MATH. SCAND. 82 (1998), 237–255

K-THEORY FOR C^*-ALGEBRAS ASSOCIATED WITH SUBSHIFTS

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Abstract.

We present K-theory formula for C^*-algebras associated with subshifts. The formula is a generalization of K-theory formula for Cuntz-Krieger algebras, which are associated with topological Markov shifts. The dimension group for a general subshift is introduced to be the dimension group for the associated AF-algebra.

1. Introduction.

In [Ma], the author has introduced and studied a class of C^*-algebras associated with subshifts in the theory of symbolic dynamics. The class of C^*-algebras is a generalized one of the Cuntz-Krieger algebras which are associated with topological Markov shifts. Each of the C^*-algebras associated with subshifts has generators of partial isometries with mutually orthogonal ranges. It also has universal properties subject to some operator relations ([Ma; Theorem 4.9 and 5.2]) so that it becomes purely infinite and simple in many cases including Cuntz-Krieger algebras. It is an analogy to the Cuntz-Krieger algebras that AF-subalgebras are appeared inside of the C^*-algebras as the algebras of all fixed points of certain one-parameter group actions, called gauge actions. However, these AF-subalgebras have more complicated structure than the AF-subalgebras appeared inside of the Cuntz-Krieger algebras.

For a subshift (Λ, σ), we denote by $\mathcal{O}_\Lambda$ and $\mathcal{F}_\Lambda^\infty$ the C^*-algebra associated with the subshift (Λ, σ) and the corresponding AF-subalgebra inside of it respectively. If a subshift is a topological Markov shift, then the K_0-group of the AF-subalgebra, as an ordered group, becomes the dimension group for the topological Markov shift considered in [Kr1] and [Kr2]. Hence for a general subshift, it seems to be natural to define "the dimension group" for a

Received Janury 8, 1996.

subshift (Λ, σ) as the K_0-group $K_0(\mathscr{F}_\Lambda^\infty)$ of the AF-algebra $\mathscr{F}_\Lambda^\infty$ as an ordered group.

In this paper, we present K-theory formula of these C^*-algebras $\mathcal{O}_\Lambda$ and $\mathscr{F}_\Lambda^\infty$ (Theorem 3.11 and Theorem 4.9). We first compute the K_0-group $K_0(\mathscr{F}_\Lambda^\infty)$ of the AF-algebra $\mathscr{F}_\Lambda^\infty$ inside of it and show that the K_0-group is realized as an inductive limit of a sequence of the K_0-groups of the finite dimensional and commutative C^*-algebras generated by support projections of canonical generators of partial isometries (Theorem 3.11). We will next show that the AF-algebra $\mathscr{F}_\Lambda^\infty$ is stably isomorphic to the crossed product of the C^*-algebra $\mathcal{O}_\Lambda$ by the gauge action. Hence, $\mathcal{O}_\Lambda$ is stably isomorphic to the crossed product of the tensor product C^*-algebra of $\mathscr{F}_\Lambda^\infty$ and the C^*-algebra of all compact operators on a Hilbert space by an action of Z. Thus it becomes to be possible to compute K-groups for the C^*-algebra $\mathcal{O}_\Lambda$ by using the Pimsner-Voiculescu six-term exact sequence for K-theory. The resulting K-group formula (Theorem 4.9) includes the K-group formula of the Cuntz-Krieger algebras ([C2]).

We will finally compute the K-group for the C^*-algebra associated with a certain sofic subshift but not conjugate to a topological Markov shift. Computation of K-groups for C^*-algebras associated with other concrete subshifts will appear in some papers (cf. [KMW]).

We remark that the C^*-algebras associated with subshifts are nuclear purely infinite simple and satisfy the Universal Coefficient Theorem in many cases. Hence, by recent results of Kirchberg and Phillips in [Ki] and [Ph], they can be completely classified by their own K-theory (Corollary 4.11).

The author would like to thank the referee for some suggestions.

After submitting the first draft of this paper, the author was informed of preprints [KPRR] and [PR] by Kumjian-Pask-Raeburn-Renault and Pask-Raeburn. They study generalization of Cuntz-Krieger algebras from graph theoretic view point, but our generalization of Cuntz-Krieger algebras are different from theirs.

2. Review of the C^*-algebras associated with subshifts.

We will review the construction of the C^*-algebras associated with subshifts along [Ma].

In the throughout this paper, a finite set $\Sigma = \{1, 2, ..., n\}$ is fixed.

Let Σ^{Z}, Σ^{N} be the infinite product spaces $\prod_{i=-\infty}^{\infty} \Sigma_i$, $\prod_{i=1}^{\infty} \Sigma_i$ where $\Sigma_i = \Sigma$, endowed with the product topology respectively. The transformation σ on $\Sigma^{\mathsf{Z}}, \Sigma^{\mathsf{N}}$ given by $(\sigma(x))_i = x_{i+1}, i \in \mathsf{Z}, \mathsf{N}$ is called the (full) shift. Let Λ be a shift invariant closed subset of $\Sigma^{\mathbb{Z}}$ i.e. $\sigma(\Lambda) = \Lambda$. The topological dynamical system $(\Lambda, \sigma|_\Lambda)$ is called a subshift. We denote $\sigma|_\Lambda$ by σ for simplicity. This class of the subshifts includes the class of the topological Markov shifts (cf. [DGS]).

A finite sequence $\mu = (\mu_1, ..., \mu_k)$ of elements $\mu_j \in \Sigma$ is called a block or a word. We denote by $|\mu|$ the length k of μ. A block $\mu = (\mu_1, ..., \mu_k)$ is said to occur in $x = (x_i) \in \Sigma^{\mathbb{Z}}$ if $x_m = \mu_1, ..., x_{m+k-1} = \mu_k$ for some $m \in \mathbb{Z}$.

For a subshift (Λ, σ), set for $k \in \mathbb{N}$

$$\Lambda^k = \{\mu : \text{a block with length } k \text{ in } \Sigma^{\mathbb{Z}} \text{ occurring in some } x \in \Lambda\}$$

and $\Lambda_l = \cup_{k=0}^{l} \Lambda^k$, $\Lambda^* = \cup_{k=0}^{\infty} \Lambda^k$ where Λ^0 denotes the empty word $\emptyset$.

Let $\{e_1, ..., e_n\}$ be an orthonormal basis of n-dimensional Hilbert space $\mathbb{C}^n$. We put

$F_\Lambda^0 = \mathbb{C}e_0$ (e_0: vacuum vector)

$F_\Lambda^k =$ the Hilbert space spanned by the vectors $e_\mu = e_{\mu_1} \otimes \cdots \otimes e_{\mu_k}, \mu = (\mu_1, \ldots, \mu_k) \in \Lambda^k$,

$F_\Lambda = \oplus_{k=0}^{\infty} F_\Lambda^k$ (Hilbert space direct sum)

We denote by $T_\nu, (\nu \in \Lambda^*)$ the creation operator on F_Λ of $e_\nu, \nu \in \Lambda^* (\nu \neq \emptyset)$ defined by

$$T_\nu e_0 = e_\nu \qquad \text{and} \qquad T_\nu e_\mu = \begin{cases} e_\nu \otimes e_\mu, & (\nu\mu \in \Lambda^*) \\ 0 & \text{else} \end{cases}$$

which is a partial isometry. We put $T_\nu = 1$ for $\nu = \emptyset$. We denote by P_0 the rank one projection onto the vacuum vector e_0. It immediately follows that $\sum_{i=1}^n T_i T_i^* + \mathsf{P}_0 = 1$. We then easily see that for $\mu, \nu \in \Lambda^*$, the operator $T_\mu \mathsf{P}_0 T_\nu^*$ is the rank one partial isometry from the vector e_ν to e_μ. Hence, the C^*-algebra generated by elements of the form $T_\mu \mathsf{P}_0 T_\nu^*, \mu, \nu \in \Lambda^*$ is nothing but the C^*-algebra $\mathscr{K}(F_\Lambda)$ of all compact operators on F_Λ. Let $\mathscr{T}_\Lambda$ be the C^*-algebra on F_Λ generated by the elements $T_\nu, \nu \in \Lambda^*$.

