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EXPLICIT REPRESENTATION OF THE SOLUTION TO SOME BOUNDARY VALUE PROBLEM

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Abstract.
In the half ray the unique solution to the boundary value problem

$$L(D_t)u(t) \ = f(t),\ t > 0$$

$$B_j(D_t)u(0) = \alpha_j, \text{for } j = 0,\ 1, ..., p - 1$$

rapidly decreasing at infinity is shown to be explicitly represented in terms of Green's function and some boundary kernels, namely,

$$u(t) = \sum_{k=0}^{p-1} \mathscr{H}_k(t)\alpha_k + \int_0^\infty \mathscr{G}(t,s)f(s)\,ds$$

1. Introduction.

Let $L(z)$ be a polynomial of degree $m \geq 1$, where the coefficient of z^m is equal to 1, and let $\{B_j(z)\}_{j=0}^{p-1}$ be p polynomials of degrees $\{m_j\}_{j=0}^{p-1}$ respectively, so that $m_j < m$ for $0 \leq j \leq p-1$. We assume that $L(z)$ has at most p roots having positive imaginary parts (counting multiplicities).

Throughout this paper we denote by D_t the differential operator $-i\frac{d}{dt}$; and if $J \subseteq \mathsf{R}$ then we denote by $\mathscr{S}(J)$ the subspace of $C^\infty(\overline{J})$ containing all the functions $u(t)$ such that

$$(1 + |t|)^p |u^{(q)}(t)|$$

are bounded for all p and q in N.

Consider the following boundary value problem

$$L(D_t)u(t) = f(t), t > 0 \tag{2.1}$$

$$B_j(D_t)u(0) = \alpha_j,\ \text{for } j = 0,\ 1, ..., p - 1 \tag{2.2}$$

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where the α_j are complex constants. Set

$$L_p(z) = (z - \tau_0)(z - \tau_1)...(z - \tau_{p-1}) \tag{2.3}$$

where the τ_j's are the p roots of $L(z)$ having positive imaginary parts.

Firstly, we assume that $m_j < p$ for $j = 0,\ 1, ..., p-1$, then

$$B_j(z) = \sum_{k=0}^{p-1} b_{jk} z^k \tag{2.4}$$

for $j = 0,\ 1, ..., p-1$. Now, if the matrix (b_{jk}) is nonsingular, one can solve Eq. (2.4) for the unknown variables z^k (for fixed z).

Hence, if we denote by (b^{jk}) the inverse matrix of (b_{jk}), we get

$$z^j = \sum_{k=0}^{p-1} b^{jk} B_k(z) \tag{2.5}$$

for $j = 0,\ 1, ..., p-1$. So that, Eq. (2.2) is equivalent to

$$D_t^k u(0) = \sum_{j=0}^{p-1} b^{kj} \alpha_j \tag{2.6}$$

for $k = 0,\ 1, ..., p-1$. Thus, any solution of the Cauchy problem (2.1), (2.2) is a solution of the problem

$$L(D_t)u(t) = f(t), t > 0 \tag{2.7}$$

$$D_t^k u(0) = \sum_{j=0}^{p-1} b^{kj} \alpha_j \tag{2.8}$$

for $k = 0,\ 1, ..., p-1$. Conversely, any solution of (2.7), (2.8) is a solution of (2.1), (2.2).

REMARKS. If the matrix (b_{jk}) is singular, then there are constants c_j not all zero so that

$$\sum_{j=0}^{p-1} c_j B_j(z) = 0 \tag{2.9}$$

Consequently, a necessary condition for the problem (2.1), (2.2) to have a solution is the following

$$\sum_{j=0}^{p-1} c_j \alpha_j = 0 \tag{2.10}$$

This shows that we can not hope to get a solution for all choices of the α_j's. Furthermore, even when we can solve, the solution is generally not unique.

Now if we allow to the m_j's to be greater than p, then, by partial fractions we can write

$$B_j(z) = Q_j(z)L_p^+(z) + B_j'(z) \tag{2.11}$$

Where the degree of B_j' is less than p.

