t_0| \leq \pi$. Then $f_\gamma \in \mathscr{AC}$ and $\|f_\gamma\|_{\mathscr{AC}}$ is bounded as $\gamma \to 0$. Also,

$$\begin{aligned}\mathrm{Var}(f_\gamma \circ \tau) &\geq \textstyle\sum_{k=1}^{n} \big|(f_\gamma \circ \tau)(\theta_{k+1}) - (f_\gamma \circ \tau)(\rho_k)\big| + \big|(f_\gamma \circ \tau)(\rho_k) - (f_\gamma \circ \tau)(\theta_k)\big| \\ &= 2\sum_{k=1}^{n} |\tau(\rho_k) - t_0|^\gamma \to 2n \qquad \text{as } \gamma \to 0.\end{aligned}$$

On the other hand, since τ defines an endomorphism of $\mathscr{AC}$, it follows that $\mathrm{Var}(f_\gamma \circ \tau)$ is bounded as $\gamma \to 0$, and this proves the result.

In many ways it is more natural to consider the algebra $\mathscr{AC}([0,1])$ of absolutely continuous functions on the unit interval $[0,1]$ rather than $\mathscr{AC}$. We have chosen to focus our presentation on the latter, since it provides the link to the algebra $\mathscr{AC}^+ = \mathscr{AC} \cap \mathscr{A}(\overline{\Delta})$ (see Section 3). However, the previous result (as well as the rest of the section) of course also holds for $\mathscr{AC}([0,1])$. Furthermore, it is easily seen that the proof of the above result also applies to the algebra $\mathscr{BV}$, so we obtain the following result.

Theorem 2.2. *Let $\tau \in \mathscr{BV}$ with $\tau(\mathsf{T}) \subseteq \mathsf{T}$. Then τ defines an endomorphism of $\mathscr{BV}$ if and only if*

$$\sup_{t\in\mathsf{T}} \#\big(\partial\big(\tau^{-1}(t)\big)\big) < \infty.$$

For the automorphisms, we immediately obtain the following two corollaries.

Corollary 2.3. *Let $\tau \in \mathscr{AC}$ be a homeomorphism of* T. *Then the following conditions are equivalent.*

(a) *τ defines an automorphism of $\mathscr{AC}$.*

(b) *$\tau^{-1} \in \mathscr{AC}$.*

(c) *τ maps sets of positive measure to sets of positive measure.*

Corollary 2.4. *Every homeomorphism of* T *defines an automorphism of* $\mathscr{BV}$.

We shall now show that there exists an absolutely continuous homeomorphism τ of T for which τ^{-1} is not absolutely continuous, so that the condition $\tau^{-1} \in \mathscr{AC}$ in Corollary 2.3 is not superfluous.

Recall the definition of a *perfect symmetric set* on T from [5, Chapitre I]. Let $\underline{\xi} = (\xi_n)$ be a sequence with $0 < \xi_n < \frac{1}{2}$ for $n \in \mathsf{N}$. First, we remove an open interval V_{11} of length $2\pi(1 - 2\xi_1)$ from the middle of T (identifying T with $[0, 2\pi]$). In the nth step, from the middle of each of the remaining 2^{n-1} closed intervals, we remove an open interval V_{nk} $(k = 1, \ldots, 2^{n-1})$ of length $2\pi\xi_1 \cdots \xi_{n-1}(1 - 2\xi_n)$, so that 2^n closed intervals each of length $2\pi\xi_1 \cdots \xi_n$ remain. Let

$$V_n = \bigcup_{k=1}^{2^{n-1}} V_{nk} \qquad \text{for } n \in \mathsf{N}$$

and

$$E_{\underline{\xi}} = \mathsf{T} \setminus \bigcup_{n=1}^{\infty} V_n.$$

Then $E_{\underline{\xi}}$ is a perfect closed set with empty interior and

$$E_{\underline{\xi}} = \left\{ \sum_{n=1}^{\infty} 2\pi\varepsilon_n \, \xi_1 \cdots \xi_{n-1}(1 - \xi_n) : \varepsilon_n = 0 \text{ or } 1 \text{ for } n \in \mathsf{N} \right\}.$$

Also,

$$m(E_{\underline{\xi}}) = \lim_{n \to \infty} 2\pi \cdot 2^n \xi_1 \cdots \xi_n.$$

Note that the Cantor set on T corresponds to $\xi_n = \frac{1}{3}$ for $n \in \mathsf{N}$.