Definition ([Ma]). The C^*-algebra $\mathcal{O}_\Lambda$ associated with subshift (Λ, σ) is defined as the quotient C^*-algebra $\mathscr{T}_\Lambda / \mathscr{K}(F_\Lambda)$ of $\mathscr{T}_\Lambda$ by $\mathscr{K}(F_\Lambda)$.

We denote by S_i, S_μ the quotient image of the operator $T_i, i \in \Sigma$, $T_\mu, \mu \in \Lambda^*$. Hence $\mathcal{O}_\Lambda$ is generated by n partial isometries $S_1, \ldots, S_n$ with relation $\sum_{i=1}^n S_i S_i^* = 1$.

If (Λ, σ) is a topological Markov shift, the C^*-algebra $\mathcal{O}_\Lambda$ is nothing but the Cuntz-Krieger algebra associated with the topological Markov shift (cf. [CK],[EFW],[Ev]).

We henceforth fix an arbitrary subshift (Λ, σ) in $\Sigma^{\mathbf{Z}}$. We denote by (X_Λ, σ) the associated right one-sided subshift for (Λ, σ).

We will present notation and basic facts for studying the C^*-algebra $\mathcal{O}_\Lambda$.

Put $a_\mu = S_\mu^* S_\mu, \mu \in \Lambda^*$. Since $T_\nu T_\nu^*$ commutes with $T_\mu^* T_\mu, \mu, \nu \in \Lambda^*$, the following identities hold

$$(*) \qquad a_\mu S_\nu = S_\nu a_{\mu\nu}, \qquad \mu, \nu \in \Lambda^*.$$

We notice that for $\mu, \nu \in \Lambda^*$ with $|\mu| = |\nu|$,

$$S_\mu^* S_\nu \neq 0 \qquad \text{if and only if} \qquad \mu = \nu.$$

We will use the following notation. Let k, l be natural numbers with $k \leq l$.

A_l = The C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by $a_\mu, \mu \in \Lambda_l$.
A_Λ = The C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by $a_\mu, \mu \in \Lambda^*$.
$\mathcal{F}_k^l$ = The C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by $S_\mu a S_\nu^*, \mu, \nu \in \Lambda^k, a \in A_l$.
$\mathcal{F}_k^\infty$ = The C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by $S_\mu a S_\nu^*, \mu, \nu \in \Lambda^k, a \in A_\Lambda$.
$\mathcal{F}_\Lambda^\infty$ = The C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by $S_\mu a S_\nu^*, \mu, \nu \in \Lambda^*, |\mu| = |\nu|, a \in A_\Lambda$.

The projections $\{T_\mu^* T_\mu; \mu \in \Lambda^*\}$ are mutually commutative so that the C^*-algebras $A_l, l \in \mathbf{N}$ are commutative. Thus we easily see the following lemma (cf. [Ma; Section 3]).

Lemma 2.1.
(i) *A_l is finite dimensional and commutative.*
(ii) *A_l is naturally embedded into A_{l+1} so that $A_\Lambda = \varinjlim A_l$ is a commutative AF-algebra.*
(iii) *Each element of $\mathcal{F}_k^l$ is a finite linear combination of elements of the form $S_\mu a S_\nu^*, \mu, \nu \in \Lambda^k, a \in A_l$. Hence $\mathcal{F}_k^l$ is finite dimensional.*
(iv) *There are two embeddings in $\{\mathcal{F}_k^l\}_{k \leq l}$:*
(iv-a) *$\iota_l : \mathcal{F}_k^l \subset \mathcal{F}_k^{l+1}$ through the embedding $A_l \subset A_{l+1}$ and*
(iv-b) *$\eta_k : \mathcal{F}_k^l \subset \mathcal{F}_{k+1}^{l+1}$ through the identity*

$$S_\mu a S_\nu^* = \sum_{j=1}^n S_{\mu j} S_j^* a S_j S_{\nu j}^*, \qquad \mu, \nu \in \Lambda^k, a \in A_l.$$

(v) *Both $\mathcal{F}_k^\infty = \lim_{l \to \infty} \mathcal{F}_k^l$ and $\mathcal{F}_\Lambda^\infty = \lim_{k \to \infty} \mathcal{F}_k^\infty$ are AF-algebras.*

In the preceding Hilbert space F_Λ, the transformation $e_\mu \to z^k e_\mu$, $\mu \in \Lambda^k, z \in \mathbf{T} = \{z \in \mathbf{C}; |z| = 1\}$ on each base e_μ yields a unitary representation which leaves $\mathcal{K}(F_\Lambda)$ invariant. Thus it gives rise to an action α of $\mathbf{T}$ on the C^*-algebra $\mathcal{O}_\Lambda$. It is called the gauge action and satisfies $\alpha_z(S_i) = zS_i, i = 1, 2, , n$.

Each element X of the $*$-subalgebra of $\mathcal{O}_\Lambda$ algebraically generated by $S_\mu, \mu \in \Lambda^*$ is written as a finite sum

$$X = \sum_{|-\nu|\geq 1} X_{-\nu} S_\nu^* + X_0 + \sum_{|\mu|\geq 1} S_\mu X_\mu \qquad \text{for some} \quad X_{-\nu}, X_0, X_\mu \in \mathcal{F}_\Lambda^\infty$$

because of the relation (*). The map $E(X) = \int_{z\in\mathsf{T}} \alpha_z(X)dz, X \in \mathcal{O}_\Lambda$ defines a projection of norm one onto the fixed point algebra $\mathcal{O}_\Lambda^\alpha$ under α. We then have (cf. [Ma; Proposition 3.11])

LEMMA 2.2. $\mathcal{F}_\Lambda^\infty = \mathcal{O}_\Lambda^\alpha$.

We will next describe structure theorems for the C^*-algebra $\mathcal{O}_\Lambda$ proved in [Ma].

THEOREM A ([Ma; Theorem 4.9 and 5.2]). *Let $\mathcal{A}$ be a unital C^*-algebra. Suppose that there is a unital *-homomorphism π from A_Λ to $\mathcal{A}$ and there are n partial isometries $s_1, \ldots, s_n \in \mathcal{A}$ satisfying the following relations*

$$\text{(a)} \qquad \sum_{j=1}^n s_j s_j^* = 1, \qquad s_\mu^* s_\mu s_\nu = s_\nu s_{\mu\nu}^* s_{\mu\nu}, \qquad \mu, \nu \in \Lambda^*$$

$$\text{(b)} \qquad s_\mu^* s_\mu = \pi(S_\mu^* S_\mu), \qquad \mu \in \Lambda^*$$

*where $s_\mu = s_{\mu_1} \cdots s_{\mu_k}, \mu = (\mu_1, \ldots, \mu_k)$. Then there exists a unital *-homomorphism $\tilde{\pi}$ from $\mathcal{O}_\Lambda$ to $\mathcal{A}$ such that $\tilde{\pi}(S_i) = s_i, i = 1, \ldots, n$ and its restriction to A_Λ coincides with π. In addition, if $\mathcal{O}_\Lambda$ satisfy the condition (I_Λ) below, this extended *-homomorphism $\tilde{\pi}$ becomes injective whenever π is injective.*

Let $\mathfrak{D}_\Lambda$ be the C^*-algebra generated by $S_\mu S_\mu^*, \mu \in \Lambda^*$ which is isomorphic to the C^*-algebra $C(X_\Lambda)$ of all continuous functions on the space of the one-sided subshift X_Λ for Λ. Put

$$\phi_\Lambda(X) = \sum_{j=1}^n S_j X S_j^*, \qquad X \in \mathfrak{D}_\Lambda$$

which corresponds to the shift σ on the one-sided space X_Λ of Λ.