When $p < m$ we define the polynomial

$$L^-(z) = \frac{L(z)}{L_p^+(z)} \tag{2.12}$$

We conclude by the following Lemma [5]:

LEMMA. *Let $L(D_t)$ be any constant coefficient differential operator, and let $f(t)$ be any function in $\mathscr{S}(\mathsf{R})$. Then, there exists a function $u(t) \in \mathscr{S}(\mathsf{R})$ which satisfies the differential equation*

$$L(D_t)u(t) = f(t), \text{ for } t > 0 \tag{2.13}$$

that the differential equation

$$L^-(D_t)v(t) = f(t), t > 0 \tag{2.14}$$

has always a solution $v(t)$ belonging to $\mathscr{S}(\mathsf{R})$ for any choice of $f(t)$in $\mathscr{S}(\mathsf{R})$. Thus, the boundary value problem (2.1), (2.2) is equivalent to the following

$$L_P^+(D_t)u(t) = v(t), \ t > 0 \tag{2.15}$$

$$B_j'(D_t)u(0) = \alpha_j - Q_j(D_t)v(0), \ j = 0, \ 1, ..., p-1 \tag{2.16}$$

which is the form just treated. If we write

$$B_j'(z) = \sum_{k=0}^{p-1} b_{jk}' z^k, \quad 0 \le j \le p-1 \tag{2.17}$$

then the problem (2.15), (2.16) has a unique solution for any given $v(t)$ in $\mathscr{S}(\mathsf{R})$ and $\{\alpha_j\} \subset \mathsf{C}$ if and only if the matrix (b_{jk}') is nonsingular.

We observe from Eq. (2.9) that the matrix (b_{ij}) is nonsingular if and only if the polynomials $B_j(z)$ are linearly independent. Similarly, the matrix (b_{ij}') is nonsingular if and only if the polynomials $B_j'(z)$ are linearly independent. We say that the $\{B_j(z)\}$ are linearly independent modulo $L_p^+(z)$ if the $\{B_j'\}$ are linearly independent.

2. Main results.

Now we are able to give explicitly the solution to the boundary value problem (2.1), (2.2) in terms of Green's function; we shall follow closely Pederson's work after redefining in a convenient manner the functions $u_j^+(t)$ (see [3], [4]).

THEOREM 1. *Let $L(z)$ be a polynomial of degree m, having at most p roots $\tau_0, \tau_1, \ldots, \tau_{p-1}$ with positive imaginary parts and no real roots. Let $\{B_j(z)\}_{j=0}^{p-1}$ be p polynomials of degrees $\{m_j\}$ with $m_j < m$, which are linearly independent modulo*

$$L_p^+(z) = (z - \tau_0)(z - \tau_1)\ldots(z - \tau_{p-1})$$

Then, for any $f \in \mathscr{S}(\mathsf{R}^+)$ and for any choice of the constants $\alpha_0, \alpha_1, \ldots, \alpha_{p-1}$, there exists a unique solution $u(t) \in \mathscr{S}(\mathsf{R}^+)$ satisfying the boundary value problem (2.1), (2.2). *Furthermore, this solution can be represented as follows*

$$u(t) = \sum_{j=0}^{p-1} \mathscr{H}_j(t)\alpha_j + \int_0^\infty \mathscr{G}(t,s)f(s)\,ds \tag{2.18}$$

PROOF. Let us first consider the case $f = 0$; then, the general solution of the equation

$$L(D_t)u(t) = 0 \tag{2.19}$$

has the form

$$u(t) = \sum_{k=0}^{m-1} \beta_k \exp(i\tau_k)$$

where the τ_k's are the roots of $L(z) = 0$. It is worth to recall that the coefficients β_k become polynomials in t whenever there are multiple roots. Now, in order for $u(t)$ to be in $\mathsf{L}^2(0, \infty)$, the coefficients β_k must vanish for any k such that $\mathfrak{Im}\ \tau_k \leq 0$ otherwise $u(t)$ could not be in $\mathsf{L}^2(0,\infty)$. Therefore, the solution of (2.19) which belongs to the space $\mathsf{L}^2(0,\infty)$ is

$$u(t) = \sum_{k=0}^{p-1} \beta_k \exp(i\tau_k) \tag{2.20}$$

with $\mathfrak{Im}\tau_k > 0$.
Following AGMON, DOUGLIS and NIRENBERG [1], we define

$$L_k^+(\tau) = \sum_{j=0}^{k} a_j^+ \tau^{k-j},\ k = 0, \quad 1, \ldots, p-1 \tag{2.21}$$

where the constants a_j^+ are defined through the expression

$$L_p^+(\tau) = \sum_{j=0}^{p} a_j^+ \tau^{p-j} \tag{2.22}$$

Let Γ^+ and Γ^- be rectifiable Jordan contours in the upper and the lower half plane enclosing the roots of $L_p^+(z)$ and $L_p^-(z)$ respectively.

