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REAL INTERPOLATION OF COMPACT OPERATORS BETWEEN QUASI-BANACH SPACES

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Abstract.

Let (A_0, A_1) and (B_0, B_1) be couples of quasi-Banach spaces and let T be a linear operator. We prove that if $T : A_0 \to B_0$ is compact and $T : A_1 \to B_1$ is bounded, then $T : (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}$ is also compact.

Some results on the structure of minimal and maximal interpolation methods are also established.

0. Introduction.

Assume that T is a linear operator such that $T : L_{p_0} \to L_{q_0}$ compactly and $T : L_{p_1} \to L_{q_1}$ boundedly. Here $1 \le p_o, p_1, q_1 \le \infty$, $1 \le q_0 < \infty$. Let $0 < \theta < 1$ and put $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$. In 1960, Krasnoselskii [13] proved that under these assumptions $T : L_p \to L_q$ is also compact.

Krasnoselskii's theorem was motivated by certain compactness results for integral operators established by Kantorovich (see [14], p. 118) and it led to the study of interpolation properties of compact operators between abstract Banach spaces. These investigations has been done during two very different periods. A first one during the 60's, simultaneous to the foundation of abstract interpolation theory, and a second period developed during the last decade, where modern interpolation techniques have been successfully used to derive new compactness results.

Contributions on this subject are due to many authors. We refer to [8] and [7] for a quite complete list of references and for historical remarks on the development during the two periods. Some more recent results can be found in the papers by Cobos [4], Cwikel and Kalton [9] and Mastylo [16].

Returning to Krasnoselskii's theorem, Zabreiko and Pustylnik [20] (see also [14], Thm. 3.11) proved in 1965 that the result is still true for the full

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rank of parameters, that is for $0 < q_0, q_1 \leq \infty$. Note that the last couple is formed by quasi-Banach spaces when $0 < q_0, q_1 < 1$. It arises then the question if a similar result holds true for abstract quasi-Banach couples, not only for L_q-couples. Accordingly, we establish in this paper such a result.

We follow the approach developed in [8] and [7] (see also [5], [6]) based on the description of the real interpolation method as a maximal and a minimal interpolation method in the sense of Aronszajn-Gagliardo [1]. The main obstacle is then to find a useful extension of maximal methods for quasi- Banach couples. The natural definitions based on scalar sequence spaces, give nothing but the sum space when applying to a couple as (L_{q_0}, L_{q_1}) with $0 < q_0, q_1 < 1$, because the spaces L_{q_j} have trivial dual.

We overcome this difficulty by giving a maximal description of the real interpolation space in terms of vector valued sequence spaces involving the couple into consideration. We have then a description for each quasi-Banach couple, rather than a description for the real interpolation method. However, this will be sufficient for our purposes. Such a description is given in Section 2, where we also derive the corresponding minimal characterization.

Working in the category of Banach couples, any maximal or minimal method defined by sequence spaces satisfying certain mild conditions, can be equivalently defined by vector valued sequence spaces as we show in Section 1. This result, that we think has independent interest, is based on the Hahn-Banach theorem and applies not only to the real method, but also to the "$\pm$" method (see [18], [10]) and Ovchinnikov's ϕ_u-method [17].

In the final Section 3 we prove the announced interpolation theorem for compact operators in the quasi-Banach case.

1. Maximal and minimal methods in Banach spaces.

Let us start by recalling the construction of the Aronszajn-Gagliardo maximal functor $H[(B_0, B_1); B](\cdot, \cdot)$; for Banach spaces (see [1]; see also [12], [3]).

If $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ are Banach couples, we write $T \in \mathscr{L}(\bar{A}, \bar{B})$ to mean that T is a linear operator from $A_0 + A_1$ into $B_0 + B_1$ such that the restriction of T to each A_j defines a bounded operator from A_j into B_j $(j = 0, 1)$. We write

$$\|T\|_{\bar{A},\bar{B}} = \max_{j=0,1}\{\|T\|_{A_j,B_j}\}$$

We say that a Banach space A is an intermediate space with respect to the couple $\bar{A} = (A_0, A_1)$ if the following continuous embeddings hold

$$\Delta(\bar{A}) = A_0 \cap A_1 \hookrightarrow A \hookrightarrow A_0 + A_1 = \Sigma(\bar{A}).$$

If, in addition to the above property, whenever $T \in \mathscr{L}(\bar{A}, \bar{A})$ if follows that

$T : A \to A$ is bounded, then A is called an interpolation space with respect to the couple $\bar{A}$.

Let $\bar{B} = (B_0, B_1)$ be a fixed Banach couple and let B be a fixed intermediate space with respect to $\bar{B}$. Given any Banach couple $\bar{A} = (A_0, A_1)$, the space $H(A_0, A_1) = H[(B_0, B_1); B](A_0, A_1)$ is defined as the collection of all those elements $a \in \Sigma(\bar{A})$ such that $Ta \in B$ for all $T \in \mathscr{L}(\bar{A}, \bar{B})$. The norm in $H[(B_0, B_1); B](A_0, A_1)$ is given by

$$\|a\|_H = \sup\{\|Ta\|_B : \|T\|_{\bar{A},\bar{B}} \le 1\}.$$

In order to give some important examples of maximal methods denote by ℓ_q $(1 \le q \le \infty)$ and c_0 the usual spaces of doubly infinite scalar sequences and, given any positive sequence (ω_m), define $\ell_q(\omega_m)$ by

$$\ell_q(\omega_m) = \{\xi_m) : (\omega_m \xi_m) \in \ell_q\}.$$

We give a similar meaning to $c_0(\omega_m)$.

Example 1.1. Let $1 \le q \le \infty$ and $0 < \theta < 1$. If $\bar{B} = (\ell_\infty, \ell_\infty(2^{-m}))$ and $B = \ell_q(2^{-\theta m})$, then the interpolation method generated by this choice is

$$H[(\ell_\infty, \ell_\infty(2^{-m})); \ell_q(2^{-\theta m})](A_0, A_1) = (A_0, A_1)_{\theta,q}$$

the real interpolation method realized as a K-space (see [12]). Namely

$$(1)\quad (A_0, A_1)_{\theta,q} = \left\{ a \in \Sigma(\bar{A}) : \|a\|_{\theta,q} = \left(\sum_{m=-\infty}^{\infty} (2^{-\theta m} K(2^m, a))^{1/q} \right)^{1/q} < \infty \right\}$$

where

$$K(2^m, a) = \inf\{\|a_0\|_{A_0} + 2^m \|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\}.$$

Example 1.2. If $\bar{B} = (\ell_1, \ell_1(2^{-m}))$ and $B = \ell_1(2^{-\theta m})$, then we obtain

$$H[(\ell_1, \ell_1(2^{-m})); \ell_1(2^{-\theta m})](A_0, A_1) = H_1(A_0, A_1)$$

Ovchinnikov's ϕ_u-method (see [12] or [17]).

Note that the sequence spaces $X = \ell_q(\omega_m)$ which arise in Examples 1.1 and 1.2 satisfy the following three conditions:

a) Sequences having only a finite number of coordinates different from zero are contained in X.

b) $\|(\xi_m)\|_X = \sup_{n \ge 0} \|(\ldots, 0, 0, \xi_{-n}, \xi_{-n+1}, \ldots, \xi_{n-1}, \xi_n, 0, 0, \ldots)\|_X$.

c) If $|\eta_m| \le |\xi_m|$ for each $m \in \mathsf{Z}$ and $(\xi_m) \in X$, then $(\eta_m) \in X$ and $\|(\eta_m)\|_X \le \|(\xi_m)\|_X$.

All sequence spaces that we consider in the rest of this section are supposed to satisfy conditions a), b) and c).

Next we shall show that the behaviour of a maximal method on a couple of sequence spaces can be shifted to couples of vector valued sequence spaces.