The idea in the following proof is simply to choose sequences $\underline{\xi}$ and $\underline{\lambda}$ such that $m(E_{\underline{\xi}}) = 0$ and $m(E_{\underline{\lambda}}) > 0$, and construct an absolutely continuous homeomorphism τ of T such that $\tau(E_{\underline{\lambda}}) = E_{\underline{\xi}}$. Geometrically, it is quite obvious that such a homeomorphism τ can be constructed by letting τ be linear on each component of the complement of $E_{\underline{\lambda}}$. The proof is a formalization of this idea.

EXAMPLE 2.5. *There exists an absolutely continuous homeomorphism τ of* T *for which τ^{-1} is not absolutely continuous.*

PROOF. Choose sequences $\underline{\xi}$ and $\underline{\lambda}$ such that $m(E_{\underline{\xi}}) = 0$ and $m(E_{\underline{\lambda}}) > 0$. Denote the open sets corresponding to $E_{\underline{\xi}}$ resp. $E_{\underline{\lambda}}$ by V_{nk} resp. W_{nk} ($n \in \mathsf{N}$, $k = 1, \ldots, 2^{n-1}$). For $N \in \mathsf{N}$, let τ_N be the increasing bijection of T which is linear on each W_{nk} with$\tau_N(W_{nk}) = V_{nk}$ for $n = 1, \ldots, N$ and $k = 1, \ldots, 2^{n-1}$, and is linear on each of the contiguous intervals. Then $\tau_{N+1} = \tau_N$ on $\bigcup_{n=1}^{N} W_n$, and it is easily seen that (τ_N) is a Cauchy sequence in $C(\mathsf{T})$. With

$$\tau = \lim_{N \to \infty} \tau_N,$$

we then have $\tau(W_{nk}) = V_{nk}$ for $n \in \mathsf{N}$ and $k = 1, \ldots, 2^{n-1}$. In particular, τ is a homeomorphism of T and

$$\tau(E_{\underline{\lambda}}) = E_{\underline{\xi}},$$

which shows that τ^{-1} is not absolutely continuous. Now let $F \subseteq \mathsf{T}$ be of measure zero. For $n \in \mathsf{N}$ and $k = 1, \ldots, 2^{n-1}$, we have $m(\tau(F) \cap V_{nk}) = m(\tau(F \cap W_{nk})) = 0$, because τ is linear on W_{nk}. Hence

$$m\Big(\tau(F) \setminus E_{\underline{\xi}}\Big) = \sum_{n,k} m(\tau(F) \cap V_{nk}) = 0.$$

Also, $\tau(F) \cap E_{\underline{\xi}}$ is of measure zero, since $E_{\underline{\xi}}$ is, so we deduce that $\tau(F)$ is of measure zero. Consequently τ is absolutely continuous, and that finishes the proof.

3. Endomorphisms of Banach algebras of analytic functions.

Let $\mathscr{B}$ be a unital, semisimple, commutative Banach algebra with character space T and let

$$\mathscr{B}^+ = \{f \in \mathscr{B} : \hat{f}(n) = 0 \text{ for } n < 0\},$$

where $\hat{f}(n) = (1/2\pi) \int_{\mathsf{T}} f(e^{i\theta})\, e^{-in\theta}\, d\theta$ ($n \in \mathsf{Z}$) are the Fourier coefficients of f. Then $\mathscr{B}^+$ is a closed subalgebra of $\mathscr{B}$. Also, every function $f \in \mathscr{B}^+$ extends to a function (also denoted by f) which is analytic on the open unit disc Δ and continuous on $\overline{\Delta}$. Consequently,

$$\mathscr{B}^+ = \mathscr{B} \cap \mathscr{A}(\overline{\Delta}),$$

where $\mathscr{A}(\overline{\Delta})$ is the disc algebra. Under very general conditions, the character space, $\Phi_{\mathscr{B}^+}$, of $\mathscr{B}^+$ is $\overline{\Delta}$. By the spectral radius formula applied to $\mathscr{B}$ and $\mathscr{B}^+$, we have

$$\sup\{|\varphi(f)| : \varphi \in \Phi_{\mathscr{B}^+}\} = \sup\{|f(z)| : z \in \mathsf{T}\} \qquad \text{for } f \in \mathscr{B}^+.$$

