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THE DIOPHANTINE SYSTEM $x^2 - 6y^2 = -5, x = 2z^2 - 1$

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1. Introduction.

In a recent paper [5] showing that the system of the title has only the solutions in integers given by $\pm z = 0, 1, 2, 3, 6$ and 91, the introduction mentioned in passing that the elementary algebraic methods developed for example in [2] do not appear sufficient to solve this problem. The authors then proceeded to prove their result using the high powered analytical methods for which they, Tzanakis, de Weger, Steiner and others have become renowned. We present below a short simple proof of their result.

It is clearly sufficient to assume that $x \geq -1, y \geq 1$ and $z \geq 0$ we shall do so without further mention. The equation $v^2 - 6u^2 = 1$ has fundamental solution $\alpha = 5 + 2\sqrt{6}$, and then it is easily shown that the general solution of $x^2 - 6y^2 = -5$ is given by the two classes $x + y\sqrt{6} = (\pm 1 + \sqrt{6})\alpha^n$ with $n \geq 0$. Let $\beta = 5 - 2\sqrt{6}$ and define the sequences $u_n = (\alpha^n - \beta^n)/(\alpha - \beta)$, $v_n = (\alpha^n + \beta^n)/2$. Then $x = \pm v_n + 12u_n$, and so our problem reduces to proving that the only solutions of the equation

$$2z^2 = 1 + v_n + 12u_n \tag{1}$$

are given by $n = 0, 1$ and 4 of the equation

$$2z^2 = 1 - v_n + 12u_n \tag{2}$$

are given by $n = 0, 1$ and 2.

Here both u_n and v_n satisfy the recurrence relation $w_{n+2} = 10w_{n+1} - w_n$ and the first few values are given in the following table:

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n	u_n	v_n	$v_n + 12u_n$	$-v_n + 12u_n$
0	0	1	1	-1
1	1	5	17	7
2	10	49	169	71
3	99	485	1673	703
4	980	4801	16561	6959

We shall quote several standard identities involving the sequences u_n and v_n and results concerning the periodicity of these sequences modulo certain moduli which are easily verified, without writing them all out in detail.

2. Preliminaries.

LEMMA 1. *The equation* $3x^4 - 2y^2 = 1$ *has only the solutions in non-negative integers given by* $x = 1, 3$.

This is shown in [1].

LEMMA 2. *The equation* $6y^2 = x^4 - 1$ *has only the solutions in non-negative integers given by* $x = 1, 7$.

This is shown in [3].

LEMMA 3. *The equation* $y^2 = 24x^4 + 1$ *has only the solutions in non-negative integers given by* $x = 0, 1$.

For, $(y+1)(y-1) = 24x^4$ and here $(y+1, y-1) = 2$. Thus with $x = x_1x_2$ *either* $y \pm 1 = 2x_1^4, y \mp 1 = 12x_2^4$ in which case $\pm 1 = x_1^4 - 6x_2^4$; here the lower sign is impossible modulo 3, whereas the upper one gives only $x_1 = 1, x_2 = 0$ by Lemma 2;
or $y \pm 1 = 6x_1^4, y \mp 1 = 4x_2^4$ and now $\pm 1 = 3x_1^4 - 2x_2^4$. Again the lower sign is impossible modulo 3, and the upper sign gives only $x_1 = x_2 = 1$ in view of Lemma 2.

LEMMA 4. *The equation* $3y^2 = 2x^4 + 1$ *has only the solution in non-negative integers given by* $x = 1$.

This is proved in [4], and is the only result that we use which has not been proved by technically elementary methods.

3. Proof of result.

There are four cases.

(A) The only solutions of (1) with n even are $n = 0$ and 4.

For with $n = 2k, 2z^2 = 1 + v_{2k} + 12u_{2k} = 2v_k(v_k + 12u_k)$, and so since $(v_k, v_k + 12u_k) = 1$ we must have $v_k = z_1^2$. Then $1 = v_k^2 - 24u_k^2 = z_1^4 - 6(2u_k)^2$, and so by Lemma 2, $z_1 = 1$ or 7 whence $k = 0$ or 2 as required.