Let $\bar{s} = (s_0, s_1)$ be any Banach couple of scalar sequence spaces over Z and let s be any intermediate sequence space with respect to $\bar{s}$. Given any sequence of Banach spaces (F_m) with $F_m \neq \{0\}$ for each $m \in \mathsf{Z}$, we put

$$s(F_m) = \{(a_m) : a_m \in F_m \text{ and } \|(a_m)\|_{s(F_m)} = \|(\|a_m\|_{F_m})\|_s < \infty\}$$

and we define $s_0(F_m)$ and $s_1(F_m)$ similarly. Assumptions a), b) and c) on scalar sequence spaces guaranteee that the vector valued sequence spaces are Banach spaces. It is also clear that

$$\bar{s}(F_m) = (s_0(F_m), s_1(F_m))$$

is a Banach couple.

We are now ready to establish the announced result.

Theorem 1.3. *Let* $H[(B_0, B_1); B](.,.)$ *be any maximal method and let* (s_0, s_1)*,* s *and* (F_m) *be as above. If* $H(s_0, s_1) = s$ *then*

$$H(s_0(F_m), s_1(F_m)) = s(F_m).$$

Proof. Take any $(a_m) \in H(s_0(F_m), s_1(F_m))$ and let $T \in \mathscr{L}((s_0, s_1), (B_0, B_1))$ with $\|T\|_{\bar{s},\bar{B}} \leq 1$. Using the Hahn-Banach theorem, for each $m \in \mathsf{Z}$ we can find $f_m \in F_m^*$ such that

$$\|f_m\|_{F_m^*} = 1 \quad \text{and} \quad f_m(a_m) = \|a_m\|_{F_m}.$$

Next consider the operator $R \in \mathscr{L}((s_0(F_m), s_1(F_m)), (B_0, B_1))$ defined by

$$R(x_m) = T(f_m(x_m)).$$

It is easy to see that

$$\|R\|_{\bar{s}(F_m),\bar{B}} \leq 1.$$

Hence, according to the definition of the maximal method, we have that

$$R(a_m) = T(\|a_m\|_{F_m}) \in B \quad \text{and}$$

$$\|T(\|a_m\|_{F_m})\|_B = \|R(a_m)\|_B \leq \|(a_m)\|_{H(s_0(F_m), s_1(F_m))}.$$

Since this holds for any $T \in \mathscr{L}((s_0, s_1), (B_0, B_1))$ with $\|T\|_{\bar{s},\bar{B}} \leq 1$, we conclude that $(\|a_m\|_{F_m}) \in H(s_0, s_1) = s$ with

$$\|(\|a_m\|_{F_m})\|_s = \|(a_m)\|_{s(F_m)} \leq \|(a_m)\|_{H(s_0(F_m),s_1(F_m))}.$$

This proves that

$$H(s_0(F_m), s_1(F_m)) \hookrightarrow s(F_m).$$

Conversely, let $(a_m) \in s(F_m)$. Given any $R \in \mathscr{L}((s_0(F_m), s_1(F_m)), (B_0, B_1))$ with $\|R\|_{\bar{s}(F_m),\bar{B}} \leq 1$ put

$$T(\xi_m) = R\left(\frac{\xi_m}{\|a_m\|_{F_m}} a_m\right).$$

(If for some $m \in \mathsf{Z}$ is $a_m = 0$, then we replace $\frac{1}{\|a_m\|_{F_m}} a_m$ by 0). The operator T belongs to $\mathscr{L}((s_0, s_1),(B_0, B_1))$ and $\|T\|_{\bar{s},\bar{B}} \leq 1$. It follows from

$$(\|a_m\|_{F_m}) \in s = H(s_0, s_1)$$

that

$$T(\|a_m\|_{F_m}) = R(a_m) \in B$$

and

$$\|R(a_m)\|_B \leq \|(\|a_m\|_{F_m})\|_s = \|(a_m)\|_{s(F_m)}.$$

Consequently

$$(a_m) \in H(s_0(F_m), s_1(F_m)) \text{ and } \|(a_m)\|_{H(s_0(F_m),s_1(F_m)} \leq \|(a_m)\|_{s(F_m)}.$$

The proof is complete.

By using Theorem 1.3 we establish now a stability property of maximal methods.

THEOREM 1.4. *Let $\bar{s} = (s_0, s_1)$ be a Banach couple of sequence spaces and let s be an interpolation space with respect to (s_0, s_1). Then, given any sequence of Banach spaces (F_m) with $F_m \neq \{0\}$ for each $m \in \mathsf{Z}$, the interpolation functor $H[(s_0, s_1); s](.,.)$ coincides with the maximal functor defined by the vector valued couple $(s_0(F_m), s_1(F_m))$ and the intermediate space $s(F_m)$.*

In other words, for any Banach couple (A_0, A_1), we have that

$$H[(s_0, s_1); s](A_0, A_1) = H[(s_0(F_m), s_1(F_m)); s(F_m)](A_0, A_1).$$

PROOF. Clearly

$$H[(s_0, s_1); s](s_0, s_1) = s$$

because s is an interpolation space with respect to (s_0, s_1). Applying Theorem 1.3 we obtain that

$$H[(s_0, s_1); s](s_0(F_m), s_1(F_m)) = s(F_m).$$

Since, by construction, $H[(s_0(F_m), s_1(F_m)); s(F_m)](.,.)$ is the biggest interpolation method $\mathscr{F}$ satisfying that

$$\mathscr{F}(s_0(F_m), s_1(F_m)) \hookrightarrow s(F_m),$$

we conclude that

$$H[(s_0, s_1); s](A_0, A_1) \hookrightarrow H[(s_0(F_m), s_1(F_m)); s(F_m)](A_0, A_1)$$

for any Banach couple (A_0, A_1).

Conversely, let $a \in H[(s_0(F_m), s_1(F_m)); s(F_m)](A_0, A_1)$ and take any $T \in \mathscr{L}(\bar{A}, \bar{s})$ with $\|T\|_{\bar{A},\bar{s}} \leq 1$. Then T can be written as $Ta = (T_m a)$ for some $T_m \in \mathscr{L}((A_0, A_1), (\mathsf{K}, \mathsf{K}))$. Here K stands for the scalar field. Choose any $u_m \in F_m, u_m \neq 0$, and let R be the vector valued operator defined by

$$Ra = \left(\frac{T_m a}{\|u_m\|_{F_m}} u_m \right).$$

The estimate

$$\|Ra\|_{s_j(F_m)} = \left\| \left(|T_m a| \frac{\|u_m\|_{F_m}}{\|u_m\|_{F_m}} \right) \right\|_{s_j} = \|Ta\|_{s_j} \leq \|a\|_{A_j}$$

yields that R belongs to $\mathscr{L}(\bar{A}, \bar{s}(F_m))$ with $\|R\|_{\bar{A},\bar{s}(F_m)} \leq 1$. It follows then that

$$Ra \in s(F_m) \text{ and } \|Ra\|_{s(F_m)} \leq \|a\|_{H[s_0(F_m), s_1(F_m)); s(F_m)](A_0, A_1).}$$

But

$$\|Ra\|_{s(F_m)} = \|(T_m a)\|_s = \|Ta\|_s.$$

Whence, since $T \in \mathscr{L}(\bar{A}, \bar{s})$ was chosen arbitrarily, we get that $a \in H[(s_0, s_1); s](A_0, A_1)$ with

$$\|a\|_{H[(s_0, s_1); s](A_0, A_1)} \leq \|a\|_{H[(s_0(F_m), s_1(F_m)); s(F_m)](A_0, A_1)}.$$

Next we discuss the case of minimal methods, a construction which is "dual" to the maximal methods.