Given $\varphi \in \Phi_{\mathscr{B}^+}$, we thus have $|\varphi(f)| \leq \|f\|_\infty$ for $f \in \mathscr{B}^+$. Hence, if $\mathscr{B}^+$ is uniformly dense in $\mathscr{A}(\overline{\Delta})$ (for example, if $\mathscr{B}^+$ contains the polynomials), then φ extends by continuity to a character on $\mathscr{A}(\overline{\Delta})$. Therefore

$$\varphi(f) = f(z) \qquad \text{for } f \in \mathscr{B}^+$$

for some $z \in \overline{\Delta}$, so it follows that the character space of $\mathscr{B}^+$ is $\overline{\Delta}$.

Under these circumstances, the endomorphisms of $\mathscr{B}^+$ correspond to those functions $\tau \in \mathscr{B}^+$ with $\tau(\overline{\Delta}) \subseteq \overline{\Delta}$ for which $f \circ \tau \in \mathscr{B}^+$ for every $f \in \mathscr{B}^+$. It thus follows from the Riesz functional calculus that every function $\tau \in \mathscr{B}^+$ with $\tau(\overline{\Delta}) \subseteq \Delta$ (that is, with $\|\tau\|_\infty < 1$) defines an endomorphism of $\mathscr{B}^+$.

Furthermore, if $\tau \in \mathscr{A}(\overline{\Delta})$ is a homeomorphism of $\overline{\Delta}$, then $\tau = \lambda\tau_a$ for some $\lambda \in \mathsf{T}$ and $a \in \Delta$, where

$$\tau_a(z) = \frac{z - a}{1 - \bar{a}z} \qquad \text{for } z \in \overline{\Delta},\ a \in \Delta$$

([8, Theorem 12.6]). It follows that the automorphisms of $\mathscr{B}^+$ correspond to those functions $\lambda\tau_a$ for which $\lambda\tau_a$ and $\lambda^{-1}\tau_{-a}$ both define endomorphisms of $\mathscr{B}^+$.

The above applies, in particular, to the algebras $\mathscr{A}^+$, Λ_γ^+ $(0 < \gamma \leq 1)$ and $\mathscr{AC}^+$. It is easily seen that $\tau \in \mathscr{A}^+$ defines an endomorphism of $\mathscr{A}^+$ if and only if $\{\|\tau^n\|_{\mathscr{A}^+} : n \in \mathsf{N}\}$ is bounded. It does not seem to be known whether this condition can be characterized in terms of intrinsic properties of the function τ, although Newman ([7]) obtained partial results. However, it follows from the Beurling-Helson result mentioned in the introduction that the automorphisms of $\mathscr{A}^+$ correspond to $\lambda\tau_0 = \lambda\alpha$ with $\lambda \in \mathsf{T}$, that is, to rotations of the disc ([4, p.143]). This result easily generalizes to the Beurling algebras.

We first turn our attention to endomorphisms of the algebras Λ_γ^+ $(0 < \gamma \leq 1)$. A theorem of Hardy and Littlewood (see [2, Theorem 5.1]) implies that an analytic function f on Δ belongs to Λ_γ^+ if and only if

$$q_\gamma(f) = \sup_{z \in \Delta} |f'(z)|(1 - |z|)^{1-\gamma} < \infty,$$

and that

$$\|f\|_{\Lambda_\gamma^+} = \|f\|_\infty + q_\gamma(f), \qquad f \in \Lambda_\gamma^+,$$

defines an equivalent norm on Λ_γ^+. Using this, it follows easily that $\lambda\tau_a$ defines an automorphism of Λ_γ^+ for every $\lambda \in \mathsf{T}$ and $a \in \Delta$ and that $\|\Psi_{\lambda\tau_a}\| \sim (1 - |a|)^{-\gamma}$ as $|a| \to 1$.

For general endomorphisms of Λ_γ^+ $(0 < \gamma \le 1)$, the situation is more complex. First, we shall give a characterization of the functions defining endomorphisms of Λ_γ^+ which reflects the Hardy and Littlewood characterization of Λ_γ^+.