Define the functions

$$u_j^+(t) = \frac{1}{2\pi i} \oint_{\Gamma^+} \frac{L_{p-j-1}^+(\tau)}{L_p^+(\tau)} \exp(it\tau)\, d\tau \tag{2.23}$$

We claim that

$$L(D_t)u_j^+(t) = 0,\ t > 0 \tag{2.24}$$

and

$$D_t^k u_j^+(0) = \delta_{jk} \tag{2.25}$$

for $j,\ k = 0,\ 1,\ \ldots, p-1$, where δ_{jk} is the Kronecker Delta. Indeed, by differentiation under the integral sign (which is of course allowed), we get

$$D_t^k\ u_j^+(t) = \frac{1}{2\pi i} \oint_{\Gamma^+} \frac{L_{p-j-1}^+(\tau)}{L_p^+(\tau)} \tau^k \exp(it\tau)\, d\tau \tag{2.26}$$

Now, if we take Γ^+ to be a large circle about the origin with radius $n \in \mathsf{N}^*$ so that Γ^+ encloses $\tau_0,\ \tau_1, \ldots,\ \tau_{p-1}$, then,

$$D_t^k\ u_j^+(0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{L_{p-j-1}^+(ne^{i\omega})}{L_p^+(ne^{i\omega})} n^{k+1}\ e^{i\omega(k+1)} d\omega$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{Q(ne^{i\omega})}{L_p^+(ne^{i\omega})} d\omega$$

Where Q is a polynomial of degree $p+k-j$ in n. Since L_p^+ is of degree p, then, by letting n go to infinity, we obviously get

$$D_t^k u_j^+(0) = \delta_{jk},\ \text{for } k-j \leq 0$$

For the case $k-j > 0$, we note that the polynomial $\tau^k\ L_{p-j-1}^+$ differs from $\tau^{k-j-1} L_p^+$ by a polynomial Q of degree at most equals to $k-1$. Thus,

$$D_t^k u_j^+(0) = \frac{1}{2\pi i} \oint_{\Gamma^+} \frac{L^+_{p-j-i}(\tau)}{L_p^+(\tau)} \tau^k \, d\tau$$

$$= \frac{1}{2\pi i} \oint_{\Gamma^+} \frac{\tau^{k-j-1} L_p^+(\tau) + Q(\tau)}{L_p^+(\tau)} d\tau$$

$$= \frac{1}{2\pi i} \oint_{\Gamma^+} \tau^{k-j-i} d\tau \frac{1}{2\pi i} \oint_{\Gamma^+} \frac{Q(\tau)}{L_p^+(\tau)} d\tau$$

$$= \frac{1}{2\pi i} \oint_{\Gamma^+} \frac{Q(\tau)}{L_p^+(\tau)} d\tau$$

By the same argument as before, since the degree of $Q(z)$ is equal to $k-1 < p-1$, we can observe that the last integral is zero. As a consequence, we obtain

$$D_t^k u_j^+(0) = \delta_{jk}, \quad j,k = 0,1,...,p-1$$

On the other hand we have

$$L(D_t) u_j^+(t) = \frac{1}{2\pi i} \oint_{\Gamma^+} \frac{L^+_{p-j-i}(\tau) L_p^+(\tau) L^-(\tau)}{L_P^+(\tau)} e^{it\tau} d\tau = 0.$$

It follows that the set $\{u_j^+(t)\}$ spans the negative exponential solutions of the homogeneous boundary value problem associated to (2.1), (2.2). Now, in order to obtain a solution to the inhomogeneous boundary value problem (2.1), (2.2), we define the functions

$$v^{\pm}(t) = \frac{1}{2\pi} \oint_{\Gamma^{\pm}} \frac{e^{it\tau}}{L(\tau)} d\tau \tag{2.27}$$

It follows from the fact that the contour $\Gamma^+ \cup \Gamma^-$ can be deformed into a large circle that

$$D_t^k (v^+(0) + v^-(0)) = i\delta_{m-1,k} \tag{2.28}$$

where $k = 0, ..., m-1$. As a consequence, the function

$$\int_0^t (v^+(t-s) + v^-(t-s)) f(s) \, ds \tag{2.29}$$

is a solution of the Eq. (2.1) with zero Cauchy Data.