Let again $\bar{B} = (B_0, B_1)$ be a fixed Banach couple and let B a fixed intermediate space with respect to $\bar{B}$. The Aronszajn-Gagliardo minimal functor [1] defined by these spaces associates to each Banach couple $\bar{A} = (A_0, A_1)$ the space $G(A_0, A_1) = G[(B_0, B_1); B](A_0, A_1)$ formed by all sums $\sum_{n=1}^{\infty} T_n b_n \in \Sigma(\bar{A})$

where $b_n \in B, T_n \in \mathscr{L}(\bar{B}, \bar{A})$ and $\sum_{n=1}^{\infty} \|T_n\|_{\bar{B},\bar{A}} \|b_n\|_B < \infty$. We provide $G[(B_0, B_1); B](A_0, A_1)$ with the norm

$$\|a\|_G = \inf\left\{ \sum_{n=1}^{\infty} \|T_n\|_{\bar{B},\bar{A}} \|b_n\|_B : a = \sum_{n=1}^{\infty} T_n b_n \right\}.$$

EXAMPLE 1.5. In (1) we introduced the real interpolation space $(A_0, A_1)_{\theta,q}$ by means of the K-functional. It can be equivalently defined by means of the J-functional

$$J(t, a) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}.$$

That is to say,

$$(2) \quad (A_0, A_1)_{\theta,q} = \left\{ a \in \Sigma(\bar{A}) : a = \sum_{m=-\infty}^{\infty} u_m \text{ (convergence in } \Sigma(\bar{A})) \text{ with} \right.$$
$$\left. (u_m) \subset \Delta(\bar{A}) \text{ and } \left(\sum_{m=-\infty}^{\infty} (2^{-\theta m} J(2^m, u_m))^q \right)^{1/q} < \infty \right\}$$

see [2], [19]). Moreover, the norm of $(A_0, A_1)_{\theta,q}$ is equivalent to

$$\|a\|_{\theta,q} = \inf\left\{ \left(\sum_{m=-\infty}^{\infty} (2^{-\theta m} J(2^m, u_m))^q \right)^{1/q} : a = \sum_{m=-\infty}^{\infty} u_m \right\}$$

(although we denote the two norms by the same letter, this will not cause any confusion).

Regarded as a J-space $(A_0, A_1)_{\theta,q}$, if $1 \le q \le \infty$ and $0 < \theta < 1$, coincides with

$$G[(\ell_1, \ell_1(2^{-m})); \ell_q(2^{-\theta m})](A_0, A_1) \quad \text{(see [12]).}$$

EXAMPLE 1.6. Take again $0 < \theta < 1$ and given any Banach couple $\bar{A} = (A_0, A_1)$ put

$$(3) \qquad \langle A_0, A_1 \rangle_\theta = \{ a \in \Sigma(\bar{A}) : a = \sum_{m=-\infty}^{\infty} w_m \text{ (convergence in } \Sigma(\bar{A}))$$

with $(w_m) \subset \Delta(\bar{A})$ satisfying that for every finite set $E \subset \mathsf{Z}$ and every scalar sequence $\xi = (\xi_m)$ with $\|\xi\|_\infty \le 1$ it holds

$$\left\| \sum_{m \in E} \xi_m 2^{(j-\theta)m} w_m \right\|_{A_j} \leq C \quad (j = 0, 1)$$

with C independent of E and $\xi\}$.

The norm of $\langle A_0, A_1 \rangle_\theta$ is given by

$$\|a\|_{\langle \theta \rangle} = \inf \left\{ C : a = \sum_{m=-\infty}^{\infty} w_m \right\}.$$

This interpolation method was introduced by Gustavsson and Peetre [11] and it admits the following minimal description

$$\langle A_0, A_1 \rangle_\theta = G[(c_0(2^{\theta m}), c_0(2^{(\theta - 1)m})); \ell_\infty](A_0, A_1)$$

(see [12]).

Again the sequence spaces in Examples 1.5 and 1.6 satisfy conditions a), b) and c). Next we show that minimal methods defined by sequence spaces fulfiling such conditions also enjoy the stability property with respect to vector valued sequences. We start with a result in the line of Theorem 1.3.

Theorem 1.7. *Let $G[(B_0, B_1); B](.,.)$ be any minimal method and let $\bar{s} = (s_0, s_1), s$ and (F_m) be as in Theorem 1.3. If $G(s_0, s_1) = s$ then*

$$G(s_0(F_m), s_1(F_m)) = s(F_m)$$

Proof. Let $(a_m) \in s(F_m)$. Since any $T \in \mathscr{L}(\bar{B}, \bar{s})$ is given by $Tb = (T_m b)$ for some $T_m \in \mathscr{L}((B_0, B_1), (\mathsf{K}, \mathsf{K}))$, we can define an operator $R \in \mathscr{L}((B_0, B_1), (s_0(F_m), s_1(F_m)))$ by the formula

$$Rb = \left(\frac{T_m b}{\|a_m\|_{F_m}} a_m \right).$$

Clearly $\|R\|_{\bar{B}, \bar{s}(F_m)} \leq 1$ if $\|T\|_{\bar{B}, \bar{s}} \leq 1$.

Take now any representation of $(\|a_m\|_{F_m})$ in $s = G(s_0, s_1)$, say $(\|a_m\|_{F_m}) = \sum_{j=1}^{\infty} T_j b_j$ with $\sum_{j=1}^{\infty} \|T_j\|_{\bar{B}, \bar{s}} \|b_j\|_B < \infty$. Then $(a_m) = \sum_{j=1}^{\infty} R_j b_j$ and

$$\sum_{j=1}^{\infty} \|R_j\|_{\bar{B}, \bar{s}(F_m)} \|b_j\|_B \leq \sum_{j=1}^{\infty} \|T_j\|_{\bar{B}, \bar{s}} \|b_j\|_B.$$

Hence

$$s(F_m) \hookrightarrow G(s_0(F_m), s_1(F_m)).$$

Conversely, any operator $R \in \mathscr{L}(\bar{B}, \bar{s}(F_m))$ can be written as $Rb = (R_m b)$ for some $R_m \in \mathscr{L}((B_0, B_1), (F_m, F_m))$. Then, for any functionals $f_m \in F_m^*$ with $\|f_m\|_{F_m^*} = 1$, the formula

$$Tx = (f_m(R_m x))$$

defines an operator from $\bar{B}$ into $\bar{s}$, and

$$\|T\|_{\bar{B}, \bar{s}} \leq \|R\|_{\bar{B}, \bar{s}(F_m)}.$$

Take now any $(a_m) \in G(s_0(F_m), s_1(F_m))$ and any representation $(a_m) = \sum_{j=1}^{\infty} R_j b_j$ with $\sum_{j=1}^{\infty} \|R_j\|_{\bar{B}, \bar{s}(F_m)} \|b_j\|_B < \infty$. Find $f_m \in F_m^*$ such that $f_m(a_m) = \|a_m\|_{F_m}$ and $\|f_m\|_{F_m^*} = 1$, and let $T_j \in \mathscr{L}(\bar{B}, \bar{s})$ be the operators obtained from the R_j's composing with functionals (f_m). Then

$$(\|a_m\|_{F_m}) = \sum_{j=1}^{\infty} T_j b_j$$

and

$$\sum_{j=1}^{\infty} \|T_j\|_{\bar{B}, \bar{s}} \|b_j\|_B \leq \sum_{j=1}^{\infty} \|R_j\|_{\bar{B}, \bar{s}(F_m)} \|b_j\|_B.$$

This shows that

$$G(s_0(F_m), s_1(F_m)) \hookrightarrow s(F_m)$$

and ends the proof.

We are now ready for the stability result in the minimal case.

THEOREM 1.8. *Let $\bar{s} = (s_0, s_1)$ be a Banach couple of sequence spaces and let s be an interpolation space with respect to $\bar{s}$. Then, given any sequence of Banach spaces (F_m) with $F_m \neq \{0\}$ for each $m \in \mathsf{Z}$ and given any Banach couple (A_0, A_1), we have*

$$G[(s_0, s_1); s](A_0, A_1) = G[(s_0(F_m), s_1(F_m)); s(F_m)](A_0, A_1).$$

PROOF. Using that s is an interpolation space with respect to $\bar{s}$, it is easy to see that

$$G[(s_0, s_1); s](s_0, s_1) = s.$$

It follows then from Theorem 1.7 that

$$G[(s_0, s_1); s](s_0(F_m), s_1(F_m)) = s(F_m).$$

On the other hand, the definition of $G[(s_0(F_m), s_1(F_m)); s(F_m)](.,.)$ yields that it is the smallest interpolation method $\mathscr{F}$ such that

$$s(F_m) \hookrightarrow \mathscr{F}(s_0(F_m), s_1(F_m)).$$

Whence

$$G[(s_0(F_m), s_1(F_m)); s(F_m)](A_0, A_1) \hookrightarrow G[(s_0, s_1); s](A_0, A_1)$$

for any Banach couple $\bar{A} = (A_0, A_1)$.