Consider the following condition called (I_Λ) in [Ma].

(I_Λ) : For any $l, k \in \mathsf{N}$ with $l \geq k$, there exists a projection q_k^l in $\mathfrak{D}_\Lambda$ such that

(i) $q_k^l a \neq 0$ for any nonzero $a \in A_l$,

(ii) $q_k^l \phi_\Lambda^m(q_k^l) = 0, \quad 1 \leq m \leq k$.

Put

$$\lambda_\Lambda(X) = \sum_{j=1}^{n} S_j^* X S_j, \qquad X \in A_\Lambda.$$

We call λ_Λ the adjancy operator on A_Λ. It is said to be irreducible if there is no λ_Λ-invariant ideal in A_Λ. In addition, it is said to be aperiodic, if for any $l \in \mathsf{N}$, there exists $N \in \mathsf{N}$ such that $\lambda_\Lambda^N(p) \geq 1$ for any minimal projection p in A_l. We thus have

Theorem B ([Ma; Theorem 6.3 and Corollary 7.4]). *If the C^*-algebra $\mathcal{O}_\Lambda$ satisfy the condition (I_Λ) and λ_Λ is irreducible, then $\mathcal{O}_\Lambda$ is simple. In addition, if λ_Λ is aperiodic (or if $\mathcal{F}_\Lambda^\infty$ is simple), $\mathcal{O}_\Lambda$ is purely infinite.*

We notice the following proposition.

Proposition C ([Ma; Proposition 5.8], cf. [CK; 2.17 Proposition]). *Let (Λ_1, σ) and (Λ_2, σ) be subshifts such that both the associated C^*-algebras $\mathcal{O}_{\Lambda_1}$ and $\mathcal{O}_{\Lambda_2}$ satisfy the condition (I_Λ). If the associated one-sided subshifts (X_{Λ_1}, σ) and (X_{Λ_2}, σ) are topologically conjugate, then there exists an isomorphism from $\mathcal{O}_{\Lambda_1}$ onto $\mathcal{O}_{\Lambda_2}$ such that $\Phi \circ \alpha_z^1 = \alpha_z^2 \circ \Phi, z \in \mathsf{T}$ where α^i is the gauge action on $\mathcal{O}_{\Lambda_i}, i = 1, 2$ respectively.*

3. $K_0(\mathcal{F}_\Lambda^\infty)$.

In this section, we will compute K_0-group for the AF-algebra $\mathcal{F}_\Lambda^\infty$.

Let $m(l)$ be the dimension of the commutative finite dimensional C^*-algebra $A_l, l \in \mathsf{N}$. Take a unique basis $\{E_l^1, \dots, E_l^{m(l)}\}$ of A_l as vector space consisting of minimal projections in A_l with orthogonal ranges so that $\sum_{h=1}^{m(l)} E_l^h = 1$.

We fix $k \leq l$ for a while.

Lemma 3.1. $\sum_{\mu \in \Lambda^k} S_\mu^* S_\mu \geq 1$

Proof. For any $\nu \in \Lambda^*$, there is a block $\mu \in \Lambda^k$ such that $\mu\nu \in \Lambda^*$ and hence $T_\mu^* T_\mu e_\nu = e_\nu$. Thus one has $\sum_{\mu \in \Lambda^k} T_\mu^* T_\mu \geq 1$ on the Hilbert space F_Λ.

Hence we have

Lemma 3.2. *For $i = 1, 2, \dots, m(l)$, there exists $\mu \in \Lambda^k$ such that $S_\mu E_l^i S_\mu^* \neq 0$.*

Let $\mathcal{F}_k^{l,i}$ be the C^*-subalgebra of $\mathcal{F}_k^l$ generated by elements $S_\mu E_l^i S_\nu^*$, $\mu, \nu \in \Lambda^k$. Since $\mathcal{F}_k^{l,i}$ is isomorphic to a full matrix algebra $M_{n(k,l,i)}(\mathsf{C})$, one has

$$\mathscr{F}_k^l \cong M_{n(k,l,1)}(\mathsf{C}) \oplus \cdots \oplus M_{n(k,l,m(l))}(\mathsf{C}).$$

Put

$$\Lambda_l^{k,i} = \{\mu \in \Lambda^k | E_l^i \le S_\mu^* S_\mu\}.$$

Lemma 3.2 implies $\Lambda_l^{k,i} \neq \emptyset, i = 1, 2, \ldots, m(l)$ and $n(k,l,i) = |\Lambda_l^{k,i}|$ the cardinal number of $\Lambda_l^{k,i}$.

COROLLARY 3.3. $K_0(\mathscr{F}_k^l) \cong K_0(A_l) \cong \mathsf{Z}^{m(l)}$.

The above isomorphism between $K_0(\mathscr{F}_k^l)$ and $K_0(A_l)$ is given by the map

$$\Phi_k^l : [S_\mu E_l^i S_\mu^*] \in K_0(\mathscr{F}_k^l) \to [E_l^i] \in K_0(A_l), \qquad i = 1, 2, \ldots, m(l), \quad \mu \in \Lambda_l^{k,i}.$$

We next study $K_0(\mathscr{F}_k^\infty)$. We denote by ι_l the inclusion from A_l into A_{l+1}. It yields the inclusion from $\mathscr{F}_k^l$ into $\mathscr{F}_k^{l+1}$ which is also denoted by ι_l. One write E_l^i as

$$E_l^i = \sum_{h=1}^{m(l+1)} \iota_l(i,h) E_{l+1}^h$$

for some $\{0,1\}$-valued map $\iota_l(i,h), i = 1, 2, \ldots, m(l)$, $h = 1, 2, \ldots, m(l+1)$.

LEMMA 3.4. *The diagram*

$$\begin{array}{ccc} K_0(\mathscr{F}_k^l) & \xrightarrow{\iota_{l*}} & K_0(\mathscr{F}_k^{l+1}) \\ \Phi_k^l \big\downarrow & & \big\downarrow \Phi_k^{l+1} \\ K_0(A_l) & \xrightarrow[\iota_{l*}]{} & K_0(A_{l+1}) \end{array}$$

is commutative.

PROOF. If $S_\mu E_l^i S_\mu^* \neq 0$ and $\iota_l(i,h) \neq 0$, then $S_\mu E_{l+1}^j S_\mu^* \neq 0$. Namely $\Lambda_l^{k,i} \subset \Lambda_{l+1}^{k,j}$ if $\iota_l(i,j) \neq 0$. Hence the commutativity of the above diagram is clear.

Thus one obtains an isomorphism $\Phi_k = \varinjlim \Phi_k^l$ from $\varinjlim K_0(\mathscr{F}_k^l) = K_0(\mathscr{F}_k^\infty)$ onto $\varinjlim K_0(A_l) = K_0(A_\Lambda)$. Namely, one has

PROPOSITION 3.5. $K_0(\mathscr{F}_k^\infty) \cong K_0(A_\Lambda) \cong \varinjlim (\mathsf{Z}^{m(l)}, \iota_l)$ where the inclusion ι_l of $\mathsf{Z}^{m(l)}$ into $\mathsf{Z}^{m(l+1)}$ is given by

$$[E_l^i] = \sum_{h=1}^{m(l+1)} \iota_l(i,h)[E_{l+1}^h], \qquad i = 1,2,\dots,m(l)$$

and

$$\mathsf{Z}^{m(l)} = \mathsf{Z}[E_l^1] \oplus \cdots \oplus \mathsf{Z}[E_l^{(m(l))}].$$

We denote by Z_Λ the above abelian group $\varinjlim (\mathsf{Z}^{m(l)}, \iota_l)$ and so that

$$\mathsf{Z}_\Lambda \cong K_0(\mathscr{F}_k^\infty) \cong K_0(A_\Lambda), \qquad k \in \mathsf{N}.$$

We next study $K_0(\mathscr{F}_\Lambda^\infty)$ as the inductive limit $\varinjlim K_0(\mathscr{F}_k^\infty)$.