THEOREM 3.1. *Let $0 < \gamma \le 1$ and let $\tau \in \Lambda_\gamma^+$ with $\tau(\overline{\Delta}) \subseteq \overline{\Delta}$. Then τ defines an endomorphism of Λ_γ^+ if and only if*

$$\sup_{z\in\Delta} |\tau'(z)| \left(\frac{1-|z|}{1-|\tau(z)|}\right)^{1-\gamma} < \infty.$$

PROOF. Suppose $C = \sup\{|\tau'(z)|\,((1-|z|)/(1-|\tau(z)|))^{1-\gamma} : z \in \Delta\} < \infty$ and let $f \in \Lambda_\gamma^+$. Then

$$\begin{aligned} \left|(f\circ\tau)'(z)\right|(1-|z|)^{1-\gamma} &= |f'(\tau(z))|(1-|\tau(z)|)^{1-\gamma}|\tau'(z)|\left(\frac{1-|z|}{1-|\tau(z)|}\right)^{1-\gamma} \\ &\le Cq_\gamma(f) \qquad \text{for } z \in \Delta. \end{aligned}$$

Hence $f\circ\tau \in \Lambda_\gamma^+$, so τ defines an endomorphism of Λ_γ^+.

Conversely, suppose that τ defines an endomorphism of Λ_γ^+ with norm c. Let $z \in \Delta$ and write $\tau(z) = re^{i\theta}$ with $r \ge 0$ and $\theta \in \mathsf{R}$. Let $f = (1 - e^{-i\theta}\alpha)^\gamma$. Then $f \in \Lambda_\gamma^+$ with $\|f\|_{\Lambda_\gamma^+} \le 3$. Furthermore, $(f\circ\tau)'(z) = -\gamma e^{-i\theta}(1-r)^{\gamma-1}\tau'(z)$, so we deduce that

$$|\tau'(z)|\left(\frac{1-|z|}{1-|\tau(z)|}\right)^{1-\gamma} \le \frac{1}{\gamma}\, q_\gamma(f\circ\tau) \le \frac{3c}{\gamma},$$

and the result follows.

For $0 < \gamma < 1$, it is well known that the algebra λ_γ^+ is generated by the polynomials. Hence, if $\tau \in \lambda_\gamma^+$ defines an endomorphism of Λ_γ^+, then $p\circ\tau \in \lambda_\gamma^+$ for every polynomial p and thus $f\circ\tau \in \lambda_\gamma^+$ for every $f \in \lambda_\gamma^+$, so τ defines an endomorphism of λ_γ^+. Conversely, if τ defines an endomorphism of λ_γ^+, then by considering the functions $(1-e^{-i\theta}\alpha)^\beta$ for $\beta > \gamma$ and letting $\beta \to \gamma$, we deduce as in the above proof that τ satisfies the condition in the theorem. Consequently, this condition also characterizes the endomorphisms of λ_γ^+.

In the light of Sherbert's result mentioned in the introduction, we would be more interested in a characterization of the endomorphisms of Λ_γ^+ in terms of Lipschitz conditions. As remarked earlier, if $\tau \in \Lambda_\gamma^+$ and $\tau(\overline{\Delta}) \subseteq \Delta$, then τ defines an endomorphism of Λ_γ^+. Also, by Sherbert's result, if $\tau \in \Lambda_1^+$ with $\tau(\overline{\Delta}) \subseteq \overline{\Delta}$, then τ defines an endomorphism of Λ_γ^+. These facts suggest that $\tau \in \Lambda_\gamma^+$ defines an endomorphism of Λ_γ^+ if and only if τ, in some sense, satisfies a Lipschitz condition of order 1 close to the set where $|\tau(z)| = 1$. This idea is pursued in the following result.

PROPOSITION 3.2. *Let* $0 < \gamma \leq 1$. *Let* $\tau \in \Lambda_\gamma^+$ *with* $\tau(\overline{\Delta}) \subseteq \overline{\Delta}$ *and let* $E = \{z \in \mathsf{T} : |\tau(z)| = 1\}$.