To check the reverse embedding, assume that $T \in \mathscr{L}(\bar{s}, \bar{A})$ and $\xi = (\xi_m) \in s$, and put $a = T\xi$. For each $m \in \mathsf{Z}$, choose $a_m \in F_m$ with $a_m \neq 0$. Using the Hahn-Banach theorem, we can find $f_m \in F_m^*$ with $\|f_m\|_{F_m^*} = 1$ and $f_m(a_m) = \|a_m\|_{F_m}$. Consider then the operator R defined by

$$R(x_m) = T\left(\frac{f_m(x_m)}{|\xi_m|}\xi_m\right).$$

It is not hard to see that $R \in \mathscr{L}(\bar{s}(F_m), \bar{A})$ with

$$\|R\|_{\bar{s}(F_m),\bar{A}} \leq \|T\|_{\bar{s},\bar{A}}.$$

Moreover, for $b = \left(\dfrac{|\xi_m|}{\|a_m\|_{F_m}} a_m\right) \in s(F_m)$, we have $Rb = T\xi = a$.

Take now any $a \in G[(s_0, s_1); s](A_0, A_1)$. Given any representation

$$a = \sum_{j=1}^{\infty} T_j\xi_j \quad \text{of } a \text{ with} \quad \sum_{j=1}^{\infty} \|T_j\|_{\bar{s},\bar{A}} \, \|\xi_j\|_s < \infty,$$

let $R_j \in \mathscr{L}(\bar{s}(F_m), \bar{A})$ be the operators associated to the T_j's and let $b_j \in s(F_m)$ be the vectors corresponding to the ξ_j's. We know that

$$R_j b_j = T_j \xi_j.$$

Hence

$$a = \sum_{j=1}^{\infty} R_j b_j$$

and since

$$\sum_{j=1}^{\infty} \|R_j\|_{\bar{s}(F_m),\bar{A}} \, \|b_j\|_{s(F_m)} \leq \sum_{j=1}^{\infty} \|T_j\|_{\bar{s},\bar{A}} \, \|\xi_j\|_s$$

we conclude that $a \in G[(s_0(F_m), s_1(F_m)); s(F_m)](A_0, A_1)$ with

$$\|a\|_{G[(s_0(F_m),s_1(F_m));s(F_m)](A_0,A_1)} \leq \|a\|_{G[(s_0,s_1);s](A_0,A_1)}.$$

The proof is complete.

2. Some remarks on the quasi-Banach case.

It is well-known that definitions (1) and (2) of the real interpolation space $(A_0, A_1)_{\theta,q}$ make sense also if (A_0, A_1) is only a couple of quasi-Banach spaces (see [2], [19]). The rank of the parameter q can then be enlarged, allowing any $0 < q \leq \infty$. The K- and the J-definitions are still equivalent, but the functional $\|\cdot\|_{\theta,q}$ (defined by the K- or by the J-construction) is no longer a norm, it is only a quasi-norm.

Next we study the possibility of a maximal description in the quasi-Banach case.

As a first idea, one might try to replace in Example 1.1 the spaces $[(\ell_\infty, \ell_\infty(2^{-m})); \ell_q(2^{-\theta m})]$ by some other quasi-Banach sequence spaces, say $[(w_0, w_1); w]$. But as soon as functionals

$$f_k : w_j \to \mathsf{K}$$

$$(\xi_m) \to \xi_k$$

are continuous on w_j ($j = 0, 1, k \in \mathsf{Z}$), then

$$\mathscr{L}((L_{p_0}, L_{p_1}), (w_0, w_1)) = \{0\} \tag{4}$$

where $0 < p_0, p_1 < 1$ and $L_{p_j} = L_{p_j}([0,1])$. Indeed, any $T \in \mathscr{L}((L_{p_0}, L_{p_1}), (w_0, w_1))$ can be written as $Tf = (T_m f)$ with $T_m \in L^*_{p_0} \cap L^*_{p_1}$. Since every continuous functional on L_{p_j} vanishes identically, equality (4) follows. Therefore

$$H[(w_0, w_1); w](L_{p_0}, L_{p_1}) = L_{p_0} + L_{p_1}$$

with

$$\|f\|_H = 0 \quad \text{for every} \quad f \in L_{p_0} + L_{p_1},$$

while

$$(L_{p_0}, L_{p_1})_{\theta,p} = L_p \quad \text{if} \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

We can remedy in part this situation by passing to vector valued sequence spaces. According to Theorem 1.4, in the category of Banach spaces it holds

$$(.,.)_{\theta,q} = H[(\ell_\infty(F_m), \ell_\infty(2^{-m}F_m)); \ell_q(2^{-\theta m}F_m)](.,.)$$

where (F_m) is any sequence of Banach spaces. Our plan for the quasi-Banach case is to choose F_m depending on the quasi-Banach couple (A_0, A_1) under consideration. The resulting maximal desciption is then going to be different for each couple. However this will be sufficient to establish the compactness theorem.

Definition 2.1. Given any quasi-Banach couple $\bar{A} = (A_0, A_1)$ we put

$$F_m = (A_0 + A_1, K(2^m, .)), \quad m \in \mathsf{Z},$$

and for $0 < \theta < 1, 0 < q \leq \infty$ we define $H_{\theta,q}(A_0, A_1)$ as the collection of all $a \in A_0 + A_1$ such that $Ta \in \ell_q(2^{-\theta m} F_m)$ for every $T \in \mathscr{L}((A_0, A_1), (\ell_\infty(F_m), \ell_\infty(2^{-m} F_m))$. We endow $H_{\theta,q}(A_0, A_1)$ with the quasi-norm

$$\|a\|_{H_{\theta,q}} = \sup\{\|Ta\|_{\ell_q(2^{-\theta m} F_m)} : \|T\|_{\bar{A}, \bar{\ell}_\infty(F_m)} \leq 1\}$$

Note that if c_j is the constant in the triangle inequality of A_j, then $c = \max\{c_0, c_1\}$ is a constant for the triangle inequality of F_m. Since c does not depend on m, it is not hard to check that vector valued sequence spaces involved in Definition 2.1 are quasi-Banach spaces.

Theorem 2.2. *Let* $\bar{A} = (A_0, A_1)$ *be a quasi-Banach couple, let* $0 < \theta < 1$ *and* $0 < q \leq \infty$. *Then*

$$(A_0, A_1)_{\theta,q} = H_{\theta,q}(A_0, A_1).$$

Proof. Let $T \in \mathscr{L}(\bar{A}, \bar{\ell}_\infty(F_m))$ with $\|T\|_{\bar{A}, \bar{\ell}_\infty(F_m)} \leq 1$. Then $Tx = (T_m x)$ for some $T_m \in \mathscr{L}((A_0, A_1), (F_m, F_m))$ such that

$$\|T_m\|_{A_0, F_m} \leq 1 \quad \text{and} \quad \|T_m\|_{A_1, 2^{-m} F_m} \leq 1.$$

Hence, given any $a \in (A_0, A_1)_{\theta,q}$ and any decomposition $a = a_0 + a_1$ with $a_j \in A_j$, we obtain

$$\|T_m a\|_{F_m} \leq c(\|T_m a_0\|_{F_m} + \|T_m a_1\|_{F_m}) \leq c(\|a_0\|_{F_m} + 2^m \|a_1\|_{F_m}).$$