The embedding η_k of $\mathscr{F}_k^\infty$ into $\mathscr{F}_{k+1}^\infty$ is given, through the embedding of $\mathscr{F}_k^l$ into $\mathscr{F}_{k+1}^{l+1}$, by the identity

$$S_\mu E_l^i S_\nu^* = \sum_{j=1}^n S_{\mu j} S_j^* E_l^i S_j S_{\nu j}^*, \qquad \mu,\nu \in \Lambda^k, \quad i = 1,2,\dots,m(l)$$

so that the induced homomorphism η_{k*} from $K_0(\mathscr{F}_k^\infty)$ to $K_0(\mathscr{F}_{k+1}^\infty)$ is given by

$$\eta_{k*}[S_\mu E_l^i S_\mu^*] = \sum_{j=1}^n [S_{\mu j} S_j^* E_l^i S_j S_{\mu j}^*], \qquad \mu \in \Lambda_l^{k,i}, \quad i = 1,2,\dots,m(l).$$

As the projection $S_j^* E_l^i S_j$ belongs to A_{l+1}, it can be written as

$$S_j^* E_l^i S_j = \sum_{h=1}^{m(l+1)} A_l(i,j,h) E_{l+1}^h$$

for some $\{0,1\}$-valued map $A_l(i,j,h)$, $i = 1,2,\dots,m(l)$, $j = 1,2,\dots,n$, $h = 1,2,\dots,m(l+1)$. Hence one has

$$S_\mu E_l^i S_\mu^* = \sum_{j=1}^n \sum_{h=1}^{m(l+1)} A_l(i,j,h) S_{\mu j} E_{l+1}^h S_{\mu j}^*, \qquad \mu \in \Lambda^k, \quad i = 1,2,\dots,m(l).$$

Lemma 3.6. *If $S_\mu E_l^i S_\mu^* \neq 0$, one has $S_{\mu j} E_{l+1}^h S_{\mu j}^* \neq 0$ for $A_l(i,j,h) \neq 0$.*

Proof. Since $A_l(i,j,h) \neq 0$, one has $S_j^* E_l^i S_j \geq E_{l+1}^h$. We also have $S_j^* a_\mu S_j \geq S_j^* E_l^i S_j$ because $S_\mu E_l^i S_\mu^* \neq 0$. Hence we obtain $S_j^* a_\mu S_j \geq E_{l+1}^h$ which implies $S_{\mu j} E_{l+1}^h S_{\mu j}^* \neq 0$.

LEMMA 3.7. *If $\Lambda_l(i,j_1,h) \neq 0$ and $\Lambda_l(i,j_2,h) \neq 0$, one has for $\mu \in \Lambda^k$*

$$[S_{\mu j_1} E_{l+1}^h S_{\mu j_1}^*] = [S_{\mu j_2} E_{l+1}^h S_{\mu j_2}^*] \quad \text{in } K_0(\mathscr{F}_{k+1}^{l+1}).$$

Put

$$\Lambda_l(i,h) = \sum_{j=1}^{n} \Lambda_l(i,j,h) \in \mathsf{Z}_+, \qquad i = 1,2,\ldots,m(l), \quad h = 1,2,\ldots,m(l+1).$$

We then define a homomorphism λ_l from $K_0(A_l)$ to $K_0(A_{l+1})$ by

$$\lambda_l([E_l^i]) = \sum_{h=1}^{m(l+1)} \Lambda_l(i,h)[E_{l+1}^h]$$

where

$$K_0(A_l) = \sum_{i=1}^{m(l)} \oplus \mathsf{Z}[E_l^i], \quad K_0(A_{l+1}) = \sum_{h=1}^{m(l+1)} \oplus \mathsf{Z}[E_{l+1}^h].$$

we indeed have

LEMMA 3.8. $\lambda_l([P]) = \sum_{j=1}^{n}[S_j^* P S_j]$ *for a projection P in A_l.*

Hence one has

LEMMA 3.9. *The diagram*

$$\begin{array}{ccc} K_0(A_l) & \xrightarrow{\iota_{l*}} & K_0(A_{l+1}) \\ \lambda_l \downarrow & & \downarrow \lambda_{l+1} \\ K_0(A_{l+1}) & \xrightarrow[\iota_{l+1*}]{} & K_0(A_{l+2}) \end{array}$$

is commutative.

Since $K_0(A_\Lambda) = \varinjlim(K_0(A_l), \iota_{l*})$, one can define a homomorphism $\lambda_\Lambda = \lambda_l$ on $K_0(A_\Lambda)$ induced by the sequence of homomorphisms $\lambda_l : K_0(A_l) \to K_0(A_{l+1}), l \in \mathsf{N}$. Namely, we obtain a homomorphism λ_Λ on $\mathsf{Z}_\Lambda(\cong K_0(A_\Lambda) \cong K_0(\mathscr{F}_k^\infty))$. We remark that it is exactly regarded as the induced homomorphism on $K_0(A_\Lambda)$ from the adjancy operator λ_Λ defined in the previous section. Hence we use the same notation λ_Λ without confusion.

Lemma 3.10. *The diagram*

$$\begin{array}{ccc} K_0(\mathscr{F}_k^\infty) & \xrightarrow{\eta_{k*}} & K_0(\mathscr{F}_{k+1}^\infty) \\ {\scriptstyle \Phi_k}\downarrow & & \downarrow{\scriptstyle \Phi_{k+1}} \\ K_0(A_\Lambda) & \xrightarrow[\lambda_\Lambda]{} & K_0(A_\Lambda) \end{array}$$

is commutative.

Proof. By Lemma 3.7, it follows that

$$\begin{aligned} \Phi_{k+1} \circ \eta_{k*}([S_\mu E_l^i S_\mu^*]) &= \Phi_{k+1}\left(\sum_{j=1}^{n}\left[S_{\mu j}\left(\sum_{h=1}^{m(l+1)} \Lambda_l(i,j,h)E_{l+1}^h\right)S_{\mu j}^*\right]\right) \\ &= \sum_{h=1}^{m(l+1)} \Phi_{k+1}\left(\sum_{j=1}^{n} \Lambda_l(i,j,h)[S_{\mu j}E_{l+1}^h S_{\mu j}^*]\right) \\ &= \sum_{h=1}^{m(l+1)} \Lambda_l(i,h)[E_{l+1}^h] = \lambda_l[E_l^i] = \lambda_\Lambda \circ \Phi_k([S_\mu E_l^i S_\mu^*]). \end{aligned}$$

Therefore we conclude

Theorem 3.11. $K_0(\mathscr{F}_\Lambda^\infty) = \varinjlim(\mathsf{Z}_\Lambda, \lambda_\Lambda)$.