Thus, the general solution of the inhomogeneous boundary value problem (2.1), (2.2) which is bounded must have the form:

$$u(t) = \sum_{j=0}^{p-1} \beta_j u_j^+(t) + \int_0^t (v^+(t-s) + v^-(t-s))f(s)\,ds - \int_0^\infty v^-(t-s)f(s)\,ds \tag{2.30}$$

This is a consequence of the facts that $v^+(t)$ is a sum of the negative exponentials when $t > 0$, and $v^-(t)$ is a sum of negative exponentials when $t < 0$. Now, by virtue of the complementing condition (linear independence of the $B_j's$), we conclude that

$$B_k(D_t)\int_0^t (v^+(t-s) + v^-(t-s))f(s)\,ds \bigg|_{t=0} = 0$$

Since the function (2.30) is a formal solution of (2.1), (2.2) then, it must satisfy the following

$$\alpha_k = B_k(D_t)u(0) = \sum_{j=0}^{p-1} \beta_j B_k(D_t)u_j^+(0) +$$

$$+ \; B_k(D_t)\int_o^t (v^+(t-s) + v^-(t-s))f(s)\,ds \bigg|_{t=0} -$$

$$- \int_0^\infty f(s)B_k(D_t)v^-(t-s)\,ds \bigg|_{t=0}$$

$$= \sum_{j=0}^{p-1} \beta_j \{Q_k(D_t)L_p^+(D_t)u_j^+(0) + B'_k(D_t)u_j^+(0)\} -$$

$$- \int_0^\infty f(s)B_k(D_t)v^-(t-s)\,ds \bigg|_{t=0}$$

$$= \sum_{j=0}^{p-1} \beta_j \, b_{kj}^+ - \int_0^\infty f(s)B_k(D_t)v^-(t-s)\,ds \bigg|_{t=0}$$

where $B_k(z) = Q_k(z)L_p^+(z) + B'_k(z) = B'_k(z) \bmod(L_p^+)$ and

$$B'_k(z) = \sum_{j=0}^{p-1} b^+_{kj} z^j$$

We get an algebraic system of equations with unknown variables β_j

$$= \sum_{j=0}^{p-1} b^+_{kl}\beta_j = \alpha_k + \int_0^\infty f(s) B_k(D_t) v^-(t-s)\, ds \Bigg|_{t=0}, \quad k = 0, 1, ..., p-1$$

We deduce from the complementing condition that the determinant of the matrix (b^+_{kj}) is not zero; so, as a consequence, the above set of equations has a unique solution $\{\beta_0, \dots, \beta_{p-1}\}$.

Define the inverse matrix

$$(b^{kj}_+) = (b^+_{kj})^{-1}$$

Hence,

$$\beta_j = \sum_{k=0}^{p-1} b^{jk}_+ \alpha_k + \int_0^\infty \sum_{k=0}^{p-1} b^{jk}_+ f(s) B_k(D_t) v^-(t-s) \Bigg|_{t=0}, \quad j = 0, 1, ..., p-1$$

and upon substitution of the β_j's into (2.30) we obtain the bounded solution of the given BVP,

$$u(t) = \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} b^{jk}_+ \alpha_k \, u^+_j(t) +$$

$$+ \left\{ \int_0^\infty \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} b^{jk}_+ B_k(D_t) v^-(t-s) \Bigg|_{t=0} f(s) ds \right\} u^+_j(t) +$$

$$+ \int_0^t v^+(t-s) f(s) ds - \int_t^\infty v^-(t-s) f(s) ds$$

If we set

$$\mathscr{H}_k(t) = \frac{1}{2\pi i} \oint_{\Gamma^+} \frac{\sum_{j=0}^{p-1} b^{jk}_+ L^+_{p-j-1}(\tau)}{L^+_p(\tau)} e^{it\tau} d\tau$$