It follows that

$$\|T_m a\|_{F_m} \leq cK(2^m, a) = c\|a\|_{F_m}$$

and therefore $Ta \in \ell_q(2^{-\theta m} F_m)$ with

$$\|Ta\|_{\ell_q(2^{-\theta m} F_m)} = \|(\|T_m a\|_{F_m})\|_{\ell_q(2^{-\theta m})} \leq c\|(K(2^m, a))\|_{\ell_q(2^{-\theta m})} = c\|a\|_{\theta,q}.$$

In other words,

$$(A_0, A_1)_{\theta,q} \hookrightarrow H_{\theta,q}(A_0, A_1).$$

Conversely, the operator

$$Ja = (..., a, a, a, ...)$$

belongs to $\mathscr{L}((A_0, A_1), (\ell_\infty(F_m), \ell_\infty(2^{-m}F_m)))$ and $\|J\|_{\bar{A},\bar{\ell}_\infty(F_m)} \leq 1$. Whence, given any $a \in H_{\theta,q}(A_0, A_1)$, we derive that $a \in (A_0, A_1)_{\theta,q}$ with

$$\|a\|_{\theta,q} = \|(K(2^m, a))\|_{\ell_q(2^{-\theta m})} = \|Ja\|_{\ell_q(2^{-\theta m}F_m} \leq \|a\|_{H\theta.q}.$$

The key of Theorem 2.2 is the characterization of $(A_0, A_1)_{\theta,q}$ by means of the K-functional. Using the equivalent definition with the J-functional, we shall establish next a minimal description for $(A_0, A_1)_{\theta,q}$ in terms of vector valued sequence spaces related to (A_0, A_1).

Recall that a quasi-norm $\|\cdot\|$ is said to be a p-norm $(0 < p \leq 1)$ if

$$\|a + b\|^p \leq \|a\|^p + \|b\|^p.$$

Given any quasi-normed space $(A, \|\cdot\|)$, the functional

$$|||a||| = \inf\left\{\left(\sum_{j=1}^{n} \|a_j\|^p\right)^{1/p} : a = \sum_{j=1}^{n} a_j,\ n \geq 1\right\}$$

defines a p-norm equivalent to $\|\cdot\|$. Here p is given by the equation $(2c)^p = 2$ where c is the constant in the triangle inequality of $\|\cdot\|$ (see, for example, [2], Lemma 3.10.1).

Note also that if $\|\cdot\|$ is a p-norm then it is also an r-norm for any $0 < r \leq p$.

Consequently, without loss of generality we may and do work with p-Banach spaces.

Given any couple (A_0, A_1) of p-Banach spaces, we put,

$$G_m = (A_0 \cap A_1, J(2^m, .)), \quad m \in \mathsf{Z}.$$

Clearly G_m is also a p-Banach space. It is not difficult to check that the vector valued sequence space $\ell_q(2^{-\theta m}G_m)$ is then a $\min(p, q)$-Banach space. Since the space $(A_0, A_1)_{\theta,q}$, realized as a J-space, is a quotient of $\ell_q(2^{-\theta m}G_m)$ it turns out that $(A_0, A_1)_{\theta,q}$ is a $\min(p, q)$-Banach space as well. This property will be useful for our latter computations.

Definition 2.3. Let $0 < \theta < 1$ and $0 < q \leq \infty$. Assume that $\bar{A} = (A_0, A_1)$ is a couple of p-Banach spaces $(0 < p \leq 1)$ and let G_m be as above. Put $r = \min(p, q)$ and define $G_\theta, q; r(A_0, A_1)$ as the collection of all those $a \in \Sigma(\bar{A})$ which can be represented as a convergent series $a = \sum_{j=1}^{\infty} T_j v_j$ in $\Sigma(\bar{A})$ with $v_j \in \ell_q(2^{-\theta m}G_m)$, $T_j \in \mathscr{L}((\ell_p(G_m),\ \ell_p(2^{-m}G_m)),$ $(A_0, A_1))$ and

$$\left(\sum_{j=1}^{\infty} \|T_j\|^r_{\bar{\ell}_p(G_m),\bar{A}} \|v_j\|^r_{\ell_q(2^{-\theta m}G_m)}\right)^{1/r} \leq \infty$$

This space becomes an r-Banach space endowed with the functional

$$\|a\|_{G_{\theta,q;r}} = \inf\left\{ \left(\sum_{j=1}^{\infty} \|T_j\|^r_{\bar{\ell}_p(G_m),\bar{A}} \, \|v_j\|^r_{\ell_q(2^{-\theta m}G_m)}\right)^{1/r} : a = \sum_{j=1}^{\infty} T_j v_j \right\}$$

Theorem 2.4. *Let $\bar{A} = (A_0, A_1)$ be a couple of p-Banach spaces, let $0 < \theta < 1$, $0 < q \leq \infty$ and write $r = \min(p, q)$. Then*

$$(A_0, A_1)_{\theta,q} = G_{\theta,q;r}(A_0, A_1).$$

Proof. Let Π be the operator defined by $\Pi(u_m) = \sum_{m=-\infty}^{\infty} u_m$. Since A_0 is a p-Banach space, we have

$$\|\Pi(u_m)\|_{A_0} \leq \left(\sum_{m=-\infty}^{\infty} \|u_m\|^p_{A_0}\right)^{1/p} \leq \left(\sum_{m=-\infty}^{\infty} J(2^m, u_m)^p\right)^{1/p} = \|(u_m)\|_{\ell_p(G_m)}.$$

Similarly

$$\|\Pi(u_m)\|_{A_1} \leq \|(u_m)\|_{\ell p(2^{-m}G_m)}.$$

Thus

$$\Pi \in \mathscr{L}((\ell_p(G_m), \ell_p(2^{-m}G_m)), (A_0, A_1)) \text{ and } \|\Pi\|_{\bar{\ell}_p(G_m),\bar{A}} \leq 1.$$

Given any $a \in (A_0, A_1)_{\theta,q}$ and any J-representation $a = \sum_{m=-\infty}^{\infty} u_m$ of a, it follows from

$$\Pi(u_m) = \sum_{m=-\infty}^{\infty} u_m = a \text{ and } \|\Pi\|_{\bar{\ell}_p(G_m),\bar{A}} \, \|(u_m)\|_{\ell_q(2^{-\theta m}G_m)}$$

$$\leq \left(\sum_{m=-\infty}^{\infty} (2^{-\theta m} J(2^m, u_m))^q\right)^{1/q}$$

that

$$a \in G_{\theta,q;r}(A_0, A_1) \text{ with } \|a\|_{G_{\theta,q;r}} \leq \|a\|_{\theta,q}.$$

Reciprocally, let $a = Tv$ with $v = (v_m) \in \ell_q(2^{-\theta m}G_m)$ and $T \in \mathscr{L}(\bar{\ell}_p(G_m), \bar{A})$. Write $\bar{v}_m$ for the vector valued sequence having all coordinates equal to zero except for the mth one that is v_m. It is clear that $T\bar{v}_m \in A_0 \cap A_1$ and

$$J(2^m, T\bar{v}_m) = \max\{\|T\bar{v}_m\|_{A_0}, 2^m\|T\bar{v}_m\|_{A_1}\}\| \leq \|T\|_{\bar{\ell}_p(G_m),\bar{a}}\|v_m\|_{G_m}.$$

Hence $a = \sum_{m=-\infty}^{\infty} T\bar{v}_m$ is a J-representation of a with

$$\begin{aligned}
\|a\|_{\theta,q} &\leq \left(\sum_{m=-\infty}^{\infty} (2^{-\theta m} J(2^m, T\bar{v}_m))^q\right)^{1/q} \\
&\leq \|T\|_{\bar{\ell}(G_m),\bar{A}} \left(\sum_{m=-\infty}^{\infty} (2^{-\theta m}\ \|v_m\|_{G_m})^q\right)^{1/q} \\
&= \|T\|_{\bar{\ell}_p(G_m),\bar{A}}\ \|v\|_{\ell_q(2^{-\theta m}G_m)}.
\end{aligned}$$

If a is now any element of $G_{\theta,q;r}(A_0, A_1)$ and $a = \sum_{j=1}^{\infty} T_j v_j$ is an arbitrary representation of a with

$$\left(\sum_{j=1}^{\infty} \|T_j\|^r_{\bar{\ell}_p(G_m),\bar{A}}\ \|v_j\|^r_{\ell_q(2^{-\theta m}G_m)}\right)^{1/r} < \infty,$$

then using that $(A_0, A_1)_{\theta,q}$ is r-normed and our previous estimate we derive that

$$\begin{aligned}
\|a\|_{\theta,q} &\leq \left(\sum_{j=1}^{\infty}\ \|T_j v_j\|^r_{\theta,q}\right)^{1/r} \\
&\leq \left(\sum_{j=1}^{\infty} \|T_j\|^r_{\bar{\ell}_p(G_m),\bar{A}}\ \|v_j\|^r_{\ell_q(2^{-\theta m}G_m)}\right)^{1/r}.
\end{aligned}$$

Consequently, $a \in (A_0, A_1)_{\theta,q}$ and $\|a\|_{\theta,q} \leq \|a\|_{G_{\theta q;r}}$.