Corollary 3.12. *If Λ is a sofic subshift, $K_0(\mathscr{F}_\Lambda^\infty) = \varinjlim(\mathsf{Z}^{m(l)}, \lambda_l)$.*

Proof. Let j_l be the canonical inclusion of $\mathsf{Z}^{m(l)}(= K_0(A_l))$ into $\mathsf{Z}_\Lambda(= K_0(A_\Lambda))$, which is induced by the natural inclusion of A_l into A_Λ. Since the following diagram

$$\begin{array}{ccc} \mathsf{Z}_\Lambda & \xrightarrow{\lambda_\Lambda} & \mathsf{Z}_\Lambda \\ {\scriptstyle j_l}\uparrow & & \uparrow{\scriptstyle j_{l+1}} \\ \mathsf{Z}^{m(l)} & \xrightarrow[\lambda_l]{} & \mathsf{Z}^{m(l+1)} \end{array}$$

is commutative, there is a homomorphism π from $\varinjlim(\mathsf{Z}^{m(l)}, \lambda_l)$ to $\varinjlim(\mathsf{Z}_\Lambda, \lambda_\Lambda)$. It is easy to see that it is indeed a surjective isomorphism because $\mathsf{Z}_\Lambda = \mathsf{Z}^{m(l)}$ for some large enough l by [Ma; Proposition 8.2]

Before ending this section, we define the dimension group $DG(\Lambda)$ for a general subshift (Λ, σ) as the dimension group for the AF-algebra $\mathscr{F}_\Lambda^\infty$, namely,

$$G(\Lambda) = K_0(\mathscr{F}_\Lambda^\infty): \quad \text{as an ordered group.}$$

The notion of the dimension group for a topological Markov shift (Λ_A, σ) determined by a matrix A with entries in $\{0, 1\}$ has been introduced by W. Krieger in [Kr1] and [Kr2]. It is realized as the dimension group for the canonical AF-algebra $\mathscr{F}_A$ appeared inside of the Cuntz-Krieger algebra $\mathcal{O}_A$ associated with the topological Markov shift (Λ_A, σ). If we restrict our construction of C^*-algebras $\mathcal{O}_\Lambda$ and $\mathscr{F}_\Lambda^\infty$ to a topological Markov shift (Λ_A, σ), they coincide with the Cuntz-Krieger algebra $\mathcal{O}_A$ and the canonical AF-algebra $\mathscr{F}_A$ respectively. Hence our above definition of the dimension group for general subshifts is a generalization of the case of topological Markov shifts. By Proposition C, we see

PROPOSITION 3.13. *The dimension group $DG(\Lambda)$ for subshift (Λ, σ) is an invariant under topological conjugacy for the associated one-sided subshift (X_Λ, σ) among the class of all subshifts such that the associated C^*-algebra $\mathcal{O}_\Lambda$ satisfies the condition (I_Λ).*

4. $K_*(\mathcal{O}_\Lambda)$.

We will, in this section, present K-theory formula for the C^*-algebra $\mathcal{O}_\Lambda$. We denote by $\mathscr{K}$ the C^*-algebra of all compact operators on a separable infinite dimensional Hilbert space. We will notice that the crossed product $\mathcal{O}_\Lambda \times_\alpha \mathsf{T}$ of $\mathcal{O}_\Lambda$ by the gauge action α of T is stably isomorphic to the associated AF-algebra $\mathscr{F}_\Lambda^\infty$. Since $\mathcal{O}_\Lambda$ is stably isomorphic to the crossed product $(\mathcal{O}_\Lambda \times_\alpha \mathsf{T}) \times_{\hat{\alpha}} \mathsf{Z}$ of $\mathcal{O}_\Lambda \times_\alpha \mathsf{T}$ by the dual action $\hat{\alpha}$, it will be possible to present K-theory formula for the C^*-algebra $\mathcal{O}_\Lambda$ by using the previous K-theory formula for the AF-algebra $\mathscr{F}_\Lambda^\infty$ and by applying the Pimsner-Voiculescu's six-term exact sequence of the K-theory for the crossed products by Z ([PV]).

We will first see that the crossed product $\mathcal{O}_\Lambda \times_\alpha \mathsf{T}$ is stably isomorphic to the AF-algebra $\mathscr{F}_\Lambda^\infty$.

Let $p_0 : \mathsf{T} \to \mathcal{O}_\Lambda$ be the constant function whose value everywhere is the unit 1 of $\mathcal{O}_\Lambda$. Hence p_0 belongs to the algebra $L^1(\mathsf{T}, \mathcal{O}_\Lambda)$ and hence to the crossed product $\mathcal{O}_\Lambda \times_\alpha \mathsf{T}$. By [Ro], the fixed point algebra ${\mathcal{O}_\Lambda}^\alpha$ is canonically isomorphic to the algebra $p_0(\mathcal{O}_\Lambda \times_\alpha \mathsf{T})p_0$. The isomorphism between them is given by the correspondence : $x \in {\mathcal{O}_\Lambda}^\alpha \to \hat{x} \in L^1(\mathsf{T}, \mathcal{O}_\Lambda) \subset \mathcal{O}_\Lambda \times_\alpha \mathsf{T}$ where the function $\hat{x}$ is defined by $\hat{x}(t) = x, t \in \mathsf{T}$.

LEMMA 4.1. *The projection p_0 is full in $\mathcal{O}_\Lambda \times_\alpha \mathsf{T}$.*

PROOF. Suppose that there exists a nondegenerate representation π of $\mathcal{O}_\Lambda \times_\alpha \mathsf{T}$ such that $\pi(p_0) = 0$. For any element S in $\mathcal{O}_\Lambda$, put $\widehat{S}(z) = S, z \in \mathsf{T}$, which belongs to $L^1(\mathsf{T}, \mathcal{O}_\Lambda)$. We denote by $*$ the α-twisted convolution product in $L^1(\mathsf{T}, \mathcal{O}_\Lambda)$ (the usual product as elements of $\mathcal{O}_\Lambda \times_\alpha \mathsf{T}$). It then follows that $\widehat{S} * p_0 = \widehat{S}$. Hence $\widehat{S}$ belongs to the ideal $\ker(\pi)$ in $\mathcal{O}_\Lambda \times_\alpha \mathsf{T}$. For $S, T \in \mathcal{O}_\Lambda$, one has $(\widehat{S} * \widehat{T}^*)(z) = S\alpha_z(T^*)$ by using the identity $(\widehat{T}^*)(z) = \alpha_z(T^*)$. For any $X \in \mathcal{O}_\Lambda$ and $\mu \in \Lambda^k$, we have

$$(\widehat{XS_\mu} * \widehat{S_\mu}^*)(z) = z^{-k} X S_\mu S_\mu^*$$

and hence

$$\left(\sum_{|\mu|=k} \widehat{XS_\mu} * \widehat{S_\mu}^* \right)(z) = z^{-k} X, \qquad k \in \mathsf{N}.$$

We denote by B_k the commutative C^*-algebra generated by $a_\mu, \mu \in \Lambda^k$. Let $F_k^i, i = 1, 2, \dots, n(k)$ be the set of all minimal projections in B_k. Since one sees for $\mu \in \Lambda^k$,

$$(\widehat{XF_k^i S_\mu^*} * \widehat{S_\mu^*}^*)(z) = z^k X F_k^i,$$

one has

$$\left(\sum_{i=1}^{n(k)} \widehat{XF_k^i S_\mu^*} * \widehat{S_\mu^*}^* \right)(z) = z^k X, \qquad k \in \mathsf{N}.$$

Hence any $\mathcal{O}_\Lambda$-valued function of the form

$$z \in \mathsf{T} \to z^k X \in \mathcal{O}_\Lambda, \qquad k \in \mathsf{Z}, \quad X \in \mathcal{O}_\Lambda$$

is contained in the ideal $\ker(\pi)$. Thus we conclude $\pi \equiv 0$ on $\mathcal{O}_\Lambda \times_\alpha \mathsf{T}$. This implies that p_0 is a full projection in $\mathcal{O}_\Lambda \times_\alpha \mathsf{T}$.