for $k = 0, \dots, p-1$

$$\mathcal{G}_1(t) = \frac{1}{2\pi} \oint_{\Gamma^+} \frac{e^{it\tau}}{L(\tau)} d\tau, \text{ if } t > 0$$

$$= \frac{-1}{2\pi} \oint_{\Gamma^-} \frac{e^{it\tau}}{L(\tau)} d\tau, \text{ if } t < 0$$

$$\mathcal{G}_2(t,s) = \frac{-i}{4\pi^2} \oint_{\Gamma^+} \oint_{\Gamma^-} \frac{\sum_{j=0}^{p-1} \sum_{k=0}^{p-1} L^+_{p-j-i}(\tau) B_k(\delta)}{L^+_p(\tau) L(\delta)} e^{i(t\tau - s\delta)} d\tau\, d\delta$$

and

$$\mathcal{G}(t,s) = \mathcal{G}_1(t-s) + \mathcal{G}_2(t,s)$$

then, the solution of the boundary value problem (2.1), (2.2) takes the final form

$$u(t) = \sum_{j=0}^{p-1} \mathcal{H}_j(t)\alpha_j + \int_0^\infty \mathcal{G}(t,s) f(s)\, ds$$

An immediate computation shows that the above kernels satisfy the following estimates

$$|D_t^k \mathcal{H}_j(t)| \le C_0 \exp(-r_0 t), \qquad \forall t > 0, \forall k = 0,\ 1,\ \ldots$$

$$|\ D_t^k \mathcal{G}_1(t)| \le C_1 \exp(-r_1 \mid t \mid), \qquad \forall t \in \mathsf{R}^*, \forall k = 0,\ 1,\ \ldots$$

$$|D_t^k \mathcal{G}_2(t,s)| \le C_2 \exp(-r_2(t+s)), \quad \forall t > 0, \forall s > 0, \forall k = 0,\ 1,\ \ldots$$

for some positive constants C_0, C_1, C_2, r_0, r_1, and r_2, depending only on $L(z)$ and $\{B_j\}$.

Now to see that the expression (2.18) is rapidly decreasing at infinity it suffices to show that the function

$$v_j(t) = t^j \int_0^\infty |f(s)|\ (C_1 \exp\ (-r_1\ |t-s|) + C_2 \exp\ (-r_2(t+s))ds$$

is bounded for each nonnegative integer j. Since $f \in \mathcal{S}(\mathsf{R}^+)$ there is a constant $C > 0$ such that

$$|f(s)| \le \frac{C}{(1+s)^{j+2}}, \forall s > 0$$

it then follows that

$$v_j(t) \leq C' t^j \left(\frac{2}{t}\right)^j \int_0^{\frac{t}{2}} \frac{ds}{(1+s)^{j+2}} + C_1 \frac{(2t)^j}{(2+t)^j} \int_{\frac{t}{2}}^{t} \frac{ds}{(1+s)^2}$$

$$+C_1 \left(\frac{t}{1+t}\right)^j \int_t^{\infty} \frac{ds}{(1+s)^2} + C_2' t^j \int_0^{\infty} \exp(-r_2(t+s))\, ds$$

$$\leq C(j) \int_0^{\infty} \frac{ds}{(1+s)^2} + C_3 t^j \exp(-r_2 t) < +\infty$$

Hence $v_j(t)$ is bounded in R^+ and consequently $u(t) \in \mathscr{S}(\mathsf{R}^+)$.
Finally, using classical techniques we can easily prove the uniqueness of this solution. This establishes the proof of the given theorem.

Let us denote by $H^k(\mathsf{R}^+), k \geq 0$ the completion of the space $\mathscr{S}(\mathsf{R}^+)$ with respect to the norm

$$\|u\|_k^2 = \sum_{j=0}^{k} \int_0^{\infty} |u^{(k)}(t)|^2\, dt \tag{2.31}$$

and we define the subspace

$$V^k = H^k(\mathsf{R}^+) \cap C^k[0, \infty]$$

As a consequence of the previous representation theorem and Theorem 6–9 [5] we obtain the estimate of the solution to the problem (2.1)–(2.2) in terms of the Data f and $\alpha_0, \ \dots, \ \alpha_{p-1}$:

THEOREM 2. *Under the same assumptions of Theorem 1, we conclude that for each* $k \in \mathsf{N}$, *there is a constant* $C > 0$ *(depending only on* $L(z), B_j(z)$ *and* k*) such that, for each* $f \in V^k$ *and* $\alpha_0, \ \dots, \alpha_{p-1} \in \mathsf{C}^p$, *the solution* $u \in V^{m+k}$ *to the* BVP (2.1) – (2.2) *satisfies the estimate*

$$\|u\|_{m+k} \leq C\left(\sum_{j=0}^{p-1} |\alpha_j| + \|f\|_k\right)$$

and has the representation

$$u(t) = \sum_{j=0}^{p-1} \mathscr{H}_j(t)\alpha_j + \int_0^{\infty} \mathscr{G}(t,s) f(s)\, ds \tag{2.32}$$

(where $\mathscr{H}_j(t)$ and $\mathscr{G}(t,s)$ are the same as in Theorem 1.).

PROOF. We deduce form the density of $\mathscr{S}(\mathsf{R}^+)$ in V^k that there is a sequence $(f_n) \subset \mathscr{S}(\mathsf{R}^+)$ converging to f in V^k. On the other hand there corresponds to each f_n at most one solution $u_n \in \mathscr{S}(\mathsf{R}^+)$ satisfying:

$$L(D_t)u_n(t) = f_n(t),\ (t > 0)$$

$$B_j(D_t)u_n(0) = \alpha_j,\ j = 0,\ \dots, p-1$$

and given by

$$u_n(t) = \sum_{j=0}^{p-1} \mathscr{H}_j(t)\alpha_j + \int_0^\infty \mathscr{G}(t,s)f_n(s)\, ds \tag{2.33}$$

We conclude by Theorem 6–9 [5] that there is a constant $C_0 > 0$ depending only on $L(z)$ and k such that

$$\| u_n - \sum_{j=0}^{p-1} \mathscr{H}_j(t)\alpha_j \|_{m+k} \leq C_0 \| f_n \|_k \tag{2.34}$$

and

$$\| u_n - u \|_{m+k} \leq C_0 \| f_n - f \|_k \tag{2.35}$$

Hence,

$$\| u \|_{m+k} \leq \sum_{j=0}^{p-1} \| \mathscr{H}_j \|_{m+k} \cdot |\alpha_j| + C_0 \| f \|_k \leq C \left(\sum_{j=0}^{p-1} |\alpha_j| + \| f \|_k \right) \tag{2.36}$$

where $C = \max\{C_0, \| \mathscr{H}_j \|_{m+k};\ j = 0,\ \dots,\ p-1\}$.

The estimate (2.36) shows that the isomorphism

$$\mathfrak{P} : (\alpha_0,\ \dots, \alpha_{p-1}, f) \to u$$

is continuous from $\mathsf{C}^p \times V^k$ onto V^{m+k}. Consequently, by letting $n \to +\infty$ in (2.33) we obtain

$$u(t) = \sum_{j=0}^{p-1} \mathscr{H}_j(t)\alpha_j + \int_0^\infty \mathscr{G}(t,s)f(s)\, ds,\ (t > 0)$$

$$= \mathfrak{P}(\alpha_o,\ \dots, \alpha_{p-1}, f)$$

This proves the theorem.

REMARKS. 1) If $L(z)$ admits a real root then we cannot hope to get an estimate of the form (2.36) even under smooth Data as shows the following example:

$$\frac{du}{dt} = \frac{1}{t+1} \in \mathsf{L}^2(\mathsf{R}^+) \cap C^\infty(\mathsf{R}^+),$$

$$u(0) = \alpha_0$$

whose unique solution is

$$u(t) = \alpha_0 + \ln(1+t),\ (t > 0)$$

which is not in $\mathsf{L}^2(0, \infty)$ whatsoever the value of the constant α_0.

2) The best constant C in (2.36) is equal to the norm of the isomorphism $\mathfrak{P}$ defined by

$$\mathrm{Sup}\ |\sum_{j=0}^{p-1} \mathscr{H}_j(t)\beta_j + \int_0^\infty \mathscr{G}(t,s)h(s)\ ds|$$

where the supremum is taken over all $h \in V^k$ and $\beta_0, \ldots, \beta_{p-1} \in \mathsf{C}$ such that

$$\sum_{j=0}^{p-1} |\beta_j| + \|h\|_k = 1$$

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