(i) *Suppose that there exists an open neighbourhood* $U \subseteq \mathsf{T}$ *of* E *such that*

$$\sup\left\{\left|\frac{\tau(e^{i\theta}) - \tau(e^{i\rho})}{\theta - \rho}\right| : e^{i\theta},\, e^{i\rho} \in U,\, e^{i\theta} \neq e^{i\rho}\right\} < \infty.$$

Then τ *defines an endomorphism of* Λ_γ^+.

(ii) *Suppose that* τ *defines an endomorphism of* Λ_γ^+. *Then*

$$\sup\left\{\left|\frac{\tau(e^{i\theta}) - \tau(e^{i\rho})}{\theta - \rho}\right| : e^{i\theta} \in E,\, e^{i\rho} \in \mathsf{T},\, e^{i\theta} \neq e^{i\rho}\right\} < \infty.$$

PROOF. (i) The set U is a union of open intervals (or open arcs to be more precise) and finitely many of these cover E. Also, if necessary, by considering smaller intervals, we may assume that $|\tau(e^{i\theta}) - \tau(e^{i\rho})| \leq C|\theta - \rho|$ for $e^{i\theta}, e^{i\rho} \in \overline{U}$ for some constant C, where $U = \bigcup_{n=1}^N U_n$ is a finite union of open intervals in T with $E \subseteq U$. Let $f \in \Lambda_\gamma^+$. For $e^{i\theta}, e^{i\rho} \in \overline{U}$, we have

$$\left|(f \circ \tau)(e^{i\theta}) - (f \circ \tau)(e^{i\rho})\right| \leq \|f\|_{\Lambda_\gamma} C^\gamma |\theta - \rho|^\gamma.$$

Let $R = \sup\{|\tau(z)| : z \in \mathsf{T} \setminus U\} < 1$ and let $e^{i\theta}, e^{i\rho} \in \mathsf{T} \setminus U$. Then the line segment from $\tau(e^{i\theta})$ to $\tau(e^{i\rho})$ lies in $\{w : |w| \leq R\}$, so

$$\left|(f \circ \tau)(e^{i\theta}) - (f \circ \tau)(e^{i\rho})\right| \leq \sup\{|f'(w)| : |w| \leq R\} \|\tau\|_{\Lambda_\gamma} |\theta - \rho|^\gamma.$$

Finally, let $e^{i\theta} \in U$ and $e^{i\rho} \in \mathsf{T} \setminus U$. We may assume that $0 \leq \rho < \theta < 2\pi$ with $\theta - \rho \leq \pi$. Choose $e^{it} \in \partial U$ with $\rho \leq t < \theta$. Then

$$\left|(f \circ \tau)(e^{i\theta}) - (f \circ \tau)(e^{i\rho})\right| \leq \widetilde{C}(|\theta - t|^\gamma + |t - \rho|^\gamma) \leq 2\widetilde{C}|\theta - \rho|^\gamma$$

for some constant $\widetilde{C}$. We thus conclude that $f \circ \tau \in \Lambda_\gamma^+$, so τ defines an endomorphism of Λ_γ^+.

(ii) Let $e^{i\theta} \in E$. As mentioned in the proof of the previous theorem, the function $f = (1 - \tau(e^{i\theta})^{-1}\alpha)^\gamma$ belongs to Λ_γ^+ with $\|f\|_{\Lambda_\gamma^+} \leq 3$. Hence there exists a constant c such that

$$\left|\tau(e^{i\theta}) - \tau(e^{i\rho})\right|^\gamma = \left|(f \circ \tau)(e^{i\theta}) - (f \circ \tau)(e^{i\rho})\right| \leq c|\theta - \rho|^\gamma \qquad \text{for } e^{i\rho} \in \mathsf{T},$$

and the result follows.

We now give an example showing that the Lipschitz condition in (ii) in the previous proposition is not sufficient to ensure that a function τ defines an endomorphism of Λ_γ^+. We do not know whether τ necessarily satisfies the condition in (i), when τ defines an endomorphism of Λ_γ^+.

EXAMPLE 3.3. *Let* $0 < \gamma < 1$. *There exists* $\tau \in \Lambda_\gamma^+$ *with* $\tau(1) = 1$, $|\tau(z)| < 1$

for $z \in \overline{\Delta} \setminus \{1\}$ *and* $|1 - \tau(z)| = O(|1 - z|)$ *as* $z \to 1$ *in* $\overline{\Delta}$ *which does not define an endomorphism of* Λ_γ^+.