Since the AF-algebra $\mathcal{F}_\Lambda^\infty$ is realized as the fixed point algebra ${\mathcal{O}_\Lambda}^\alpha$, one sees, by [Bro;Corollary 2.6]

COROLLARY 4.2. *$\mathcal{O}_\Lambda \times_\alpha \mathsf{T}$ is stably isomorphic to $\mathcal{F}_\Lambda^\infty$.*

The Pimsner-Voiculescu's six term exact sequence of the K-theory for the crossed product $(\mathcal{O}_\Lambda \times_\alpha \mathsf{T}) \times_{\hat{\alpha}} \mathsf{Z}$ says that the following sequence becomes exact:

$$\begin{array}{ccccc} K_0(\mathcal{O}_\Lambda \times_\alpha \mathsf{T}) & \xrightarrow{\mathrm{id}-\hat{\alpha}_*^{-1}} & K_0(\mathcal{O}_\Lambda \times_\alpha \mathsf{T}) & \xrightarrow{\iota_*} & K_0(\mathcal{O}_\Lambda \times_\alpha \mathsf{T}) \times_{\hat{\alpha}} \mathsf{Z}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{O}_\Lambda \times_\alpha \mathsf{T}) \times_{\hat{\alpha}} \mathsf{Z}) & \xleftarrow[\iota_*]{} & K_1(\mathcal{O}_\Lambda \times_\alpha \mathsf{T}) & \xleftarrow[\mathrm{id}-\hat{\alpha}_*^{-1}]{} & K_1(\mathcal{O}_\Lambda \times_\alpha \mathsf{T}). \end{array}$$

Since the double crossed product $(\mathcal{O}_\Lambda \times_\alpha \mathsf{T}) \times_{\hat{\alpha}} \mathsf{Z}$ is stably isomorphic to $\mathcal{O}_\Lambda$, one has

LEMMA 4.3.
(i) $K_0(\mathcal{O}_\Lambda) \cong K_0(\mathcal{O}_\Lambda \times_\alpha \mathsf{T})/(\mathrm{id} - \hat{\alpha}_*^{-1})K_0(\mathcal{O}_\Lambda \times_\alpha \mathsf{T})$
(ii) $K_1(\mathcal{O}_\Lambda) \cong \mathrm{Ker}\,(\mathrm{id} - \hat{\alpha}_*^{-1})$ on $K_0(\mathcal{O}_\Lambda \times_\alpha \mathsf{T})$.

We will next study the group $K_0(\mathcal{O}_\Lambda \times_\alpha \mathsf{T})$ and the action $\hat{\alpha}_*$ on it. The next lemma follows from Lemma 4.1 and [Ri; Proposition 2.4].

LEMMA 4.4. *The inclusion* $\iota : p_0(\mathcal{O}_\Lambda \times_\alpha \mathsf{T})p_0 \to \mathcal{O}_\Lambda \times_\alpha \mathsf{T}$ *induces an isomorphism* $\iota_* : K_0(p_0(\mathcal{O}_\Lambda \times_\alpha \mathsf{T})p_0) \to K_0(\mathcal{O}_\Lambda \times_\alpha \mathsf{T})$ *on K-theory.*

Under the identification, $\mathcal{F}_\Lambda^\infty = {\mathcal{O}_\Lambda}^\alpha = p_0(\mathcal{O}_\Lambda \times_\alpha \mathsf{T})p_0$, we define an isomorphism β on $K_0(\mathcal{F}_\Lambda^\infty)$ as $\beta = {\iota_*}^{-1} \circ \hat{\alpha}_* \circ \iota_*$. Namely the diagram

$$\begin{array}{ccc} K_0(\mathcal{O}_\Lambda \times_\alpha \mathsf{T}) & \xrightarrow{\hat{\alpha}_*} & K_0(\mathcal{O}_\Lambda \times_\alpha \mathsf{T}) \\ \iota_* \downarrow & & \downarrow \iota_* \\ K_0(\mathcal{F}_\Lambda^\infty) & \xrightarrow[\beta]{} & K_0(\mathcal{F}_\Lambda^\infty) \end{array}$$

is commutative.

The following lemma is a key.

LEMMA 4.5. *For a projection P in* $\mathcal{F}_\Lambda^\infty$ *and a partial isometry S in* $\mathcal{O}_\Lambda$ *with* $\alpha_z(S) = zS, z \in \mathsf{T}$ *and* $P \le S^*S$, *we have* $\beta[P] = [SPS^*]$ in $K_0(\mathcal{F}_\Lambda^\infty)$.

PROOF. Let $j : \mathcal{F}_\Lambda^\infty \to p_0(\mathcal{O}_\Lambda \times_\alpha \mathsf{T})p_0$ be the canonical isomorphism and $\iota : p_0(\mathcal{O}_\Lambda \times_\alpha \mathsf{T})p_0 \hookrightarrow \mathcal{O}_\Lambda \times_\alpha \mathsf{T}$ the inclusion. For $P \in \mathcal{F}_\Lambda^\infty$, we denote by $\widehat{P} = \iota \circ j(P) \in L^1(\mathsf{T}, \mathcal{O}_\Lambda) \subset \mathcal{O}_\Lambda \times_\alpha \mathsf{T}$ the constant P-valued function: $\widehat{P}(z) = P, z \in \mathsf{T}$. As $SPS^* \in \mathcal{F}_\Lambda^\infty$, we similarly denote by $\widehat{SPS^*} = \iota \circ j(SPS^*) \in L^1(\mathsf{T}, \mathcal{O}_\Lambda)$ the constant SPS^*-valued function. It suffices to show $[\widehat{SPS^*}] = \hat{\alpha}_*[\widehat{P}]$ in $K_0(\mathcal{O}_\Lambda \times_\alpha \mathsf{T})$. Let $\widehat{S} \in L^1(\mathsf{T}, \mathcal{O}_\Lambda)$ be the constant S-valued function : $\widehat{S}(z) = S, z \in \mathsf{T}$. We denote by $*$ the twisted convolution product (usual product) in $\mathcal{O}_\Lambda \times_\alpha \mathsf{T}$. It then follows that $(\widehat{S} * \widehat{P})(z) = SP, z \in \mathsf{T}$ and $(\widehat{S} * \widehat{P} * \widehat{S}^*)(z) = z^{-1}SPS^*, z \in \mathsf{T}$. Thus we have $\hat{\alpha}(\widehat{S} * \widehat{P} * \widehat{S}^*) = \widehat{SPS^*}$. As $(\widehat{S}^* * \widehat{S})(z) = S^*S \in \mathcal{F}_\Lambda^\infty$ and hence $\widehat{S}^* * \widehat{S} = \widehat{S^*S}$. Since the inclusion $\widehat{\ } = \iota \circ j : \mathcal{F}_\Lambda^\infty = {\mathcal{O}_\Lambda}^\alpha \hookrightarrow \mathcal{O}_\Lambda \times_\alpha \mathsf{T}$ is a homomorphism, one has $\widehat{P} \le \widehat{S^*S}$ because $P \le S^*S$. Thus one sees

$$[\widehat{S} * \widehat{P} * \widehat{S}^*] = [\widehat{P}] \qquad \text{in} \quad K_0(\mathcal{O}_\Lambda \times_\alpha \mathsf{T})$$

so that we conclude

$$\hat{\alpha}_*[\widehat{P}] = [\widehat{SPS^*}] \quad \text{in } K_0(\mathcal{O}_\Lambda \times_\alpha \mathsf{T}) \quad \text{and} \quad \beta[P] = [SPS^*] \quad \text{in } K_0(\mathcal{F}_\Lambda^\infty).$$

Lemma 4.6. *For a nonzero projection* $S_\mu E_l^i S_\mu^*$ *in* $\mathcal{F}_k^l$ *with* $\mu = j\nu \in \Lambda^k$, *one has* $\beta^{-1}[S_\mu E_l^i S_\mu^*] = [S_\nu E_l^i S_\nu^*]$ *in* $K_0(\mathcal{F}_{k-1}^l)$.