This completes the proof.

Remark 2.5. In the Banach case definition of minimal interpolation method sets the fixed spaces $[(B_0, B_1); B]$ as domains of operators. Then the obstruction that we found at the beginning of this section does not arise now when extending definition of minimal methods to quasi-Banach couples. In fact, it is possible to give a minimal description for the real method in the category of quasi-Banach spaces. Namely, if we take all G_m equal to the scalar field K in Definition 2.3, then Theorem 2.4 remains true (see [15], Thm. 3.2).

Remark 2.6. Gustavsson-Peetre method defined in (3) also makes sense for quasi-Banach couples (see [10]). If (A_0, A_1) is a couple of p-Banach spaces, it is easy to see that $\langle A_0, A_1\rangle_\theta$ is also a p-Banach space. In that case $\langle A_0, A_1\rangle_\theta$ admits the following minimal description

$$\langle A_0, A_1\rangle_\theta = \left\{ a = \sum_{j=1}^{\infty} T_j v_j : T_j \in \mathscr{L}((c_0(2^{\theta m}), c_0(2^{(\theta-1)m})), (A_0, A_1)), \right.$$

$$\left. v_j \in \ell_\infty \text{ and } \left(\sum_{j=1}^{\infty} \|T_j\|^p_{\bar{c}_0, \bar{A}} \, \|v_j\|^p_{\ell_\infty} \right)^{1/p} < \infty \right\}.$$

We skip details because the main idea of the proof is the same as in the Banach case (see [12], Thm. 5).

3. The compactness theorem.

In this section we establish the compactness theorem in the quasi-Banach case.

Recall that a linear operator between quasi-Banach spaces is called compact if it transforms each bounded set into a set whose closure is compact or, equivalently, if it transforms each bounded sequence into a sequence having a convergent subsequence.

Our arguments are based on properties of operators J and Π introduced in Theorems 2.2 and 2.4.

Theorem 3.1. *Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be quasi-Banach couples and let $T : \bar{A} \to \bar{B}$ such that*

$$T : A_0 \to B_0 \quad \textit{compactly}.$$

Then for any $0 < \theta < 1$ and $0 < q \leq \infty$,

$$T : (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}$$

is compact.

Proof. Since any quasi-Banach space is p-Banach for some $0 < p \leq 1$, we may assume without loss of generality that the four spaces A_0, A_1, B_0, B_1 are p-Banach for some $0 < p \leq 1$. As we have seen in Theorem 2.4, the map $\Pi(u_m) = \sum_{m=-\infty}^{\infty} u_m$ is then bounded acting from $\ell_p(2^{-jm} G_m)$ into A_j for $j = 0, 1$. Here again

$$G_m = (A_0 \cap A_1, J(2^m, \cdot)).$$

Moreover, when we endow $(A_0, A_1)_{\theta,q}$ with the quasi-norm defined by the J-

functional, the map Π is surjective from $\ell_q(2^{-\theta m}G_m)$ into $(A_0, A_1)_{\theta,q}$ and it induces the quasi-norm of $(A_0, A_1)_{\theta,q}$. Thus

$$T : (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q} \text{ is compact}$$

if and only if

$$T\Pi : \ell_q(2^{-\theta m}G_m) \to (B_0, B_1)_{\theta,q} \text{ is compact.}$$

On the other hand, $J(a) = (\dots, a, a, a, \dots)$ is bounded acting from B_j into $\ell_\infty(2^{-jm}F_m)$ for $j = 0, 1$. We put as in Theorem 2.2

$$F_m = (B_0 + B_1, K(2^m, \cdot)).$$

If we consider now on $(B_0, B_1)_{\theta,q}$ the quasi-norm given by the K-functional, then

$$J : (B_0, B_1)_{\theta,q} \to \ell_q(2^{-\theta m}F_m)$$

is a metric injection. Hence a necessary and sufficient condition for

$$T\Pi : \ell_q(2^{-\theta m}G_m) \to (B_0, B_1)_{\theta,q}$$

to be compact is that

$$\hat{T} = JT\Pi : \ell_q(2^{-\theta m}G_m) \to \ell_q(2^{-\theta m}F_m)$$

is compact.

So, we have the following diagram of bounded operators

$$\begin{array}{l}
\ell_p(G_m) \xrightarrow{\Pi} A_0 \xrightarrow{T} B_0 \xrightarrow{J} \ell_\infty(F_m) \\
\ell_p(2^{-m}G_m) \xrightarrow{\Pi} A_1 \xrightarrow{T} B_1 \xrightarrow{J} \ell_\infty(2^{-m}F_m) \\
\hline
\ell_q(2^{-\theta m}G_m) \xrightarrow{\Pi} (A_0, A_1)_{\theta,q} \xrightarrow{T} (B_0, B_1)_{\theta,q} \xrightarrow{J} \ell_q(2^{-\theta m}F_m)
\end{array}$$

and our task is to show that $\hat{T} = JT\Pi$ is compact.

The advantage of working with $\hat{T}$ instead of T is that we can use certain families of projections on the couples of vector valued sequences. Indeed, write $\bar{\ell}_p(G) = (\ell_p(G_m), \ell_p(2^{-m}G_m))$ and for each positive integer $n \in \mathsf{N}$ consider mappings $P_n, Q_n^+, Q_n^- \in \mathscr{L}(\bar{\ell}_p(G_m), \ell_p(G_m))$ defined by

$P_n(u_m) = (\dots, 0, 0, u_{-n}, u_{-n+1}, \dots, u_{n-1}, u_n, 0, 0, \dots)$

$Q_n^+(u_m) = (\dots, 0, 0, u_{n+1}, u_{n+2}, \dots)$

$Q_n^-(u_m) = (\dots, u_{-n-2}, u_{-n-1}, 0, 0, \dots)$.

These mappings satisfy the following four conditions:

(I) They are uniformly bounded in $\bar{\ell}_p(G_m)$,

$$\sup_{n\in\mathsf{N}}\Big\{\|P_n\|_{\bar{\ell}_p(G_m),\bar{\ell}_p(G_m)},\ \|Q_n^+\|_{\bar{\ell}_p(G_m),\bar{\ell}_p(G_m)},\ (\|Q_n^-\|_{\bar{\ell}_p(G_m),\bar{\ell}_p(G_m)}\Big\}=1;$$

(II) If I stands for the identity operator in $\Sigma(\bar{\ell}_p(G_m))$, then

$$I=P_n+Q_n^++Q_n^- \quad \text{for each} \quad n\in\mathsf{N};$$

(III) For each $n\in\mathsf{N}$,

$$\|Q_n^+\|_{\ell_p(G_m),\ell_p(2^{-m}G_m)}=2^{-(n+1)}=\|Q_n^-\|_{\ell_p(2^{-m}G_m),\ell_p(G_m)}$$

and

$$\|P_n\|_{\Sigma(\bar{\ell}_p(G_m)),\Delta(\bar{\ell}_p(G_m))}\le c2^n;$$

(IV) If $(u_m)\in\ell_p(G_m)$ then

$$\|Q_n^-(u_m)\|_{\ell_p(G_m)}\to 0 \text{ as } n\to\infty,$$

while if $(u_m)\in\ell_p(2^{-m}G_m)$ then

$$\|Q_n^+(u_m)\|_{\ell_p(2^{-m}G_m)}\to 0 \text{ as } n\to\infty.$$

Properties (I), (II) and (III) are direct consequence of definitions of projections P_n, Q_n^+, Q_n^-. Property (IV) follows from (II), (I) and the fact that sequences having only a finite number of coordinates different from 0 are dense in $\ell_p(G_m)$ (resp. in $\ell_p(2^{-m}G_m)$).