Proof. Let S be the singular, inner function defined by

$$S(z) = \exp\left(-\frac{1+z}{1-z}\right), \qquad z \in \overline{\Delta} \setminus \{1\}.$$

(For $t > 0$, we could use S^t instead.) Since $S' = -2(1-\alpha)^{-2}S$, it follows that

$$h = 1 + \tfrac{1}{4}(1-\alpha)^{\gamma+1} S \in \Lambda_\gamma^+.$$

Also, $h(z) \neq 0$ for $z \in \overline{\Delta}$ and $|h(e^{i\theta})| \leq 1 + 2^{\gamma-1}(\sin(|\theta|/2))^{\gamma+1}$ for $|\theta| \leq \pi$. Let $k(\theta) = (\sin(|\theta|/2))^{\gamma+1}$ for $|\theta| \leq \pi$, and let

$$G(z) = \frac{1}{2\pi}\int_{\mathsf{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, k(\theta)\, d\theta$$

and $F(z) = \exp(G(z))$ for $z \in \Delta$. Since Re G is the Poisson integral of k, we deduce that $|F(e^{i\theta})| = \exp(k(\theta))$ for $|\theta| \leq \pi$. If necessary, by multiplying F with a constant of modulus 1, we may assume that $F(1) = 1$. Since $k' \in \Lambda_\gamma$, it follows from [10, Theorem III.13.27] that $G' \in \Lambda_\gamma^+$. In particular, $G \in \Lambda_1^+$ and thus $F \in \Lambda_1^+$. Now let

$$\tau = \frac{h}{F}.$$

Then $\tau \in \Lambda_\gamma^+$ with $\tau(1) = 1$ and $|\tau(z)| < 1$ for $z \in \overline{\Delta} \setminus \{1\}$. Furthermore,

$$|1 - \tau(z)| \leq |1 - h(z)| + |\tau(z)|\,|F(z) - 1| = O(|1 - z|) \qquad \text{as } z \to 1 \text{ in } \overline{\Delta}.$$

Finally, we have

$$\tau' = -\frac{\frac{1}{2}S}{F(1-\alpha)^{1-\gamma}} - \frac{\frac{1}{4}(\gamma+1)(1-\alpha)^{\gamma}S}{F} - \frac{hF'}{F^2},$$

and the last two terms are bounded, so $1/\tau'(z) = O(|1-z|^{1-\gamma})$ as $z \to 1$ in Δ. Since $((1-\tau)^\gamma)' = -\gamma(1-\tau)^{\gamma-1}\tau'$ and since $\tau(1) = 1$, it follows that $1/((1-\tau)^\gamma)' = o(|1-z|^{1-\gamma})$ as $z \to 1$ in Δ. Hence $(1-\tau)^\gamma = (1-\alpha)^\gamma \circ \tau \notin \Lambda_\gamma^+$, so τ does not define an endomorphism of Λ_γ^+.

Finally, we shall discuss endomorphisms of the algebra $\mathscr{AC}^+$. Privalov (see [2, Theorem 3.11]) gave a useful characterization of $\mathscr{AC}^+$ by showing that an analytic function f on Δ belongs to $\mathscr{AC}^+$ if and only if $f' \in \mathscr{H}^1$, where $\mathscr{H}^1$ is the Hardy space of analytic functions f on Δ satisfying

$$\|f\|_{\mathscr{H}^1} = \sup_{r<1} \frac{1}{2\pi} \int_{\mathsf{T}} |f(re^{i\theta})|\, d\theta < \infty.$$

Also, it follows from [10, Theorem VII.8.2] that $\mathscr{BV}^+ = \mathscr{AC}^+$.

We first mention that the proof of Theorem 2.1 shows that every $\lambda\tau_a$ with $\lambda \in \mathsf{T}$ and $a \in \Delta$ defines an isometric automorphism of $\mathscr{AC}^+$. Also, we have already mentioned that if $\tau \in \mathscr{AC}^+$ with $\tau(\overline{\Delta}) \subseteq \Delta$, then τ defines an endomorphism of $\mathscr{AC}^+$. We shall now prove an analogue of Proposition 3.2 for $\mathscr{AC}^+$.