Proof. Since $S_\mu E_l^i S_\mu^* \neq 0$, we see that $S_\nu E_l^i S_\nu^* \leq S_j^* S_j$ because of the identity $S_j^* S_j S_\nu E_l^i S_\nu^* = S_\nu a_\mu E_l^i S_\nu^* = S_\nu E_l^i S_\nu^*$. Hence we have the conclusion by the previous lemma.

Corollary 4.7. *The homomorphism* $\beta^{-1} : K_0(\mathcal{F}_\Lambda^\infty) \to K_0(\mathcal{F}_\Lambda^\infty)$ *corresponds to the shift* σ *in* $\varinjlim K_0(\mathcal{F}_k^\infty) = \varinjlim \mathsf{Z}_\Lambda$. *Namely, if* $x = (x_1, x_2, \dots)$ *is a sequence representing an element of* $\varinjlim K_0(\mathcal{F}_k^\infty)$, *then* $\beta^{-1}x$ *is represented by* $\sigma(x) = (x_2, x_3, \dots)$.

Since the diagram

$$\begin{array}{ccc} K_0(\mathcal{F}_\Lambda^\infty) & \xrightarrow{\mathrm{id}-\beta^{-1}} & K_0(\mathcal{F}_\Lambda^\infty) \\ \Phi \downarrow & & \downarrow \Phi \\ \varinjlim \mathsf{Z}_\Lambda & \xrightarrow[\mathrm{id}-\sigma]{} & \varinjlim \mathsf{Z}_\Lambda \end{array}$$

is commutative, one has

Corollary 4.8.
(i) $K_0(\mathcal{O}_\Lambda) \cong \varinjlim \mathsf{Z}_\Lambda / (\mathrm{id} - \sigma) \varinjlim \mathsf{Z}_\Lambda$
(ii) $K_1(\mathcal{O}_\Lambda) \cong \mathrm{Ker}\,(\mathrm{id} - \sigma)$ *on* $\varinjlim \mathsf{Z}_\Lambda$.

Let j be the homomorphism from $K_0(\mathcal{F}_0^\infty) = \mathsf{Z}_\Lambda$ to $K_0(\mathcal{F}_\Lambda^\infty) = \varinjlim \mathsf{Z}_\Lambda$ induced by the inclusion : $\mathcal{F}_0^\infty \hookrightarrow \mathcal{F}_\Lambda^\infty$.

As in the proof of [C2; 3.1 Proposition], we see that every element in $\varinjlim \mathsf{Z}_\Lambda$ is equivalent modulo $(\mathrm{id} - \sigma) \varinjlim \mathsf{Z}_\Lambda$ to an element in Z_Λ. Since the diagram

$$\begin{array}{ccc} \mathsf{Z}_\Lambda & \xrightarrow{\mathrm{id}-\lambda_\Lambda} & \mathsf{Z}_\Lambda \\ j \downarrow & & \downarrow j \\ \varinjlim \mathsf{Z}_\Lambda & \xrightarrow[\mathrm{id}-\sigma]{} & \varinjlim \mathsf{Z}_\Lambda \end{array}$$

is commutative and $j(x) \in (\mathrm{id} - \sigma) \varinjlim \mathsf{Z}_\Lambda, x \in \mathsf{Z}_\Lambda$ implies $x \in (\mathrm{id} - \lambda_\Lambda)\mathsf{Z}_\Lambda$, we then have

$$K_0(\mathcal{O}_\Lambda) \cong j(\mathsf{Z}_\Lambda)/(\mathrm{id} - \sigma) \varinjlim \mathsf{Z}_\Lambda = \mathsf{Z}_\Lambda/(\mathrm{id} - \lambda_\Lambda)\mathsf{Z}_\Lambda.$$

Similarly as in the same argument in [C2; 3.1.Proposition], we have

$$K_1(\mathcal{O}_\Lambda) \cong \text{ Ker } (\mathrm{id} - \lambda_\Lambda) \text{ on } \mathsf{Z}_\Lambda.$$

Thus we present the K-theory formula for the C^*-algebra $\mathcal{O}_\Lambda$

THEOREM 4.9.
(i) $K_0(\mathcal{O}_\Lambda) \cong \mathsf{Z}_\Lambda/(\mathrm{id} - \lambda_\Lambda)\mathsf{Z}_\Lambda \cong \varinjlim(\mathsf{Z}^{m(l+1)}/(\iota_{l*} - \lambda_l)\mathsf{Z}^{m(l)})$
(ii) $K_1(\mathcal{O}_\Lambda) \cong$ Ker $(\mathrm{id} - \lambda_\Lambda)$ *in* $\mathsf{Z}_\Lambda \cong \varinjlim$ (Ker $(\iota_{l*} - \lambda_l)$ *in* $\mathsf{Z}^{m(l)}$)
where

$$\mathsf{Z}_\Lambda = \varinjlim (\mathsf{Z}^{m(l)}, \iota_{l*}), \qquad m(l) = \dim A_l$$

and

$$\lambda_\Lambda = \varinjlim \lambda_l, \quad \lambda_l : \mathsf{Z}^{m(l)} = K_0(A_l) \to \mathsf{Z}^{m(l+1)} = K_0(A_{l+1})$$

is defined by

$$\lambda_l([P]) = \sum_{j=1}^{n} [S_j^* P S_j] \quad \text{for a projection } P \text{ in } A_l.$$

More precisely, for the minimal projections $E_l^1, \dots, E_l^{m(l)}$ *of* A_l *with* $\sum_{i=1}^{m(l)} E_l^i = 1$ *and the canonical basis* $e_l^1, \dots, e_l^{m(l)}$ *of* $\mathsf{Z}^{m(l)}$, *the map* $[E_l^i] \to e_l^i$ *extends to an isomorphism of* $K_0(\mathcal{O}_\Lambda)$ *onto* $\varinjlim(\mathsf{Z}^{m(l+1)}/(\iota_l - \lambda_l)\mathsf{Z}^{m(l)})$.

Before ending this section, we note the following lemma.

LEMMA 4.10. *The* C^**-algebra* $\mathcal{O}_\Lambda$ *is nuclear and satisfies the Universal Coefficient Theorem in the sense of Rosenberg and Schocet.*

PROOF. Since the double crossed product $(\mathcal{O}_\Lambda \times_\alpha \mathsf{T}) \times_{\hat{\alpha}} \mathsf{Z}$ is stably isomorphic to $\mathcal{O}_\Lambda$, the assertion is immediate from Corollary 4.2 (cf. [RS], [Bl; p. 287]).

Hence, as in Theorem B, one sees by [Ki] and [Ph]

COROLLARY 4.11. *If the* C^**-algebra* $\mathcal{O}_\Lambda$ *satisfies the condition* (I_Λ) *and the adjancy operator* λ_Λ *is aperiodic, then* $\mathcal{O}_\Lambda$ *is a separable nuclear purely infinite simple* C^**-algebra satisfying the Universal Coefficient Theorem. Thus, these* C^**-algebras are completely classified by their own K-theory up to isomorphism.*

5. Sofic subshifts and examples.

There is a class of subshifts called sofic subshifts. It is truly wider, up to conjugate, than the class of subshifts of finite type and hence that of topological Markov shifts. Hence the C^*-algebras associated with sofic subshifts which are not conjugate to topological Markov shifts can not be dealt with within the Cuntz-Krieger's approach. For a subshift (Λ, σ) and words $\mu, \nu \in \Lambda^*$, we write $\mu \sim \nu$ if

$$\{\gamma \in \Lambda^* \mid \mu\gamma \in \Lambda^*\} = \{\gamma \in \Lambda^* \mid \nu\gamma \in \Lambda^*\}.$$

If the cardinality of the equivalence classes $\Lambda^*/\sim$ is finite, the subshift (Λ, σ) is said to be sofic (cf. [DGS], [W]). Hence a subshift (Λ, σ) is sofic if and only if the commutative C^*-subalgebra A_Λ of $\mathcal{O}_\Lambda$ is finite dimensional (cf. [Ma; Proposition 8.2]).