The same families of mappings can be defined on the couple $\bar{\ell}_\infty(F_m)=(\ell_\infty(F_m),\ell_\infty(2^{-m}F_m))$. Call them R_n, S_n^+, S_n^- . They satisfy the corresponding versions of (I), (II) and (III). Furthermore, the argument pointed out for proving (IV) yields that

(IV') If (v_m) belongs to the closure of $\Delta(\bar{\ell}_\infty(F_m))$ in $\ell_\infty(F_m)$ then

$$\|S_n^-(v_m)\|_{\ell_\infty(F_m})\to 0 \text{ as } n\to\infty.$$

On the other hand,

$$\|S_n^+(v_m)\|_{\ell_\infty(2^{-m}F_m)}\to 0 \text{ as } n\to\infty$$

provided that (v_m) belongs to the closure of $\Delta(\bar{\ell}_\infty(F_m))$ in $\ell_\infty(2^{-m}F_m)$.

We shall also need the following interpolation results

$$(\ell_p(G_m),\ell_p(2^{-m}G_m))_{\theta,q}=\ell_q(2^{-\theta m}G_m) \tag{5}$$

$$(\ell_\infty(F_m),\ell_\infty(2^{-m}F_m))_{\theta,q}=\ell_q(2^{-\theta m}F_m). \tag{6}$$

We only prove the embedding

$$\ell_q(2^{-\theta m}G_m) \hookrightarrow (\ell_p(G_m), \ell_p(2^{-m}G_m))_{\theta,q} \tag{7}$$

because the rest of the proof of (5) and (6) can be carried out by doing minor modifications in the arguments of [5], Lemma 2.1.

Take any $u = (u_m) \in \ell_q(2^{-\theta m}G_m)$ and let w_m be the sequence having all coordinates equal to zero except for the mth one which is u_m. Since

$$J(2^m, w_m) = \max\{\|w_m\|_{\ell_p(G_m)}, 2^m\|w_m\|_{\ell_p(2^{-m}G_m)}\} = \|u_m\|_{G_m},$$

we see that

$$u = \sum_{m=-\infty}^{\infty} w_m \quad (\text{convergence in } \Sigma(\bar{\ell}_p(G_m)))$$

with $w_m \in \Delta(\bar{\ell}_p(G_m))$ and

$$\|u\|_{(\ell_p(G_m), \ell_p(2^{-m}(G_m))_{\theta,q}} \le \left(\sum_{m=-\infty}^{\infty} (2^{-\theta m}J(2^m, w_m))^q\right)^{1/q}$$

$$\le \left(\sum_{m=-\infty}^{\infty} (2^{-\theta m}\|u_m\|_{G_m})^q\right)^{1/q} = \|u\|_{\ell_q(2^{-\theta m}G_m)}.$$

This establishes (7).

We are now in a position to use the approach developed in [7], Thm. 1.3, to show the compactness of $\hat{T}$.

Using (II) we have

$$\begin{aligned}\hat{T} = \hat{T}(P_n + Q_n^+ + Q_n^-) &= \hat{T}P_n + \hat{T}Q_n^+ + (R_n + S_n^+ + S_n^-)\hat{T}Q_n^- \\ &= \hat{T}P_n + R_n\hat{T}Q_n^- + S_n^-\hat{T}Q_n^- + S_n^+\hat{T}Q_n^- + \hat{T}Q_n^+.\end{aligned}$$

Our plan is to check that $\hat{T}P_n$ and $R_n\hat{T}Q_n^-$ are compact and then that the remaining three operators have norms converging to 0 as $n \to \infty$.

For $\hat{T}P_n$ we have the following diagram

$$\begin{tikzcd}
& \ell_p(G_m) \arrow[r, "\hat{T}"] & \ell_\infty(F_m) \\
\ell_q(2^{-\theta m}G_m) \arrow[ur, "P_n"] \arrow[dr, "P_n"'] & & \\
& \ell_p(2^{-m}G_m) \arrow[r, "\hat{T}"] & \ell_\infty(2^{-m}F_m)
\end{tikzcd}$$

where $\hat{T}P_n : \ell_q(2^{-\theta m}G_m) \to \ell_\infty(F_m)$ is compact. Therefore we can apply Lions- Peetre compactness theorem (see [2], Thm. 3.8.1) to derive that

$$\hat{T}P_n : \ell_q(2^{-\theta m}G_m) \to (\ell_\infty(F_m), \ell_\infty(2^{-m}F_m))_{\theta,q} = \ell_q(2^{-\theta m}F_m)$$

is compact. Here we have used (6) to identify the interpolation space.

Note that the Lions-Peetre theorem as stated in [2] refers to Banach couples, but it is not difficult to adapt the proof for the present quasi- Banach case.

For the operator $R_n\hat{T}Q_n^-$ the relevant diagram is

with $R_n\hat{T}Q_n^- : \ell_p(G_m) \to \ell_q(2^{-\theta m}F_m)$ compactly. Whence, another application of the Lions-Peetre theorem and formula (5) yield that

$$R_n\hat{T}Q_n^- : \ell_q(2^{-\theta m}G_m) \to \ell_q(2^{-\theta m}F_m)$$

is also compact.

We pass now to show that the norm of

$$S_n^-\hat{T}Q_n^- : \ell_q(2^{-\theta m}G_m) \to \ell_q(2^{-\theta m}F_m)$$

tends to 0 as $n \to \infty$.

It follows from formulae (5) and (6), and estimate for the norm of an interpolated operator by the real method that

$$\begin{aligned}
&\|S_n^-\hat{T}Q_n^-\|_{\ell_q(2^{-\theta m}G_m),\ell_q(2^{-\theta m}F_m)} \\
&\le C\|S_n^-\hat{T}Q_n^-\|^{1-\theta}_{\ell_q(G_m),\ell_\infty(F_m)}\ \|S_n^-\hat{T}Q_n^-\|^{\theta}_{\ell_q(2^{-m}G_m),\ell_\infty(2^{-m}F_m)} \\
&\le C\|S_n^-\hat{T}Q_n^-\|^{1-\theta}_{\ell_q(G_m),\ell_\infty(F_m)}\ \|\hat{T}\|^{\theta}_{\ell_q(2^{-m}G_m),\ell_\infty(2^{-m}F_m)}
\end{aligned}$$

where we have used (I) to get the last inequality. So, it suffices to check that

$$\|S_n^-\hat{T}Q_n^-\|_{\ell_p(G_m),\ell_\infty(F_m)} \to 0 \text{ as } n \to \infty. \tag{8}$$

Take any $\epsilon > 0$ and let U be the closed unit ball of $\ell_p(G_m)$. Since $\hat{T} : \ell_p(G_m) \to \ell_\infty(F_m)$ is compact, we can find vectors $w_1, ..., w_r$ in $\ell_p(G_m)$ having only a finite number of coordinates different from zero and such that for any $u \in U$

$$\min_{1\le j\le r}\{\|\hat{T}u - \hat{T}w_j\|_{\ell_\infty(F_m)}\} \le (\epsilon^p/2)^{1/p}.$$

Vectors $\hat{T}w_1, ..., \hat{T}w_r$ belong to $\Delta(\bar{\ell}_\infty(F_m))$. According to (IV'), there exists $N \in \mathsf{N}$ such that for any $n \ge N$

$$\|S_n^-\hat{T}w_j\|_{\ell_\infty(F_m)} \le (\epsilon^p/2)^{1/p} \text{ for } j = 1, ..., r.$$

Take next $v \in U$ and $n \geq N$. Using that $\ell_\infty(F_m)$ is p-Banach, we get that

$$\|S_n^- \hat{T} Q_n^- v\|^p_{\ell_\infty(F_m)} \leq \|S_n^- \hat{T} Q_n^- v - S_n^- \hat{T} w_j\|^p_{\ell_\infty(F_m)} + \|S_n^- \hat{T} w_j\|^p_{\ell_\infty(F_m)}$$

$$\leq \|\hat{T} u - \hat{T} w_j\|^p_{\ell_\infty(F_m)} + \epsilon^p/2.$$

Here $u = Q_n^- v$ which belongs also to U by (I). The choice of j so that

$$\|\hat{T} u - \hat{T} w_j\|^p_{\ell_\infty(F_m)} \leq \epsilon^p/2$$

now implies that

$$\|S_n^- \hat{T} Q_n^- v\|_{\ell_\infty(F_m)} \leq \epsilon \text{ for any } n \geq N$$

and establishes (8).