Proposition 3.4. *Let $\tau \in \mathscr{AC}^+$ with $\tau(\overline{\Delta}) \subseteq \overline{\Delta}$ and let $E = \{z \in \mathsf{T} : |\tau(z)| = 1\}$. Suppose that there exists an open neighbourhood $U \subseteq \overline{\Delta}$ of E such that τ' is bounded on $U \setminus \mathsf{T}$ (for example, if $\tau \in \Lambda_1^+$). Then τ defines an endomorphism of $\mathscr{AC}^+$.*

Proof. Let $f \in \mathscr{AC}^+$ and let $R = \sup\{|\tau(z)| : z \in \overline{\Delta} \setminus U\} < 1$. Then

$$\left|(f \circ \tau)'(z)\right| \le \sup\{|f'(w)| : |w| \le R\}\, |\tau'(z)| \qquad \text{for } z \in \Delta \setminus U.$$

Also,

$$\left|(f \circ \tau)'(z)\right| \le \sup\{|\tau'(z)| : z \in U \setminus \mathsf{T}\}\, |f'(\tau(z))| \qquad \text{for } z \in U \setminus \mathsf{T},$$

and $f' \circ \tau \in \mathscr{H}^1$ by [2, Corollary, p.29]. We thus deduce that $(f \circ \tau)' \in \mathscr{H}^1$. Hence $f \circ \tau \in \mathscr{AC}^+$, which proves the result.

This result should be contrasted with the fact that there exists $\tau \in \Lambda_1$ ($\subseteq \mathscr{AC}$) with $\tau(\mathsf{T}) \subseteq \mathsf{T}$ such that τ does not define an endomorphism of $\mathscr{AC}$ (this follows from Theorem 2.1). We shall now see that the sufficient condition $\tau \in \Lambda_1^+$ in the previous proposition cannot be relaxed to $\tau \in \Lambda_\gamma^+$ for some $\gamma < 1$.

Example 3.5. *Let $0 < \gamma < 1$. There exists $\tau \in \Lambda_\gamma^+ \cap \mathscr{AC}^+$ with $\tau(1) = 1$, $|\tau(z)| < 1$ for $z \in \overline{\Delta} \setminus \{1\}$ and $|1 - \tau(z)| = O(|1 - z|)$ as $z \to 1$ in $\overline{\Delta}$ which does not define an endomorphism of $\mathscr{AC}^+$.*

Proof. Let τ be as in Example 3.3. Then $\tau \in \mathscr{AC}^+$, so we only have to prove that τ does not define an endomorphism of $\mathscr{AC}^+$. Let $f = (1 - \alpha)^{1-\gamma} \in \mathscr{AC}^+$. Then

$$|(f' \circ \tau)(z)| = (1 - \gamma)|1 - \tau(z)|^{-\gamma} \ge c_1 |1 - z|^{-\gamma}$$

for $z \in \overline{\Delta} \setminus \{1\}$ for some constant $c_1 > 0$. Since $1/\tau'(z) = O(|1 - z|^{1-\gamma})$ as $z \to 1$ in Δ, it follows that

$$\left|(f \circ \tau)'(z)\right| \ge c_2 |1 - z|^{-1}$$

in a neighbourhood of $z = 1$ for some constant $c_2 > 0$. Hence $f \circ \tau \notin \mathscr{A}\mathscr{C}$, so τ does not define an endomorphism of $\mathscr{A}\mathscr{C}^+$.

By applying the method used in the proof of Theorem 2.1 to the functions $(1-\lambda\alpha)^\gamma$ ($\lambda \in \mathsf{T}$, $\gamma > 0$), it can be seen that, if a non-constant function $\tau \in \mathscr{A}\mathscr{C}^+$ with $\tau(\overline{\Delta}) \subseteq \overline{\Delta}$ defines an endomorphism of $\mathscr{A}\mathscr{C}^+$, then $\sup_{z\in\mathsf{T}} \#(\tau^{-1}(z)) < \infty$. Unfortunately, we do not know whether there exists a function τ which does not satisfy this condition, but it follows from Proposition 3.4 that the condition holds when $\tau \in \Lambda_1^+$.

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