(B) The only solution of (1) with n odd is $n = 1$.

For with $n = 2k + 1$,

$$\begin{aligned} 2z^2 &= 1 + v_{2k+1} + 12u_{2k+1} \\ &= 1 + (5v_{2k} + 24u_{2k}) + 12(v_{2k} + 5u_{2k}) \\ &= (v_k^2 - 24u_k^2) + 17(v_k^2 + 24u_k^2) + 168u_k v_k \\ &= 18v_k^2 + 168u_k v_k + 384u_k^2 \\ &= 6(v_k + 4u_k)(3v_k + 16u_k). \end{aligned}$$

Here the final two factors on the right have no common factor, and since $3v_k + 16u_k \equiv 3 \pmod 4$, we must have $v_k + 4u_k = z_1^2, 3v_k + 16u_k = 3z_2^2$. Thus $4u_k = 3z_2^2 - 3z_1^2, v_k = 4z_1^2 - 3z_2^2$, and so

$$\begin{aligned} 1 &= v_k^2 - 24u_k^2 = (4z_1^2 - 3z_2^2)^2 - \tfrac{27}{2}(z_1^2 - z_2^2)^2, \text{ whence} \\ 2 &= 5z_1^4 + 6z_1^2 z_2^2 - 9z_2^4 = 6z_1^4 - (z_1^2 - 3z_2^2)^2, \text{ or} \\ 1 &= 3z_1^4 - 2(\tfrac{1}{2}(z_1^2 - 3z_2^2))^2, \end{aligned}$$

and so by Lemma 1, the only possibilities are $z_1 = 1$ or 3. The former gives $k = 0$ whence $n = 1$ and the latter no solution.

(C) The only solutions of (2) with n even are $n = 0$ and 2.

For with $n = 2k, 2z^2 = 1 - v_{2k} + 12u_{2k} = 24u_k(-2u_k + v_k)$, and so since $v_k - 2u_k$ is odd and has no factor in common with u_k we must have

either $u_k = z_1^2, v_k - 2u_k = 3z_2^2$; in this case $1 = v_k^2 - 24u_k^2 = v_k^2 - 24z_1^4$ implies $z_1 = 0$ or 1 in view of Lemma 3; the latter yields $n = 2$ whilst the former gives no solution;

or $u_k = 3z_1^2, v_k - 2u_k = z_2^2$. Since $u_k \equiv 0 \pmod 3$, it follows that $3|k$, and with $k = 3m$ we find that $z_1^2 = \frac{1}{3}u_{3m} = u_m(32u_m^2 + 1)$. This implies that u_m is a square, and just as above this is possible only u_m if equals 0 or 1. The former gives $n = 0$ and the latter no solution.

(D) The only solution of (2) with n odd is $n = 1$.

For with $n = 2k + 1$,

$$\begin{aligned}2z^2 &= 1 - v_{2k+1} + 12u_{2k+1}\\ &= 1 - (5v_{2k} + 24u_{2k}) + 12(v_{2k} + 5u_{2k})\\ &= (v_k^2 - 24u_k^2) + 7(v_k^2 + 24u_k^2) + 72u_kv_k\\ &= 8(v_k + 3u_k)(v_k + 6u_k)\end{aligned}$$

and this is possible only if $z_1^2 = v_k + 6u_k$. But then

$$\begin{aligned}2z_1^4 + 1 &= 2(v_k + 6u_k)^2 + 1\\ &= 2v_k^2 + 24u_kv_k + 72u_k^2 + v_k^2 - 24u_k^2\\ &= 3(v_k + 4u_k)^2\end{aligned}$$

and by Lemma 4, this is possible only for $k = 0$.

This concludes the proof.

NOTE ADDED IN PROOF. By an extraordinary coincidence this same problem was considered by R. J. Stroeker & B. M. M. de Weger in "On a quartic Diophantine Equation", Proc. Edinburgh Math. Soc. 39 (1996), 97–114. Their method is as long and similar in conception to that of [5] although the details are totally different.

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