It remains to check that the norms of $S_n^+ \hat{T} Q_n^-$ and $\hat{T} Q_n^+$ converge to 0 as $n \to \infty$. For this we can use the same argument as in [7], Thm. 1.3.

Factorization

$$\begin{tikzcd}
\ell_p(2^{-m}G_m) \arrow[rrr, "S_n^+ \hat{T} Q_n^-"] \arrow[dr, "Q_n^-"'] & & & \ell_\infty(2^{-m}F_m) \\
& \ell_p(G_m) \arrow[r, "\hat{T}"] & \ell_\infty(F_m) \arrow[ur, "S_n^+"'] &
\end{tikzcd}$$

gives that

$$\|S_n^+ \hat{T} Q_n^-\|_{\ell_p(2^{-m}G_m), \ell_\infty(2^{-m}F_m)} \leq 2^{-n} \, \|\hat{T}\|_{\ell_p(G_m), \ell_\infty(F_m)} 2^{-n} \to 0 \text{ as } n \to \infty.$$

Therefore

$$\lim_{n\to\infty} \|S_n^+ \hat{T} Q_n^-\|_{\ell_q(2^{-\theta m}G_m), \ell_q(2^{-\theta m}F_m)} = 0.$$

For the remaining operator $\hat{T} Q_n^+$ we have that

(9) $$\lim_{n\to\infty} \|\hat{T} Q_n^+\|_{\ell_p(G_m), \ell_\infty(F_m)} = 0.$$

Let us establish this fact proceeding by contradiction. If (9) does not hold then there is $\lambda > 0$, a subsequence (n_1) and vectors u_{n_1} in the closed unit ball of $\ell_p(G_m)$ such that

$$\lim_{n_1\to\infty} \|\hat{T} Q_{n_1}^+ u_{n_1}\|_{\ell_\infty(F_m)} = \lambda.$$

According to (I), the sequence $(Q_{n_1}^+ u_{n_1})$ is bounded in $\ell_p(G_m)$. Hence, compactness of $\hat{T} : \ell_p(G_m) \to \ell_\infty(F_m)$ implies, passing to another subsequence if necessary, that $(\hat{T} Q_{n_2}^+ u_{n_2})$ converges to some w in $\ell_\infty(F_m)$. So $\|w\|_{\ell_\infty(F_m)} = \lambda > 0$. On the other hand, $(\hat{T} Q_{n_2}^+ u_{n_2})$ is a null sequence in $\ell_\infty(2^{-m} F_m)$ because, due to (III), the sequence $(Q_{n_2}^+ u_{n_2})$ converges to zero in $\ell_p(2^{-m} G_m)$. By compatibility we conclude that $w = 0$ contradicting $w \neq 0$.

The fact that

$$\|\hat{T} Q_n^+\|_{\ell_q(2^{-\theta m} G_m), \ell_q(2^{-\theta m} F_m)} \to 0 \text{ as } n \to \infty$$

follows now from (9).

The proof is complete.

Since Lorentz function spaces $L_{p,q}$ arise by real interpolation between L_p spaces, namely (see [2], [3] or [19])

$$(L_{p_0}, L_{p_1})_{\theta,q} = L_{p,q} \text{ for } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \text{ and } p_0 \neq p_1,$$

as a direct consequence of Theorem 3.1 we have the following complement of the result by Zabreiko and Pustylnik [20].

Corollary 3.2. *Let (U, μ) and (V, ν) be σ-finite measure spaces, with μ, ν being positive measures. Assume that $1 \leq p_0 \neq p_1 \leq \infty$, $0 < q_0 \neq q_1 \leq \infty$ and let T be a linear operator such that*

$$T : L_{p_0}(U, d\mu) \to L_{q_0}(V, d\nu) \textit{ compactly}$$

and

$$T : L_{p_1}(U, d\mu) \to L_{q_1}(V, d\nu) \textit{ boundedly.}$$

If $0 < \theta < 1$, $0 < r \leq \infty$, $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$, then

$$T : L_{p,r}(U, d\mu) \to L_{q,r}(V, d\nu)$$

is also compact.

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REFERENCES

1. N. Aronszajn and E. Gagliardo, *Interpolation spaces and interpolation methods*, Ann. Mat. Pura Appl. 68, (1965), 51–118.
2. J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

3. Yu. A. Brudnyi and N. Ya. Kruglyak, *Interpolation Functors and Interpolation Spaces*, Vol. 1, North-Holland Amsterdam, 1991.
4. F. Cobos, *On the optimality of compactness results for interpolation methods associated to polygons*, Indag. Math., N.S., 5 (1994), 397–402.
5. F. Cobos, D. E. Edmunds and A. J. B. Potter, *Real interpolation and compact linear operators*, J. Funct. Anal. 88 (1990), 351–365.
6. F. Cobos and D. L. Fernândez, *On interpolation of compact operators*, Ark. Mat. 27 (1989), 211–217.
7. F. Cobos, T. Kühn and T. Schonbek, *One-sided compactness results for Aronszajn-Gagliardo functors*, J. Funct. Anal. 106 (1992), 274–313.
8. F. Cobos and J. Peetre, *Interpolation of compactness using Aronszajn- Gagliardo functors*, Israel J. Math. 68 (1989), 220–240.
9. M. Cwikel and N. J. Kalton, *Interpolation of compact operators by the methods of Calderón and Gustavsson-Peetre*, Proc. Edinburgh Math. Soc. 38 (1995), 261–276.
10. J. Gustavsson, *On interpolation of weighted L^p-spaces and Ovchinnikov's theorem*, Studia Math. 72 (1982), 237–251.
11. J. Gustavsson and J. Peetre, *Interpolation of Orlicz spaces*, Studia Math. 60 (1977), 33–59.
12. S. Janson, *Minimal and maximal methods of interpolation*, J. Funct. Anal. 44 (1981) 50-73.
13. M. A. Krasnoselskii, *On a theorem of M. Riesz*, Soviet Math. Dokl. 1 (1960), 229–231.
14. M. A. Krasnoselskii, P. P. Zabreiko, E. I. Pustylnik and P. E. Sbolevskii, *Integral operators in spaces of summable functions*, Noordhoff, Leyden, 1976.
15. M. Mastylo, *On interpolation of some quasi-Banach spaces*, J. Math. Anal. Appl. 147 (1990), 403–419.
16. M. Mastylo, *On interpolation of compact operators*, preprint.
17. V.I. Ovchinnikov, *The method of orbits in interpolation theory*, in Mathematical Reports, Vol. 1, Part 2 (Harwood Academic Publishers 1984), 349–516.
18. J. Peetre, *Sur l'utilisation des suites inconditionellement sommables dans la théorie des espaces d'interpolation*, Rend. Sem. Mat. Univ. Padova 46 (1971), 173–190.
19. H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.
20 P. P. Zabreiko, and E. I. Pustylnik, *On interpolation properties of compactness*, Functional analysis and theory of functions 2, Ucen. Zap. Kazan Univ. 124 (1965), 114–118.

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