Suppose that a subshift (Λ, σ) is sofic. Put $N = \dim A_\Lambda < \infty$. Hence the adjancy operator λ_Λ on A_Λ is realized as an $N \times N$ matrix with entries in non-negative integers. We then notice that λ_Λ is irreducible (resp. aperiodic) in the sense of Section 2 if and only if it is irreducible (resp. aperiodic) in the sense of non-negative matrix.

It is well-known that if λ_Λ is aperiodic, the AF-algebra $\mathcal{F}_\Lambda^\infty$ is simple and has a unique tracial state τ_Λ (cf. [Bra], [Ef], [Ev2]). Thus we can summarize the previous discussions on K-theory for the C^*-algebras $\mathcal{O}_\Lambda$ and $\mathcal{F}_\Lambda^\infty$ as in the following way.

Proposition 5.1. *Suppose that a subshift (Λ, σ) is sofic. If the C^*-algebra $\mathcal{O}_\Lambda$ satisfies the condition (I_Λ) and the adjancy operator λ_Λ is aperiodic, then we have*

(i) *$\mathcal{O}_\Lambda$ is simple and purely infinite.*

(ii) $K_0(\mathcal{O}_\Lambda) \cong \mathsf{Z}^N/(1-\lambda_\Lambda)\mathsf{Z}^N$ *and* $K_1(\mathcal{O}_\Lambda) \cong \mathrm{Ker}\,(1-\lambda_\Lambda)$ *in* Z^N.

(iii) $DG(\Lambda) \cong \varinjlim(\mathsf{Z}^N, \lambda_\Lambda) \cong \tau_\Lambda(\mathcal{F}_\Lambda^\infty)$ *in* R.

where τ_Λ is a unique tracial state on $\mathcal{F}_\Lambda^\infty$.

Thus by Corollary 4.11 we see that if a subshift (Λ, σ) is sofic, the C^*-algebra $\mathcal{O}_\Lambda$ is stably isomorphic to some Cuntz-Krieger algebra $\mathcal{O}_{\lambda_\Lambda}$ associated with a matrix λ_Λ with entries in non-negative integers.

We present examples of the C^*-algebras associated with sofic subshifts.

Example 1 (Cuntz algebras $\mathcal{O}_n$, [C], [C2], [C3]).

Let (Λ_n, σ) be the full shift over $\Sigma = \{1, 2, \dots, n\}$. The C^*-algebra $\mathcal{O}_{\Lambda_n}$ associated with it is the Cuntz algebra $\mathcal{O}_n$ of order n. Then the commutative C^*-algebras A_l are reduced to the scalar C so that $m(l) = 1, l \in \mathsf{N}$. It is easy to see that the adjancy operator λ_Λ is the n-multiplication on $\mathbb{Z} = K_0(A_l) = K_0(\mathbb{C})$. Hence we see

$$K_0(\mathcal{F}_{\Lambda_n}^{\infty}) = \mathbb{Z}[\frac{1}{n}], \qquad K_0(\mathcal{O}_n) = \mathbb{Z}/(1-n)\mathbb{Z}, \qquad K_1(\mathcal{O}_n) = 0.$$

EXAMPLE 2 (Cuntz-Krieger algebras $\mathcal{O}_A$, [CK], [C2], [C3]).

Let (Λ_A, σ) be the topological Markov shift determined by an $n \times n$-matrix A with $\{0,1\}$-entries. The C^*-algebra $\mathcal{O}_{\Lambda_A}$ associated with it is the Cuntz-Krieger algebra $\mathcal{O}_A$. Suppose that A is an irreducible but not permutation matrix with rank n. Hence one sees that $A_l = \mathbb{C}S_1S_1^* \oplus \cdots \oplus \mathbb{C}S_nS_n^*, l \in \mathbb{N}$ so that $m(l) = n, l \in \mathbb{N}$. It is easy to see that the adjancy operator $\lambda_\Lambda (= \lambda_l)$ is given by operating the transpose of the matrix A from $\mathbb{Z}^n = K_0(A_l)$ to $\mathbb{Z}^n = K_0(A_{l+1})$. Hence we see

$$K_0(\mathcal{F}_{\Lambda_A}^{\infty}) = \varinjlim(\mathbb{Z}^n, A^t), \quad K_0(\mathcal{O}_A) = \mathbb{Z}^n/(1-A^t)\mathbb{Z}^n,$$

$$K_1(\mathcal{O}_A) = \mathrm{Ker}(1-A^t) \text{ in } \mathbb{Z}^n.$$

EXAMPLE 3.

Suppose $\Sigma = \{1,2\}$. Let Y be the subshift in $\Sigma^{\mathbb{Z}}$ defined by the condition that all blocks of 2′s which have maximal length have even length, which is called the even shift (cf. [DGS; p. 251]). It is a sofic subshift but not conjugate to a topological Markov shift. One easily sees for $\mu = (\mu_1, \ldots, \mu_k) \in Y^*$

$$S_\mu^* S_\mu = \begin{cases} 1 & \text{if} \quad \mu = (2,\ldots,2), \\ S_1^*S_1 & \text{if} \quad \mu = (*,\ldots,*,1) \text{ or } \mu = (*,\ldots,*,1,\underbrace{2,\ldots,2}_{\text{even}}) \\ S_2^*S_1^*S_1S_2 & \text{if} \quad \mu = (*,\ldots,*,1,\underbrace{2,\ldots,2}_{\text{odd}}). \end{cases}$$

Put

$$P_1 = S_1^*S_1 - P_2, \qquad P_2 = S_1^*S_1 \cdot S_2^*S_1^*S_1S_2 \quad \text{and} \quad P_3 = S_2^*S_1^*S_1S_2 - P_2$$

so that one has $P_1 + P_2 + P_3 = 1$. Hence one sees

$$A_l = A_Y = \mathbb{C}P_1 \oplus \mathbb{C}P_2 \oplus \mathbb{C}P_3, \quad l \geq 2$$

and hence $m(l) = 3, l \geq 2$. This means that

$$\mathbb{Z}_Y = K_0(A_Y) = \mathbb{Z}[P_1] \oplus \mathbb{Z}[P_2] \oplus \mathbb{Z}[P_3] \cong \mathbb{Z}^3.$$

It is easy to see that the adjancy operator $\lambda_A (= \lambda_l)$ is the homomorphism on Z^3 given by the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Thus we have

$$K_0(\mathscr{F}_Y^\infty) \cong \varinjlim \left(\mathsf{Z}^3, \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right) \cong \mathsf{Z} \oplus \mathsf{Z} + \frac{1+\sqrt{5}}{2}\mathsf{Z} \quad \text{in } \mathsf{R} \oplus \mathsf{R},$$

$$K_0(\mathscr{O}_Y) \cong \mathsf{Z}^3 \Big/ \left(1 - \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right) \mathsf{Z}^3 \cong \mathsf{Z},$$

$$K_1(\mathscr{O}_Y) \cong \text{ Ker } \left(1 - \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right) \text{ in } \mathsf{Z}^3 \cong \mathsf{Z}.$$

Other concrete examples which are not sofic subshifts will be dealt with in some papers (cf. [KMW]).

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